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On the notion of a geometric object in a Klein space

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

ABSTRACT. Let G be a group and let M be an object of the topos G-Set. Let (C) denote the following condition: X is isomorphic to some subobject of one eof the objects $M_n = \underbrace{P(\ldots(P(M)))}_{n \text{ times}}$. We prove the main result that (C) holds is and only if card X < sup{card M_n:N∈ℕ = $\chi(M)$ and {G∈G: ∀m∈M gm=m} ⊆ {g∈G: ∀x∈G: ∄x∈X gx=x}. If G is infinite then Γ(G) denote the least cardinal number α for which the power of the set of homomorphism of G into the group of bijections of the set of the power α is greater than the power of G. If G is finite then we put Γ(G)=0. We prove that under the assumption Γ(G) < $\chi(M)$ if (C) holds for all X of the power less than $\chi(M)$ then {g∈G: ∀m∈M gm=m} = {e}. In the paper we formulate the above theorems in the language of Klein's geometry.

E.J. Jasińska and M. Kucharzewski ([2], [3]), in their attempt to formulate the notion of geometry in Klein's sense more precisely, defined a Klein space as a triplet (M,G,f), where M is an arbitrary non-empty set (called a fibre), (G.,e) is a group and f defines an operation of the group G on the set M, namely
$f : M \times G \longrightarrow M$,

$$\forall\, x \in M \; f(x,e) = x, \tag{1}$$

$$\forall\, x \in M \; \forall\, g_1, g_2 \in G \quad f(f(x,g_1),g_2) = f(x,g_2g_1), \tag{2}$$

which is effective, i.e.

$$\forall\, g \in G \; ((\forall\, x \in M \; f(x,g) = x) \Rightarrow g = e).$$

They called a triplet (X,G,F), where X is an arbitrary non-empty set and F defines an operation of the group G on the set X, a geometric object of the Klein space (M,G,f). This definition gives rise to two reservations.

1) There is no "a priori" connection between the fibre of an object X and the fibre of a Klein space, as well as between the operation f of the group G on M and the operation F of this group on X.

2) The role played by the effectivity assumption is not clearly seen.

In order to solve the first problem B. Szocinski proposed in [5] some modification, namely a restricted definition of a geometric object.

That definition allowed finally to settle a connection between the fibre of a Klein space and the fibre of its geometric object as well as between the operations of the group on those fibres. This connection is established in Theorem 1 of the present paper. Theorem 2 explains the role of the effectivity assumption in the definition of a Klein space. The results contained in these theorems can be expressed in terms of notions of the topoi theory (power object, subobject). As an introduction to the topoi theory we may serve a book [1].

Let us recall the terminology of the paper [5].

By an abstract object we mean the triplet (M,G,f), where $(G,\cdot,e)$ is a group and $f\colon M \times G \longrightarrow M$ is a function satisfying (1) and (2).

If the operation f is effective, then the triplet (M,G,f) is called a Klein space.

Two abstract objects (M_1,G,f_1), (M_2,G,f_2) are called equivalent if there exists a bijective $h\colon M_1 \longrightarrow M_2$ which satisfies the following condition:

$$\forall\, m \in M\ \forall\, g \in G \qquad f_2(h(m),g) = h(f_1(m,g)).$$

An abstract object (S,G,I), where $I(s,g) := s$ for any $s \in S$, $g \in G$, is called a scalar.

If (M,G,f) is an abstract object, $\tilde{M} \subset M$, $\tilde{M} \neq \phi$, and $f(m,g) \in \tilde{M}$ for each $m \in \tilde{M}$ and $g \in G$ (in this case the set $\tilde{M}$ is called an invariant) then the triplet $(\tilde{M},G,\tilde{f})$, where $\tilde{f} = f|_{\tilde{M}\times G}$, is an abstract object. This object is called a partial object of the abstract object (M,G,f) determined by an invariant subset $\tilde{M}$.

Let (X,G,f) be a given abstract object. We denote by $P(X)$ the family of all subsets of the fibre of this object.

The transformation:

$$F^*\colon P(X) \times G \longrightarrow P(X),$$

given by the formula

$$F^*(A,g) := F(A,g) = \{F(x,g) : x \in A\},$$

is obviously an operation of the group G on the set $P(X)$ where as the triplet $(P(X),G,F^*)$ is an abstract object. The standard geometric object of rank k of Klein space (M,G,f) is an abstract object $(\Omega^{(k)}(M),G,f^{(k)})$ ($k \in \mathbb{N}$) defined by the following conditions:

a) for k = 1 this object is the object of all subsets of the fibre of the Klein space (M,G,f) i.e.

$$\Omega^{(1)}(M) := P(M) \text{ and } f^{(1)} = f^*,$$

b) the object $(\Omega^{(m+1)}(M),G,f^{(m+1)})$ is the object of all subsets of the fibre of the object $(\Omega^{(m)}(M),G,f^{(m)})$ i.e.

$$\Omega^{(m+1)}(M) := P(\Omega^{(m)}(M)) \text{ and } f^{(m+1)} = (f^{(m)})^*.$$

The abstract object (X,G,F) which is equivalent to some partial object of a standard geometric object is called the geometric object of the Klein space (M,G,f).

The following Theorem 1 strengthens the conclusion of the paper [6] in which it was showed that for the Klein space (M,G,f) the scalar (M,G,I) is its geometric object.

THEOREM 1. *If* (M,G,f) *is a Klein space,* (X,G,F) *an abstract object,* card X < χ (M) := sup {card M, card $\Omega^{(1)}(M)$, card $\Omega^{(2)}(M)$,...}, *then* (X,G,F) *is the geometric object of* (M,G,f).

Proof. Let us denote by Ord the class of ordinals numbers. We fix $\delta \in$ Ord such that : card M = card δ. It is easy to check that using the effectivity of f for any wellordering $(m_\gamma)_{\gamma<\delta}$ of M the transitive fibre* of $\Omega^2(M)$ determined by

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*)Without effectivity of f we can prove that the non-effectivity group of this fibre is equal to the non-effectivity group of the object (M,G,f) (it is obvious that this group include the non-effectivity group of the object (M,G,f)).

$$\left\{(m_\gamma : \gamma < \beta) : 0 \le \beta \le \delta\right\} \in \Omega^2(M)$$

is equivalent to the object (G,G,L) where $L(a,b) = ba$.

In this way we obtain that the object (G,G,L) is a geometric object of rank at most 2.

Z. Moszner presented the author with the following proof of the condition (W1):

For any $p \in \mathbb{N}$ and $p \ge 3$, any group G and any Klein space (M,G,f), every transitive abstract object (X,G,F) is equivalent to some partial object of the standard geometric object of rank p.

If the object (X,G,F) is transitive, then it is equivalent to the object $(\{aG_0,\ a \in G\},G,L)$, where G_0 is some subgroup G and $L(aG_0,b) = baG_0$. Although it follows from [4], for the reader's convenience we present the direct proof. Let x_0 be a fixed element of X. A set $G_0 := \{a \in G : F(x_0,a) = x_0\}$ is then a subgroup of G, because $F(x_0,e) = x_0$ implies $e \in G_0$ and when $a,b \in G_0$, then $F(x_0,ab^{-1}) = F(F(x_0,b),ab^{-1}) = F(x_0,a) = x_0$. Thus $ab^{-1} \in G_0$. The function $h(aG_0) = F(x_0,a)$ is a well defined bijection of a set $\{aG_0,\ a \in G\}$ onto X. Indeed, when $aG_0 = bG_0$ then $a^{-1}b \in G_0$. Hence $F(x_0,a^{-1}b) = x_0$, then:

$$F(x_0,a) = F(F(x_0,a^{-1}b),a) = F(x_0,b).$$

When $F(x_0,a) = F(x_0,b)$, then

$$x_0 = F(x_0,a^{-1}a) = F(F(x_0,a),a^{-1}) = F(F(x_0,b),a^{-1}) = F(x_0,a^{-1}b).$$

Hence $a^{-1}b \in G_0$ and $aG_0 = bG_0$.

By the transitivity, h is a mapping onto X.

Moreover, we have:

$$h(L(aG_0,b)) = h(baG_0) = F(x_0,ba) = F(F(x_0,a),b) = F(h(aG_0),b).$$

which proves the equivalence of objects (X,G,F) and $(\{aG_0 : a \in G\}, G,L)$. The latter object is a partial object of a standard geometric object of rank 1 of the object (G,G,L). This implies that the object $(\{aG_0 : a \in G\},G,L)$ and equivalent to it the transitive object (X,G,F) are geometric objects of rank at most 3, hence of rank at most $p \geq 3$. This means that the condition (W1) is fulfilled.

Let (X,G,F) be a nontransitive abstract object. The fibre X is a disjoint union of orbits of the operation F, that is sets $F(x,G)$ for $x \in X$, for which the object (X,G,F) is transitive. Let $\alpha \in \mathrm{Card}$ denote the number of orbits of the operation F.

We have:

$$\alpha \leq \mathrm{card}\ X < \chi\ (M). \tag{3}$$

We shall prove that there exists $k \in \mathbb{N}$ obeying the following property (W2):

for all $l \geq k$, α is less or equal to the number of orbits of the operation of the standard geometric object of rank l.

First we prove the property (W2) in the case where M is infinite.

Let us take $k \geq 2$ for which $\alpha \leq \mathrm{card}\ \Omega^k(M)$. Then for $l \geq k$ we have:

$$\alpha \leq \mathrm{card}\ \Omega^k(M) \leq \mathrm{card}\ \Omega^l(M).$$

Now it suffices to show that for $l \geq 2$ the family of orbits of the

operation of a standard geometric object of rank l is equipotent to the set $\Omega^l(M)$. We have : card $G \leq$ card M^M = card 2^M, because the operation f is effective. Hence for any operation of the group G, the power of every orbit does not exceed card 2^M.
For $l \geq 2$ we have

$$\text{card } \Omega^l(M) \geq \text{card } \Omega^2(M) > \text{card } 2^M \geq \text{card } G.$$

Since a fibre $\Omega^l(M)$ is a union of orbits, we conclude that the family of orbits of the same object cannot have the power less than card $\Omega^l(M)$. Therefore the family of orbits of the operation of the standard geometric object of rank l is equipotent to the set $\Omega^l(M)$. This completes the proof of the property (W2) for infinite M.
Now we present the proof of (W2) for finite M. In virtue of (3) α is a positive integer. Let Bij(M) denote the group of the bijections of the set M. Since the operation f is effective:

$$\text{card } G \leq \text{card Bij}(M) < \aleph_0 .$$

Thus the power of G is a positive integer. The power of every orbit is less than or equal to card G.
The positive integer k for which:

$$\frac{2^{2^{\cdot^{\cdot^{\cdot^{2^{\text{card } M}}}}}}\Big\}\ k \text{ times}}{\text{card } G} \geq \alpha,$$

will satisfy (W2).

Now let $s = \max(3,k)$. It follows that every transitive abstract object with the group G is equivalent to some partial object of the object $\Omega^s(M)$.

Let $\{X_\gamma : \gamma \in I\}$ denote the family of orbits of the operation F (I is a set of indices). We have : card $I = \alpha$. By R_s we denote the family of orbits of the object $(\Omega^s(M),G,f^{(s)})$. Since $s \geq k$ from (W2) we conclude that there exists an injection : $j : I \longrightarrow R_s$. For $\gamma \in I$ we define the injection $i_\gamma : \Omega^s(M) \longrightarrow \Omega^{s+2}(M)$ by : $i_\gamma(x) = \{\{x\},\{x\}\cup j(\gamma)\}$. The range of the injection i_γ is an invariant subset of a fibre of the object $\Omega^{s+2}(M)$. This range is given bv:

$$\left\{\{\{x\},\ \{x\} \cup j(\gamma)\} : x \in \Omega^s(M)\right\}.$$

The abstract object determined by this range is a Klein space equivalent to a standard geometric object of rank s.

If $\gamma_1 \in I$, $\gamma_2 \in I$, $\gamma_1 \neq \gamma_2$ then the Klein spaces determined by the ranges of the injections i_{γ_1}, i_{γ_2} possess disjoint fibres. A transitive abstract object determined by X_γ is equivalent to some partial object of the object

$$\left\{\{\{x\},\ \{x\} \cup j(\gamma)\} : x \in \Omega^s(M)\right\}.$$

Since the fibres are disjoint, the object (X,G,F) is equivalent to some partial object of a standard geometric object of rank s+2. This completes the proof of Theorem 1.

REMARKS.

1) The condition card $X < \chi(M)$ in Theorem 1 can be replaced by an equivalent condition stating that a number of transitive fibres of the object (X,G,F) is less than $\chi(M)$.

2) Our proof of Theorem 1 can estimate the rank of the abstract object (X,G,F) by $s+2$, where $s = \max(3,k)$.

3) We can prove Theorem 1 using the effectivity of f only to prove that (G,G,L) is a geometric object. In order to prove this, we observe that property (W2) is identical for the abstract object (M,G,f) and for the Klein space with the fibre M and group $G/\tilde{G}$, where $\tilde{G}$ denotes the non-effectivity group of the object (M,G,f). So, we can prove (W2) without effectivity, which ends the proof.

4) So, without effectivity of f we can prove that each abstract object (X,G,F) such that card $X < \chi(M)$ and its non-effectivity group include the non-effectivity group of the object (M,G,f) is a geometric object of the object (M,G,f).

Now we go on to inverting, in some sens, of Theorem 1. Let us adopt the following definition:

If the group G is infinite, then by $\Gamma(G)$ we denote the least cardinal number α for which the power of the set of homomorphisms of G into the group of bijections of set consisting of α elements is greater then the power of G. If G is finite, we take $\Gamma(G) = 0$.

Now we prove:

LEMMA. $\Gamma(G) \leq$ card G.

Proof. For the groups G_1, G_2 we denote by $\mathrm{Hom}(G_1,G_2)$ the set of homomorphisms of the group G_1 into the group G_2. It is sufficient to show that for infinite groups G: $\mathrm{card}\left(\mathrm{Hom}(G,\mathrm{Bij}(G))\right) = 2^{\mathrm{card}\ G}$, because $2^{\mathrm{card} G} > \mathrm{card}\ G$. For $g \in G$ we define left translation $L_g : G \longrightarrow G$ by equality:

$$L_g(x) = g \cdot x.$$

Let $H \subset Bij(G)$ denote the subgroup of right translations of G. We have:

$$\text{card } H = \text{card } G, \ \text{card}(Bij(G)) = 2^{\text{card } G}.$$

Hence:

$$\text{card}(Bij(G)/H) = 2^{\text{card } G}, \text{ where } Bij(G)/H := \{hH : h \in Bij(G)\}.$$

It is sufficient to observe that the mapping ϕ:

$$Bij(G)/H \ni hH \xrightarrow{\phi} \{G \ni g \longrightarrow h \circ L_g \circ h^{-1} \in Bij(G)\} \in Hom(G,Bij(G))$$

is injective.

THEOREM 2. *Let* $\Gamma(G) < \chi(M)$ *and the property* (W3) *hold: every object* (X,G,F) *for which* card $X < \chi(M)$ *is a* (W3) *geometric object.*

Then (M,G,f) *is a Klein space.*

REMARKS.

1) The condition $\Gamma(G) < \chi(M)$, is trivially fulfilled if (M,G,f) is a Klein space, because in that case our Lemma asserts that: $\Gamma(G) \le \text{card } G \le \text{card } Bij(M) < \chi(M)$.

2) Let us denote by $\tilde{G}$ the non-effectivity group of the object (M,G,f). Because we can prove Theorem 1 using effectivity of f only to the transitive objects (X,G,F), we obtain that property (W3) holds if and only if for each normal subgroup $H \subset G$: $(G:H) < \chi(M)$ implies that $\tilde{G} \subset H$.

Proof of Theorem 2. We begin by proving that card $G < \chi(M)$. For a finite group G this condition is evident. In the case of infinite groups we suppose by contradiction, that card $G \geq \chi(M)$.

Consequently:

(4) the power of the family of partial objects of the objects $\Omega^n(M)$, $n \in \mathbb{N}$, does not exceed card G.

We will analyse the number of nonequivalent abstract object with the power of the fibre equal to $\Gamma(G)$. Every abstract object with the group G and of the power the fibre $\mathfrak{m}$ is equivalent at most to $\mathrm{card}(\mathrm{Bij}(\mathfrak{m}))$ abstract objects with the same fibre.

We have:

$$\Gamma(G) < \chi(M) \leq \mathrm{card}\ G.$$

Hence:

$$\mathrm{card}\left(\mathrm{Bij}(\Gamma(G))\right) < \chi(M) \leq \mathrm{card}\ G.$$

The definition of $\Gamma(G)$ implies that:

$$\mathrm{card}\left(\mathrm{Hom}\langle G, \mathrm{Bij}(\Gamma(G))\rangle\right) > \mathrm{card}\ G$$

This inequality means that the number of abstract objects with a fixed fibre of the power $\Gamma(G)$ is greater than the number of elements of G. The class of equivalence of an abstract object in relation to the equivalence relation of objects on the family of abstract objects with the same fibre of the power equal to $\Gamma(G)$ has a power

$$\kappa \le \mathrm{card}\Big(\mathrm{Bij}(\Gamma(G))\Big) < \mathrm{card}\ G$$

Therefore the number of nonequivalent abstract objects with the power of the fibre equal to $\Gamma(G)$ is greater than the number of elements of the group G. Thus, in virtue of (4), the property (W3) does not hold, which contradicts our assumption. We have just proved that:

$$\mathrm{card}\ G < \chi(M).$$

Hence the object (G,G,L), where $L(x,g) = g \cdot x$ is for some $n \in \mathbb{N}$ equivalent to some partial object of the object $\Omega^{(n)}(M)$. Were (M,G,f) not a Klein space, it would mean that the operation f is not effective. Thus, the operation f has at least two distinct unities e_1 and e_2. If e_1 and e_2 are the unities of the operation f, then they are unities of the operation f^*, f^{**},.. as well. Consequently the operations of all objects $\Omega^{(n)}(M)$, $n \in \mathbb{N}$ have two distinct unities e_1 and e_2. Hence, operations in all partial objects of $\Omega^{(n)}(M)$, $n \in \mathbb{N}$, have two distinct unities e_1 and e_2. Since the operation L of the object (G,G,L) has precisely one unity, we arrived at contradiction. This completes the proof of Theorem 2.

The question arises:

For which groups G and abstract objects (M,G,f) does the property (W3) of Theorem 2 hold if we assume that $\Gamma(G) \ge \chi(M)$?

We present a partial answer, the property (W3) is fulfilled trivially if there exist only a trivial operation of G on the sets of the power less than $\chi(M)$. It is equivalent to the non-existence of the normal subgroup $H \subset G$ such that $1 < (G:H) < \chi(M)$.

It is easily seen that under the assumption $\Gamma(G) \geq \chi(M)$ this condition is particularly fulfilled if G is a simple group or if G is a divisible group of the power less than $\sup\{\mathrm{card}\ \aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots\}$.

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