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The stability of the Pexider equation for set–valued functions

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birtthday

ABSTRACT. The Hyers-Ulam stability of the Pexider functional equation is studied for set-valued functions.

Let $(M,+)$ be an Abelian semigroup and let $(Y,\|\cdot\|)$ be a Banach space. It is well known that if ε is a non-negative real number and a map $f\colon M \longrightarrow Y$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for $x,y \in M$, then there exists a unique additive function $a\colon M \longrightarrow Y$ such that

$$\|a(x) - f(x)\| \le \varepsilon$$

for $x \in M$ (see [2]).

In this paper we give an analogue of this result for the Pexider functional equation for set-valued functions. The stability of the Pexider equation has been studied in K. Nikodem's paper [3] for single-valued functions.

Let $(Y, \|\cdot\|)$ be a normed space. The functional

$$d(A,B) = \inf\{\lambda > 0:\ A \subset B + \lambda S,\ B \subset A + \lambda S\},$$

where S is the closed unit ball in Y, is a metric in the set of all non-empty closed bounded subsets of Y. The functional d is said to be the Hausdorff metric.

The following lemmas collect the main properties of d.

LEMMA 1.

(a) $d(A+C,B+C) = d(A,B)$,

(b) $d(\lambda A,\lambda B) = |\lambda|\ d(A,B)$

for A,B,C *from the space* cc(Y) *of all non-empty compact convex subsets of* Y *and for any real number* λ.

For the proof of (a) see [4], (b) is a simple corollary from the definition of d.

LEMMA 2. (see [1]). *If* $(Y, \|\cdot\|)$ *is a Banach space, then the metric space* (cc(Y),d) *is complete.*

Now we shall give the main theorem of the paper.

THEOREM. *Let* F,G *and* H *be some set-valued functions from an Abelian semigroup* (M,+) *with neutral element zero into the family of all non-empty compact convex subsets of a Banach space* $(Y, \|\cdot\|)$ *and let* d *denote the Hausdorff metric in* $(Y, \|\cdot\|)$. *If*

$$d(F(x+z),\ G(x) + H(z)) \le \varepsilon \tag{1}$$

for some $\varepsilon \ge 0$ *and for all* $x,z \in M$, *then there exists a unique additive set-valued function* A *with non-empty compact convex values in* Y *such that*

$$d(A(x) + F(0),\ F(x)) \le 4\varepsilon, \tag{2}$$

$$d(A(x) + G(0), G(x)) \le 4\varepsilon \tag{3}$$

and

$$d(A(x) + H(0), H(x)) \le 4\varepsilon \tag{4}$$

for every $x \in M$.

Proof. At first we shall prove that

$$d(F(x+y) + F(0), F(x) + F(y)) \le 4\varepsilon. \tag{5}$$

Setting in (1) $z = 0$ we obtain

$$d(F(x), G(x) + H(0)) \le \varepsilon. \tag{6}$$

Similarly, setting in (1) $x = 0$ and $z = y$ we obtain

$$d(F(y), G(0) + H(y)) \le \varepsilon. \tag{7}$$

Now using first the triangle inequality and next Lemma 1 (a) we can write

$$\begin{aligned} d(F(x+y) + F(0), F(x) + F(y)) \\ &\le d(F(x+y) + F(0), G(x) + H(y) + F(0)) \\ &+ d(G(x) + H(y) + F(0), G(x) + H(y) + G(0) + H(0)) \\ &+ d(G(x) + H(y) + G(0) + H(0), F(x) + H(y) + G(0)) \end{aligned}$$

$$+\ d(F(x) + H(y) + G(0),\ F(x) + F(y))$$

$$= d(F(x+y),\ G(x) + H(y)) + d(F(0),\ G(0) + H(0))$$

$$+\ d(H(0) + G(x),\ F(x)) + d(G(0) + H(y),\ F(y)).$$

Thus by (1), (6) and (7) we obtain (5).

Setting in (5) $y = x$ we obtain

$$d(F(2x) + F(0),\ 2F(x)) \le 4\varepsilon$$

whence, by Lemma 1 (b),

$$d(2^{-1}F(2x) + 2^{-1}F(0),\ F(x)) \le 2\varepsilon.$$

It can be showed by induction that for every positive integer n we have

$$d\left(2^{-n}F(2^n x) + (1-2^{-n})F(0),\ F(x)\right) \le (1-2^{-n})4\varepsilon. \qquad (8)$$

Write

$$A_n(x) := 2^{-n}F(2^n x)$$

for every $x \in S$ and for every positive integer n. It follows by (8) that

$$d(A_n(x),\ F(x)) \le d(A_n(x),\ A_n(x) + (1-2^{-n})F(0))$$

$$+ d(A_n(x) + (1-2^{-n})\ F(0),\ F(x))$$

$$\le (1-2^{-n})(\|F(0)\| + 4\varepsilon).$$

Let k be a positive integer. It is easy to see that

$$d\left(A_{n+k}(x), A_k(x)\right) = d\left(2^{-n-k}F(2^{n+k}x), 2^{-k}F(2^k x)\right)$$

$$= 2^{-k}d\left(A_n(2^k x), F(2^k x)\right)$$

$$\le 2^{-k}(1-2^{-n})(\|F(0)\| + 4\varepsilon),$$

which means that for every $x \in M$ the sequence $(A_n(x))$ is a Cauchy sequence in cc(Y) and consequently convergent. Let A(x) be its limit. With respect to (5) we have

$$d(A(x+y), A(x) + A(y)) \le d(A(x+y), A(x+y) + 2^{-n}F(0))$$

$$+ d(A(x+y) + 2^{-n}F(0),\ A_n(x+y) + 2^{-n}F(0))$$

$$+ d\left(2^{-n}F(2^{-n}(x+y)) + 2^{-n}F(0),\ 2^{-n}F(2^n x) + 2^{-n}F(2^n y)\right)$$

$$+ d(A_n(x) + A_n(y),\ A(x) + A(y))$$

$$\le 2^{-n}\|F(0)\| + d(A(x+y), A_n(x+y))$$

$$+ 2^{-n}\ 4\varepsilon + d(A_n(x),\ A(x)) + d(A_n(y),\ A(y)).$$

The above inequalities imply the relation

$$A(x+y) = A(x) + A(y).$$

Thus A is an additive set-valued function.

According to (8) we get

$$d(A(x) + F(0), F(x)) \leq d(A(x) + F(0), A_n(x) + F(0))$$

$$+ d(A_n(x) + F(0), A_n(x) + (1-2^{-n})F(0))$$

$$+ d(A_n(x) + (1-2^{-n})F(0), F(x)) = d(A(x), A_n(x))$$

$$+ 2^{-n}\|F(0)\| + (1-2^{-n})4\varepsilon$$

Hence letting $n \longrightarrow \infty$, we obtain (2).

Conditions (6) and (7) imply

$$d\left(2^{-n}F(2^nx),\ 2^{-n}G(2^nx) + 2^{-n}H(0)\right) \leq 2^{-n}\varepsilon$$

and

$$d\left(2^{-n}F(2^nx),\ 2^{-n}G(0) + 2^{-n}H(2^nx)\right) \leq 2^{-n}\varepsilon.$$

Consequently

$$d(A(x), 2^{-n}G(2^nx)) \leq d(A(x), 2^{-n}F(2^nx))$$

$$+ \; d(2^{-n}F(2^n x), 2^{-n}H(0) + 2^{-n}G(2^n x))$$

$$+ \; d(2^{-n}H(0) + 2^{-n}G(2^n x),\; 2^{-n}G(2^n x))$$

$$\le d(A(x), A_n(x)) + 2^{-n}\varepsilon + 2^{-n}\|H(0)\|$$

whence

$$A(x) = \lim_{n \to \infty} 2^{-n}G(2^n x). \tag{9}$$

Similarly we can prove that

$$A(x) = \lim_{n \to \infty} 2^{-n}H(2^n x). \tag{10}$$

Now using (1) we shall show the following inequality

$$d(G(x+y) + G(0),\; G(x) + G(y)) \le 4\varepsilon. \tag{11}$$

Indeed

$$d(G(x+y) + G(0),\; G(x) + G(y))$$

$$\le d(G(x+y) + G(0) + H(0), F(x+y) + G(0)) + d(F(x+y), G(x) + H(y))$$

$$+ \; d(G(x) + H(y) + G(0),\; F(y) + G(x))$$

$$+ \; d(F(y) + G(x),\; G(x) + G(y) + H(0)) \le 4\varepsilon.$$

Similarly, we can show that

$$d(H(x+y) + H(0),\ H(x) + H(y)) \le 4\varepsilon. \tag{12}$$

Setting in (11) and (12) y = x we obtain

$$d(G(2x) + G(0),\ 2G(x)) \le 4\varepsilon \tag{13}$$

and

$$d(H(2x) + H(0),\ 2H(x)) \le 4\varepsilon.$$

This implies that

$$d\left(2^{-1}G(2x) + 2^{-1}G(0),\ G(x)\right) \le 2\varepsilon$$

and

$$d\left(2^{-1}H(2x) + 2^{-1}H(0),\ F(x)\right) \le 2\varepsilon.$$

It can be proved by induction that

$$d\left(2^{-n}G(2^{n}x) + (1-2^{-n})G(0),\ G(x)\right) \le (1-2^{-n})4\varepsilon \tag{13}$$

and

$$d\left(2^{-n}H(2^{n}x) + (1-2^{-n})H(0),\ H(x)\right) \le (1-2^{-n})4\varepsilon. \tag{14}$$

From (9), (10) and (13) we obtain

$$d(A(x) + G(0),\ G(x)) \leq d(A(x) + G(0), 2^{-n}G(2^n x) + G(0))$$

$$+\ d(2^{-n}G(2^n x) + (1-2^{-n})G(0) + 2^{-n}G(0),\ 2^{-n}G(2^n x) + (1-2^{-n})G(0))$$

$$+\ d(2^{-n}G(2^n x) + (1-2^{-n})G(0),\ G(x)) \leq d(A(x),\ 2^{-n}G(2^n x))$$

$$+\ 2^{-n}\|G(0)\| + (1-2^{-n})4\varepsilon.$$

This implies (3). Similarly we can verify (4).

To demonstrate uniqueness of A suppose that (2) holds with an additive set-valued function $A\colon M \longrightarrow cc(Y)$. By induction we can show that

$$A(nx) = nA(x)$$

for $x \in M$ and positive integers n, whence by (2) we have

$$d(nA(x) + F(0),\ F(nx)) \leq 4\varepsilon$$

Dividing by n and passing to the limit as $n \longrightarrow \infty$ we obtain

$$A(x) = \lim_{n \to \infty} \frac{1}{n} F(nx).$$

This proves the uniqueness of A and completes the proof.

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