ANDRZEJ ZAJTZ

Lifting pseudogroups actions in fibre bundles

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

1. Introduction Let $\Pi: N \longrightarrow M$ be a fibred manifold and Γ a pseudogroup of local diffeomorphisms of M. A lifting of Γ to the fibre space N is a functorial assignement to each $f \in \Gamma$ a local diffeomorphism f' of N preserving the fibration and covering f. This means that if f maps an open set $U \subset M$ onto the open set $V \subset M$ then f' maps $\pi^{-1}(U)$ onto $\pi^{-1}(V)$ and $\pi \circ f' = f \circ \pi$. Moreover $(g \circ f)' = g' \circ f'$ and $(id_U)' = id_{\pi^{-1}(U)}$.

The case of the lifting of Lie transformation groups acting transitively on the base manifold was considered for instance by T.E. Stewart [9] and H. Kuiper - K. Yano [2]. The lifting of all local diffeomorphisms of M (Γ = Diff(M)) was a background of the notion of natural bundles introduced by A. Nijenhuis [6] and gave rise to a modern formulation of the theory of geometric objects. It is due to R.S. Palais and C.L. Terng [7] that we know that such lifings are of finite order, that is, the values of f' on a fibre $\pi^{-1}(x)$ depend only on a certain jet of f at x. An estimation of the order in question can be found in [7], improved by D.B.A. Epstein and W.P. Thurston [1]. The sharp bound on the order of natural bundles was finally given by the author in [10]. It is a function of the dimension of M and the fibre dimension of N. In the meantime quite a number of new results have been appeared concerning the finiteness of liftings in particular cases.

In this note we will be concerned with this type of problem for liftings of analytic Lie pseudogroups acting on (real) analytic manifolds. The fibre manifold N will be assumed to be of class C^∞. We shall give a certain characterization of analytic Lie pseudogroups as well as a functorial approach to the lifting problem adequate to geometrical applications.

2. Smooth pseudogroups of local analytic diffeomorphisms. Let Γ be a Lie pseudogroup of order s of local bi-analytic maps on a real analytic manifold M. For the relevant definitions we refer to the paper by A.M. Rodrigues [8]. We shall say that Γ is smooth if the set $j^r\Gamma$ of all r-jets of elements of Γ is a submanifold of j^rM - the manifold of all r-jets of local diffeomorphisms of M. Denote by $\alpha, \beta : j^rM \longrightarrow M$ respectively the source and target projections.

A local analytic vector field X is called a Γ vector field if the (analytic) transformations f_t of the one parametr local group generated by X belong to Γ for every t. By $L(\Gamma)$ we denote the sheaf over M of germs of Γ vector fields. $L(\Gamma)$ is a sheaf of Lie algebras (see [8]) by the same argument as in the case of C^∞- Γ-fields. $J^rL(\Gamma)$ will denote the set of r-jets of Γ-fields.

Let $i\colon M \longrightarrow j^rM$ be the canonical section of r-jets of the identities. Denote by $i * Vj^rM \cong j^rTM$ the pull-back of α-vertical tangent spaces to j^rM. The indicated isomorphism is established canonically via the r-th prolongation of vector fields

$$j^r_x X \longrightarrow j^r_{l(x)} p^r X$$

with $p^r X = d/dt(j^r f_t)$, $X = d/dt(f_t)$.

$J^r\Gamma$ being a submanifold of $j^r M$, it determines a vector subbundle $E^r(\Gamma)$ of $J^r TM$ defined to be $i^* V j^r \Gamma$.

By definition of a Lie pseudogroup of order s, Γ is the set of solutions of the non-linear Lie equation $j^s\Gamma$. Then $E^s(\Gamma)$ is the associated linear Lie equation and $L(\Gamma)$ is the set of solutions of $E^s(\Gamma)$. Recall that Γ is called a *regular* Lie pseudogroup of order s if $j^s L(\Gamma) = E^s(\Gamma)$. This means that the Lie equation $E^s(\Gamma)$ is completely integrable.

We define the prolongation of $J^r\Gamma$ to be

$$p^{J^r}\Gamma = J^1(J^r\Gamma) \cap J^{r+1}M.$$

2.1. PROPOSITION. *Let Γ be a smooth analytic pseudogroup acting transitively. There exists an integer k_o such that for each $k \geq k_o$ it holds*

(i) $p^{J^k}\Gamma = J^{k+1}\Gamma$,

(ii) Γ *is a regular Lie pseudogroup of order* k.

Proof. Obviously $J^{r+1}\Gamma \subset p^{J^r}\Gamma$. The natural jet-projection $J^{r+1}\Gamma \longrightarrow J^r\Gamma$ is a surjective homomorphism of Lie groupoids. Hence the projection is a submersion [5]. Thus $J^{r+1}\Gamma$ is a fibration over $J^r\Gamma$ for each r. By the prolongation theorem of Cartan-Kuranishi and in view of the transitivity of Γ there exists an integer k_o such that for each $k \geq k_o$ $J^k\Gamma$ is an involutive system of partial differential equations and $J^{k+1}\Gamma = pJ^k\Gamma$. Since Γ is analytic, it follows that Γ is a regular Lie pseudogroup of order k, Q.E.D.

Let O be a point of M to be called an origin. Denote by D_k the vertex group at O of the groupoid $J^k M$ and put $D = D_\infty$. D is the limit of the inverse system of Lie groups (D_k, j^k_l) with j^k_l being the jet-projection D_k onto D_l for $k \geq l$. We endow D with the inverse limit topology. Let d_k be the Lie algebra of all k-jets at 0 of vector fields on M vanishing at the origin and put $d = d_\infty$. d_k is the Lie algebra of D_k with the exponential map $j^k_o X \longrightarrow j^k_o(\exp X)$, where exp X denotes the flow of X on M. Passing to the inverse limit we can define the exponential map $\exp : d \longrightarrow D$.

Let $G = J^\infty_o\Gamma \cap D$ be the group of infinite jets of all Γ-transformations preserving the origin. We put also $g = j^\infty_o L(\Gamma) \cap d$. Analogously we define the (Lie) groups G_k and Lie algebras g_k. G is contained in the inverse limit of the system (G_k, j^k_l) and endows its topology from D.

The symbols H_k and h_k are defined by the following exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H_k & \longrightarrow & G & \xrightarrow{j_k} & G_k & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & h_k & \longrightarrow & g & \xrightarrow{j_k} & g_k & \longrightarrow & 0,
\end{array}
$$

where the vertical arrows stand for exponential maps.

Obviously $g_k \subset L(G_k)$. In the case where Γ is regular of order k, the equality $j^k L(\Gamma) = E^k(\Gamma)$ means that $g_k = L(G_k)$.

2.2. LEMMA. *If* $J^{k+1}\Gamma = pJ^k\Gamma$ *then* H_k/H_{k+1} *is a connected Lie group.*

Proof. H_k/H_{k+1} is isomorphic to the Lie group $\ker(G_{k+1} \longrightarrow G_k)$. We know that the prolongation $pJ^k\Gamma$ is an affine bundle over $J^k\Gamma$. Its fibre over $j^k_o\mathrm{id}$ is exactly (under our assumption) the kernel group mentioned above. Therefore it is topologically a vector space. Hence our conclusion.

Summerizing the results of this section we can conclude that the topological group G satisfies, in the case where Γ is a smooth analytic and transitive Lie pseudogroup, the following conditions:

1. G has a decreasing sequence of closed normal subgroups

$$H_o = G \supset H_1 \supset \ldots \supset H_r \supset H_{r+1} \supset \ldots$$

which constitute a fundamental system of neighborhoods of the unit element.

2. $\bigcap_{r=1}^{\infty} H_k = \langle e \rangle$

3. $G_r = G/H_r$ are finite dimensional Lie groups relative to the quotient topology and G is a subgroup of $\varprojlim\{G_r\}$.

4. There exists a subalgebra g of $\varprojlim\{L(G_r)\}$ such that g_k = the k-th projection of g in the inverse system, coincides with $L(G_k)$.

5. There exists an integer k_o such that H_k/H_{k+1} is connected for all $k \geq k_o$.

For such a type of infinite Lie groups we proved in [10] the following theorem.

2.3. THEOREM. *Suppose that* G *acts continuously on a* C^∞ *-manifold* M. *Let* $x \in M$. *Then for sufficiently large* r, H_r *stabilizes* x.

Note that if G acts transitively on M then the order of the stability of the action is the same at all points of M.

Under some additional conditions on the Lie algebra g there exists an upper bound on the stability order when x varies on M.

2.4. PROPOSITION. *Suppose that the ideals* h_r *of* g *satisfy the condition*

$$[h_r,h_s] = h_{r+s} \bmod h_{r+s+1}. \tag{2}$$

If all groups H_r/H_{r+1} *are connected then* $H_{2\dim M+1}$ *acts trivially on* M [10]. *If the last property holds for* $r \geq k_o$ *then the stability order is* $\max(k_o, 2\dim M+1)$.

The additional assertion follows easily from the proof of the first part of the proposition (see [10])

2.5. REMARK. The assumption of the continuity of the action with respect to the inverse limit topology (= filtration topology) is essential. The natural action of the group of all invertible polynomials without constant terms on R^n has no finite stability order. But this action is not continuous in the above topology.

3. The order of liftings. Let Γ be a smooth pseudogroup of local bi-analytic transformations of R^n containing translations. Suppose that $\pi: E \longrightarrow R^n$ is a C^∞-fibre bundle over R^n and the assignement $f \longrightarrow Ef$ is a lifting of Γ to E as described in the introduction. From the locality property of the lifting it follows immediately that the values of Ef on the fibre over a point $x \in \mathrm{dom}(f)$ depend only on the germs of f at x. Denote by F the fibre over the origin O and by G the group of germs of Γ-transformations preserving the origin. Notabene, since the transformations are analytic this group coincides with the group of infinite jets of its elements.

There is an action of G on F defined by

$$(\bar{f},y) \longrightarrow (Ef)(y)$$

for $\bar{f}$ being the germ of f at 0 and $y \in F$.

We impose the following regularity condition on the lifting:

(R) The above action $G \times F \longrightarrow F$ is continuous with respect to the inverse limit topology in G.

Let t_v be the translation of R^n by v. For any $f \in \Gamma$ and $z \in E$ projecting onto x from the domain of f we have

$$(Ef)(z) = \left(Et_{f(x)} \circ E\left(t_{-f(x)} \circ f \circ t_x\right) \circ Et_{-x}\right)(z). \tag{3}$$

The germ of $t_{-f(x)} \circ f \circ t_x$ is in G and $Et_{-x}(z)$ is an element of F. We see from (3) that the lifting is of finite order (that is, $Ef(z)$ depends only on a finite jet of f at x) if and only if the action of G on F has a finite stability order. So, we may apply the results of section 2 to get the following

3.1. THEOREM. *Let* Γ *be a smooth analytic pseudogroup acting on* R^n *and containing translations. Suppose that a lifting of* Γ *to a fibre bundle* E *satisfies the regularity condition* (R). Then in each of the cases below the lifting has a finite order.

1. *The action of* G *on* F *is transitive or*

2. *The Lie algebra* g *of infinite jets of* Γ*-vector fields vanishing at the origin satisfies condition* (2) *of proposition* 2.4.

4. Liftings of classical analytic pseudogroups. It turns out that all calssical Cartan pseudogroups satisfy condition (2). Moreover, all of then, except the pseudogroup of contact transformations, contain translations. Let L be one of the following infinite Lie algebras:

(I) The Lie algebra of all formal (i.e. formal power series) vector fields on R^n.

(II) The Lie algebra of formal vector fields preserving the volume form $dx^1\wedge dx^2\wedge\ldots\wedge dx^n$.

(III) The Lie algebra of formal vector fields preserving the volume form up to constant factors.

(IV) The Lie algebra of formal vector fields preserving the symplectic form $dx^1\wedge dx^{n+1}+\ldots+dx^n\wedge dx^{2n}$.

(V) The Lie algebra of formal vector fields preserving the symplectic form up to constant factors.

(VI) The Lie algebra of formal vector fields preserving the contact form $dx^0 + \sum_i x^i dx^{n+i} - x^{n+i}dx^i$ up to functional factors.

Let $\{L_k\}$ be the natural filtration of L. It was proved by T. Morimito in [4] that $[L_r,L_s] = L_{r+s}$ for all r and s, r+s > 0. Let g be the subalgebra of L consisting of all convergent power series belonging to L. It is clear that the quotient Lie algebras g/h_k and L/L_k coincide. Hence $h_k/h_{k+1} = L_k/L_{k+1}$. It follows that

$$h_k = L_k \bmod h_{k+1}.$$

This relation and the result of Morimoto give condition (2).

REFERENCES

[1] Epstein D.B.A. and Thurston W.P., *Transformation groups and natural bundles*, Proc. London Math. Soc. 38 (1979), 219-237.

[2] Kuiper N. and Yano K., *On geometric objects and Lie groups of transformations*, Ind. Math. 17 (1955), 411-420.

[3] Mikulski W.M., *Locally determined associated spaces*, J. London Math. Soc. 32 (1985), 357-364.

[4] Morimoto T., *The derivation algebras of the classical infinite Lie algebras*, J. Math. Kyoto Univ. 16 (1976), 1-24.

[5] Ngo Van Que, *Du prolongement des espaces fibres et des structures infinitesimales*, Ann.Inst. Fourier Grenoble, 17 (1967), 157-223.

[6] Nijenhuis A., *Natural bundles and their properties*, Diff. Geom.in honor of K. Yano, Tokyo 1952, pp. 317-334.

[7] Palais R.S. and Terng C.L., *Natural bundles have finite order*, Topology 16 (1977), 217-277.

[8] Rodrigues A.M., *The first and second fundamental theorems of Lie for pseudogroups*, Am. J. Math. 74 (1962), 265-282.

[9] Stewart T.E., *Lifting group actions*, Ann. Math. 74 (1961), 193-198.

[10] Zajtz A., *The sharp upper bound on the order of natural bundles of given dimensions*, Bull. Soc. Math. de Belgique, 39 (1987), 347-357.

[11] Zajtz A., *Actions of some infinite Lie groups*, Aequationes Math. 34 (1987), 139-146.