ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE
Zeszyt 189 Prace Matematyczne XIV 1997

Maria E. Pliś

Singular solutions of some nonlinear singular partial differential equations

Abstract. In this paper a family of formal solutions of the Laplace-Beltrami equation is constructed and the convergence of these formal series is proved by finding the majorant convergent series.

In [1] and [2] a family of singular solutions of the nonlinear singular partial differential equations of the form

$$\begin{aligned}\left[\left(t\frac{\partial}{\partial t}\right)^{l} + c_{l-1}(x)\left(t\frac{\partial}{\partial t}\right)^{l-1} + \cdots + c_0(x)\right] u \\ = tb(x) + G_2\left(x, t, \left\{\left(t\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial x}\right)^{\alpha} u\right\}_{(j,\alpha)\in I_m}\right)\end{aligned} \tag{1}$$

is constructed. Here $t \in \mathbf{C}$, $x \in \mathbf{C}^n$, $l \geq m$, $(j,\alpha) \in \mathbf{N}_0 \times \mathbf{N}_0^n$, $I_m = \{(j,\alpha);\ j + \mid\alpha\mid \leq m,\ j < m\}$,

$$G_2(x,t,Z) = \sum_{p+|q|\geq 2} g_{pq}(x) t^p Z^q,$$

$Z = \{Z_{j\alpha}\}_{(j,\alpha)\in I_m}$, $q = \{q_{j\alpha}\}_{(j,\alpha)\in I_m}$.

The main idea of the construction is to use a majorant equation method for a family of formal solutions.

The aim of this paper is to show that in several cases the convergence of the formal solutions can be proved directly by finding the majorant convergent series.

We follow the notations used in [2], namely $\widetilde{\mathbf{C}\setminus\{0\}}$ denotes the universal covering of $\mathbf{C}\setminus\{0\}$, $S_\theta = \{t \in \widetilde{\mathbf{C}\setminus\{0\}};\ \mid \arg t \mid < \theta\}$ a sector in $\widetilde{\mathbf{C}\setminus\{0\}}$;

AMS (1991) subject classification: 35C10.

$S(\varepsilon(s)) = \{t \in \widetilde{\mathbf{C} \setminus \{0\}};\ 0 <\mid t \mid< \varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a function defined on $\mathbf{R}$ continuous and strictly positive, $\Delta_r = \{x \in \mathbf{C}^n;\ \mid x_j \mid \leq r,\ j = 1, \dots, n\}$; and $\tilde{\mathcal{O}}_+$ is the set of functions $u(t, x)$ satisfying the following conditions:

i) there exist a function $\varepsilon(s)$ defined on $\mathbf{R}$ continuous and strictly positive and a real number $r > 0$ such that u is holomorphic in $S(\varepsilon(s)) \times \Delta_r$;

ii) there exists a real number $a > 0$ such that for any $\theta > 0$

$$\max_{x \in \Delta_r} \mid u(t, x) \mid = O(\mid t \mid^a) \text{ as } t \text{ tends to zero in } S_\theta.$$

To facilitate the presentation of the method, we prove the main result in a very special case of the Laplace-Beltrami equation

$$\left[t^2 \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) + c\right] u = 0. \tag{2}$$

It is clear that equation (2) is a particular case of equations of type (1), namely we can rewrite (2) in equivalent form

$$\left[\left(t\frac{\partial}{\partial t}\right)^2 - t\frac{\partial}{\partial t} + c\right] u = -t^2 \frac{\partial^2}{\partial x^2} u. \tag{3}$$

Thus, $m = 2$, $c_1(x) \equiv -1$, $c_0(x) \equiv c$, $b(x) \equiv 0$ and $G_2(x, t, Z) = -t^2 Z_{02}$, with $g_{pq}(x) \equiv 0$ for $(p, q) \neq (2, 1_{02})$ and $g_{21_{02}}(x) = -1$. Here 1_{02} denotes q such that $q_{02} = 1$ and $q_{j\alpha} \equiv 0$ for other (j, α).

It is worth pointing out that equation (2) which is linear in the classical sense (see e.g. [4], Chapter 2), here becomes nonlinear in the sense of [1] and it will be evident that the same method works for more general cases, although with more complicated estimates.

A characteristic polynomial of the operator on the left-hand side of (3) is

$$C(\rho) = \rho^2 - \rho + c = (\rho - s)(\rho - (1 - s))$$

with $c = s(1 - s)$.

THE MAIN THEOREM. *Assume that $C(2k + s) \neq 0$ and $C(2k + 1 - s) \neq 0$ for $k = 1, 2, \dots$. Then the equation* (3) *has a family of solutions belonging to $\tilde{\mathcal{O}}_+$ of the form*

$$u^1(t, x) = a(x)t^s + \sum_{p=1}^{\infty} (-1)^p \left[\prod_{k=1}^{p} C(2k + s)\right]^{-1} \frac{\partial^{2p} a}{\partial x^{2p}}(x) t^{2p+s}, \tag{4}$$

provided $\operatorname{Re} s > 0$*;*

$$u^2(t,x) = b(x)t^{1-s} + \sum_{p=1}^{\infty}(-1)^p \left[\prod_{k=1}^{p} C(2k+1-s)\right]^{-1} \frac{\partial^{2p} b}{\partial x^{2p}}(x)t^{2p+1-s}, \quad (5)$$

provided $\operatorname{Re} s < 1$, *and*

$$u(t,x) = u^1(t,x) + u^2(t,x) \quad (6)$$

for $0 < \operatorname{Re} s < 1$, *where* $a(x)$ *and* $b(x)$ *are arbitrary functions holomorphic in some disk centered at 0.*

This theorem is a reformulation of Theorem 3 in [1], for the particular case of (2).

We begin by proving Lemma 1, which is crucial for the proof of convergence of formal series (4). This lemma can be found in [2] without proof and our proof is adapted from [3] (Lemma 5.1.3).

LEMMA 1. *If a function* f *is holomorphic in some disk* Δ_R, *and satisfies*

$$\max_{x\in\Delta_r} | f(x) | \leq \frac{C}{(R-r)^p} \quad \text{for } 0 < r < R \quad (7)$$

for some $p \geq 0$ *and* $C > 0$, *then*

$$\max_{x\in\Delta_r} \left| \frac{\partial f}{\partial x_j}(x) \right| \leq \frac{(p+1)eC}{(R-r)^{p+1}} \quad \text{for } 0 < r < R,$$

and $j = 1, \dots, n$.

Proof. Fix r, $0 < r < R$, $x \in \Delta_r$ and $j \in \{1, \dots, n\}$. Set $\rho = R - | x_j | \geq R - r$. Now fix $\varepsilon \in (0, \rho)$. It is obvious that if $| y_j - x_j | < \varepsilon$, then $| y_j | < R - (\rho - \varepsilon)$. Thus, by (7)

$$| f(y) | \leq \frac{C}{(\rho-\varepsilon)^p}$$

for every $y = (x_1, \dots, y_j, \dots, x_n)$ with $| y_j - x_j | < \varepsilon$. By Cauchy's inequality

$$\left| \frac{\partial f}{\partial x_j}(x) \right| \leq \frac{C}{\varepsilon(\rho-\varepsilon)^p} .$$

Taking $\varepsilon = \frac{\rho}{p+1}$ we obtain $\rho - \varepsilon = \rho - \frac{\rho}{p+1} = \rho\left(1 - \frac{1}{p+1}\right) = \rho\frac{p}{p+1} = \rho\left(1 + \frac{1}{p}\right)^{-1}$, therefore $\frac{1}{\varepsilon(\rho-\varepsilon)^p} = \frac{p+1}{\rho^{p+1}}\left(1 + \frac{1}{p}\right)^p < \frac{p+1}{\rho^{p+1}}e \leq \frac{(p+1)e}{(R-r)^{p+1}}$, which proves the lemma.

We will need a purely technical

LEMMA 2. *If* $\operatorname{Re} s > 0$, *and* $q \in \mathbf{C}$, *then for some* $\sigma > 0$ *and for some* $n \in \mathbf{N}$

$$\left|\frac{k_1 + k_2 s - q}{k_1 + k_2}\right| \geq \frac{\sigma}{2}$$

for every $k_1, k_2 \in \mathbf{N}$ *with* $k_1 + k_2 \geq n$.

Proof. Since $\operatorname{Re} s > 0$, 0 does not belong to the segment $[1, s]$ in $\mathbf{C}$. For this reason $\left|\frac{k_1}{k_1+k_2} + \frac{k_2}{k_1+k_2} s\right| \geq \sigma = \operatorname{dist}\{0, [1, s]\}$, for all $k_1, k_2 \in \mathbf{N}$. We choose a large enough n satisfying $\frac{|q|}{n} \leq \frac{\sigma}{2}$, and we see that

$$\left|\frac{k_1 + k_2 s - q}{k_1 + k_2}\right| \geq \left|\frac{k_1 + k_2 s}{k_1 + k_2}\right| - \frac{|\, q \,|}{k_1 + k_2} \geq \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$$

for $k_1 + k_2 \geq n$.

Proof of The Main Theorem. The proof is naturally divided into two steps. First, by simple computation we check that the formal series (4), (5) and obviously (6) are solutions of (3), and every formal solution of the form

$$v(t, x) = \sum_{i, j_1, j_2 = 1}^{\infty} v_{i j_1 j_2}(x) t^{i + j_1 s + j_2 (1 - s)}$$

must be of the form (4), (5) or (6).

Now we pass to the second step, namely to the proof of the convergence of (4) and (5). We write $a_p(x) = (-1)^p \left[\prod_{k=1}^{p} C(2k + s)\right]^{-1} \frac{\partial^{2p} a}{\partial x^{2p}}(x)$ for $p \geq 1$ and $a_0(x) = a(x)$. By Lemma 2 with $q = s$ and $q = 1 - s$

$$|\, C(2p + s) \,| = |\, (2p + s - s)(2p + s - (1 - s)) \,| \geq \left(\frac{\sigma}{2}\right)^2 (2p + 1)^2$$

for p sufficiently large, say for $p \geq n$, and this gives

$$|\, a_p(x) t^{2p} \,| \leq B \left(\frac{\sigma}{2}\right)^{-2p} \frac{1}{((2p + 1)!!)^2} \left|\frac{\partial^{2p} a}{\partial x^{2p}}(x)\right| \,|\, t \,|^{2p} .$$

Now fix $R > 0$ such that the function a is holomorphic on Δ_R. Denote $C = \max\{a(x);\ x \in \Delta_R\}$; then $a(x)$ satisfies (7) with $p = 0$. By Lemma 1 applied $2p$ times

$$\max_{x \in \Delta_r} \left|\frac{\partial^{2p} a}{\partial x^{2p}}(x)\right| \leq \frac{(2p)! e^{2p} C}{(R - r)^{2p}}$$

for $0 < r < R$. Hence,

$$| a_p(x)t^{2p} | \leq B\left(\frac{2}{\sigma}\right)^{2p} \frac{(2p)!e^{2p}C \mid t \mid^{2p}}{((2p+1)!!)^2(R-r)^{2p}} = C'\left(\frac{2e \mid t \mid}{\sigma(R-r)}\right)^{2p} \frac{(2p)!}{((2p+1)!!)^2}$$

for $x \in \Delta_r$. Thus, fixing r we get locally uniform convergence of the formal series

$$u^1(t,x) = \sum_{p=0}^{\infty} a_p(x)t^{2p+s}$$

for $| t |$ sufficiently small, and for $x \in \Delta_r$. It is immediate that $u^1 \in \bar{\mathcal{O}}_+$.

The same proof works for u^2.

References

[1] Gérard R., Tahara H., *Solutions Holomorphes et Singulières d'Équations aux Dérivées Partielles Singulières Non Linéaires*, Publ. RIMS, Kyoto Univ. **29.1** (1993), 121-151.

[2] Gérard R., Tahara H., *Formal, holomorphic and singular solutions of non linear singular partial differential equations*, Manuscript.

[3] Hörmander L., *Linear partial differential operators*, Springer Verlag, Berlin, Göttingen, Heidelberg, 1963.

[4] Schlichtkrull H., *Hyperfunctions and harmonic analysis on symmetric spaces*, Birkhäuser, Boston, 1984.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
Poland

Manuscript received February 2, 1995