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On bounded solutions of a generalized Schilling's problem

Abstract. In this note we prove that, under some assumptions on a, b, c, h, g and Q, the zero function is the only solution $\varphi : \mathbf{R} \mapsto \mathbf{R}$ of the functional equation

$$\varphi(g(x)) = a(x)\varphi(h(x)) + b(x)\varphi(h^{-1}(x)) + c(x)\varphi(x),$$

bounded in a neighbourhood of the origin and such that

$$\varphi(x) = 0 \quad \textit{for} \ \ |x| > Q.$$

Studies of a physical problem have led R. Schilling (see [4]) to the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)], \tag{1}$$

where q is a fixed number from the open interval $(0,\ 1)$, and its solutions $f : \mathbf{R} \mapsto \mathbf{R}$ such that

$$f(x) = 0 \quad \text{for} \ \ |x| > \frac{q}{1-q}. \tag{2}$$

K. Baron in [1] proved the following theorem:

If $q \in (0,\ \sqrt{2}-1]$ *then the zero function is the only solution* $f : \mathbf{R} \mapsto \mathbf{R}$ *of equation* (1) *fulfilling condition* (2) *and bounded in a neighbourhood of the origin.*

This paper generalizes this result (the above theorem in another direction was generalized by J. Morawiec in [3]). Consider namely the functional equation

$$\varphi(g(x)) = a(x)\varphi(h(x)) + b(x)\varphi(h^{-1}(x)) + c(x)\varphi(x), \tag{3}$$

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where unknown function φ maps $\mathbf{R}$ into $\mathbf{R}$. Assume that functions a, b, c map $\mathbf{R}$ into $\mathbf{R}$ and functions h, g fulfil the following conditions:

(H) $h : \mathbf{R} \mapsto \mathbf{R}$ is an increasing bijection such that $h(x) > x$ for all $x \in \mathbf{R}$,

(G) $g : \mathbf{R} \mapsto \mathbf{R}$ is an increasing bijection such that $g(x) < x$ for $x > 0$, $g(x) > x$ for $x < 0$.

First let us note the following

REMARK 1. *Let h fulfil condition* (H). *Then, for every $Q \in (0, \min\{-h^{-1}(0), h(0)\})$, we have*

$$h^{-1}(Q) < 0 < h(-Q). \tag{4}$$

Proof. If $Q \in (0, \min\{-h^{-1}(0), h(0)\})$, then $Q < h(0)$ and $h^{-1}(0) < -Q$. Hence, by (H), we obtain $h^{-1}(Q) < 0$ and $0 < h(-Q)$. This completes the proof.

Now we shall show the following

LEMMA 1. *Suppose that $a, b, c : \mathbf{R} \mapsto \mathbf{R}$,*

$$\liminf_{x\to 0} |c(x)| > 1, \tag{5}$$

$$c(0) \neq 1 \tag{6}$$

and let h, g fulfil (H) *and* (G). *Furthermore, let $Q \in (0, \min\{-h^{-1}(0), h(0)\})$ be such that*

$$c(x) \neq 0 \quad \text{for } x \in (h^{-1}(Q), 0) \cup (0, h(-Q)). \tag{7}$$

If $\varphi : \mathbf{R} \mapsto \mathbf{R}$ is a solution of equation (3), *bounded in a neighbourhood of the origin such that*

$$\varphi(x) = 0 \quad \text{for } |x| > Q, \tag{8}$$

then

$$\varphi(x) = 0 \quad \text{for } h^{-1}(Q) < x < h(-Q). \tag{9}$$

Proof. Let A denote the set

$$(h^{-1}(Q), 0) \cup (0, h(-Q)).$$

If $h^{-1}(Q) < x < h(-Q)$ then, by (H), $Q < h(x)$ and $h^{-1}(x) < -Q$ so from (8), $\varphi(h(x)) = 0 = \varphi(h^{-1}(x))$. Hence, by (3) and (7), we obtain

$$\varphi(x) = \frac{1}{c(x)}\varphi(g(x)) \quad \text{for } x \in A. \tag{10}$$

Moreover, (3), (G) and (6) make it obvious that $\varphi(0) = 0$.
Now we note that by Remark 1, (G) and induction we have

$$g^i(x) \in A \quad \text{for } x \in A,\ i \in \mathbf{N}. \tag{11}$$

Hence and from (10) we obtain, by induction,

$$\varphi(x) = \frac{1}{\prod_{i=0}^{n-1} c(g^i(x))} \varphi(g^n(x)) \quad \text{for } x \in A,\ n \in \mathbf{N}. \tag{12}$$

Since (see [2], Th. 0.4)

$$\lim_{n\to\infty} g^n(x) = 0 \quad \text{for } x \in \mathbf{R}, \tag{13}$$

we have, by (11), (7) and (5),

$$\lim_{n\to\infty} \frac{1}{\prod_{i=0}^{n-1} c(g^i(x))} = 0 \quad \text{for } x \in A. \tag{14}$$

φ is bounded in a neighbourhood of the origin so by (13) the sequence $\varphi(g^n(x))$ is bounded. Hence, from (12) and (14), we have

$$\varphi(x) = \lim_{n\to\infty} \frac{1}{\prod_{i=0}^{n-1} c(g^i(x))} \varphi(g^n(x)) = 0 \quad \text{for } x \in A.$$

This completes the proof.

If $Q < h(-Q)$ then (H) gives $h^{-1}(Q) < -Q$. Hence, as an immediate consequence of Lemma 1, we obtain the following

COROLLARY 1. *Suppose that the functions a, b, c, h, g are the same as in Lemma 1. Furthermore, let $Q \in (0, \min\{-h^{-1}(0), h(0)\})$ fulfilling condition* (7) *be such that $Q < h(-Q)$. Then the zero function is the only solution $\varphi : \mathbf{R} \mapsto \mathbf{R}$ of equation* (3) *bounded in a neighbourhood of the origin and such that condition* (8) *is satisfied.*

Now we are going to prove

THEOREM 1. *Suppose that the functions a, b, c, h, g are the same as in Lemma 1, and that $Q > 0$ fulfils conditions* (7),

$$g(Q) \leq h(-Q) \tag{15}$$

and

$$h^{-1}(Q) \le g(-Q). \tag{16}$$

Furthermore, if $h(-Q) < Q$, *let*

$$a(x)b(h(x)) \neq c(x)c(h(x)) \quad \textit{for } x \in (-Q,\ h^{-1}(Q)) \cup (h(-Q),\ Q). \tag{17}$$

If $\varphi : \mathbf{R} \mapsto \mathbf{R}$ *is a solution of equation* (3), *bounded in a neighbourhood of the origin fulfilling* (8), *then*

$$\varphi(x) = 0 \quad \textit{for } x \in \mathbf{R} \setminus \{-Q,\ h^{-1}(Q),\ h(-Q),\ Q\}. \tag{18}$$

Proof. First, note that from (15), (16) and (G) we have

$$h^{-1}(Q) \le g(-Q) < 0 < g(Q) \le h(-Q). \tag{19}$$

Hence $Q \in (0,\ \min\{-h^{-1}(0),\ h(0)\})$.
If $Q < h(-Q)$ then our assertion results from Corollary 1. Suppose that $h(-Q) \le Q$. From Lemma 1 we obtain our assertion for $Q = h(-Q)$. Let

$$h(-Q) < Q. \tag{20}$$

If $|x| < Q$ then, by (G) and (19), we have $h^{-1}(Q) < g(x) < h(-Q)$ and from Lemma 1, $\varphi(g(x)) = 0$.
Hence and from (3),

$$a(x)\varphi(h(x)) + b(x)\varphi(h^{-1}(x)) + c(x)\varphi(x) = 0 \quad \text{for } |x| < Q.$$

Let us note that for $x \ge 0$, $Q < h(0) \le h(x)$ and for $x \le 0$, $h^{-1}(x) \le h^{-1}(0) < -Q$. Using (8) we get

$$b(x)\varphi(h^{-1}(x)) + c(x)\varphi(x) = 0 \quad \text{for } 0 \le x < Q \tag{21}$$

and

$$a(x)\varphi(h(x)) + c(x)\varphi(x) = 0 \quad \text{for } -Q < x \le 0. \tag{22}$$

Thus

$$b(h(x))\varphi(x) + c(h(x))\varphi(h(x)) = 0 \quad \text{for } h^{-1}(0) \le x < h^{-1}(Q) \tag{23}$$

and

$$a(h^{-1}(x))\varphi(x) + c(h^{-1}(x))\varphi(h^{-1}(x)) = 0 \quad \text{for } h(-Q) < x \le h(0). \tag{24}$$

(19) and (20) show that

$$[0,\ Q) \cap (h(-Q),\ h(0)] = (h(-Q),\ Q).$$

For every x in this interval, by (21) and (24), we obtain

$$c(h^{-1}(x))b(x)\varphi(h^{-1}(x)) + c(h^{-1}(x))c(x)\varphi(x) = 0$$

and

$$b(x)a(h^{-1}(x))\varphi(x) + b(x)c(h^{-1}(x))\varphi(h^{-1}(x)) = 0.$$

Hence and from (17) we see that

$$\varphi(x) = 0 \quad \text{for } h(-Q) < x < Q. \tag{25}$$

Similarly by (19), (20), (22), (23) and (17) we have

$$\varphi(x) = 0 \quad \text{for } -Q < x < h^{-1}(Q). \tag{26}$$

From (25), (26) and Lemma 1 we have

$$\varphi(x) = 0 \quad \text{for } x \in \mathbf{R} \setminus \{-Q,\ h^{-1}(Q),\ h(-Q),\ Q\},$$

which completes the proof of Theorem 1.

At present we are able to prove the following

THEOREM 2. *Suppose that the functions a, b, c, h, g satisfy the same assumptions as in Lemma 1. Furthermore, let $Q > 0$ be such that conditions* (15), (16) *and*

$$c(x) \neq 0 \quad \textit{for } x \in [h^{-1}(Q),\ 0) \cup (0,\ h(-Q)] \tag{27}$$

are satisfied and, if $h(-Q) \leq Q$, let

$$a(x)b(h(x)) \neq c(x)c(h(x)) \quad \textit{for } x \in [-Q,\ h^{-1}(Q)] \cup (h(-Q),\ Q) \tag{28}$$

and

$$0 \neq b(h(Q)) \neq 1. \tag{29}$$

Then the zero function is the only solution $\varphi : \mathbf{R} \mapsto \mathbf{R}$ of equation (3) *bounded in a neighbourhood of the origin and such that condition* (8) *is satisfied.*

Proof. If $Q < h(-Q)$ this results from Corollary 1. Assume that

$$h(-Q) \leq Q. \tag{30}$$

Let B denote the set

$$\{h(Q),\ g(h^{-1}(Q)),\ h^{-2}(Q),\ h^2(Q),\ g(h(-Q)),\ h^2(-Q),\ h^{-1}(-Q)\}.$$

First we show that

$$\varphi(x) = 0 \quad \text{for } x \in B. \tag{31}$$

By (30), (4) and (H) we have

$$-Q \leq h^{-1}(Q) < 0 < h(-Q) \leq Q. \tag{32}$$

From (H) we obtain

$$Q < h(Q) < h^2(Q) \quad \text{and} \quad h^{-1}(-Q) < -Q. \tag{33}$$

By (H) and (32) we have

$$Q < h^2(-Q) \quad \text{and} \quad h^{-2}(Q) < -Q. \tag{34}$$

Finally, (G) and (4) give

$$h^{-1}(Q) < g(h^{-1}(Q)) < 0 \quad \text{and} \quad 0 < g(h(-Q)) < h(-Q). \tag{35}$$

From (33), (34), (35), (32) and by Th. 1 we have $\varphi(x) = 0$ for $x \in B$. Putting in equation (3) in turn $x = Q$, $x = h^{-1}(Q)$, $x = h(Q)$, $x = h(-Q)$ and $x = -Q$ we get from (31),

$$\varphi(g(Q)) = b(Q)\varphi(h^{-1}(Q)) + c(Q)\varphi(Q), \tag{36}$$

$$0 = a(h^{-1}(Q))\varphi(Q) + c(h^{-1}(Q))\varphi(h^{-1}(Q)), \tag{37}$$

$$\varphi(g(h(Q))) = b(h(Q))\varphi(Q), \tag{38}$$

$$0 = b(h(-Q))\varphi(-Q) + c(h(-Q))\varphi(h(-Q)), \tag{39}$$

$$\varphi(g(-Q)) = a(-Q)\varphi(h(-Q)) + c(-Q)\varphi(-Q). \tag{40}$$

Now we must distinguish two cases:
- (i) $g(Q) \neq h(-Q)$,
- (ii) $g(Q) = h(-Q)$.

(i) From (19), $0 < g(Q) < h(-Q)$ and by (32) and Th. 1 we get $\varphi(g(Q)) = 0$. Hence from (36), (37) and (28) we have $\varphi(h^{-1}(Q)) = 0$ and $\varphi(Q) = 0$.

(ii) By (G) and (H), $0 < g(h(Q)) \neq h(-Q)$. If $g(h(Q)) \neq Q$ then from Th. 1, $\varphi(g(h(Q))) = 0$ whence, by (38) and (29), we obtain $\varphi(Q) = 0$. If $g(h(Q)) = Q$ then from (38) and (29) we get $\varphi(Q) = 0$. Finally, (37) and (27) give $\varphi(h^{-1}(Q)) = 0$.

Thus, in both cases, we have obtained

$$\varphi(Q) = 0 = \varphi(h^{-1}(Q)). \tag{41}$$

By (G) we get $-Q < g(-Q) < 0$ whence, from (41) and Th. 1, $\varphi(g(-Q)) = 0$. Consequently, taking into account properties (39) and (40) and using (28), we see that $\varphi(-Q) = 0$. Hence, by (39) and (27), we obtain $\varphi(h(-Q)) = 0$. In view of Th. 1 we have $\varphi \equiv 0$.

Finally, we give two remarks

REMARK 2. *Suppose that functions h, g fulfil* (H) *and* (G). *Then there exists exactly one $Q_0 > 0$ such that, if $Q \leq Q_0$, then Q fulfils conditions* (15) *and* (16) *and, if $Q > Q_0$, then at least one of conditions* (15) *and* (16) *is not satisfied.*

Proof. Define

$$h_1(x) := h(-x), \quad x \in \mathbf{R},$$

$$g_1(x) := g(-x), \quad x \in \mathbf{R}.$$

Functions h_1 and g_1 have the following properties:

(H_1) $h_1 : \mathbf{R} \mapsto \mathbf{R}$ is a decreasing bijection such that $h_1(x) > -x$ for all $x \in \mathbf{R}$,

(G_1) $g_1 : \mathbf{R} \mapsto \mathbf{R}$ is a decreasing bijection such that $g_1(x) > -x$ for $x > 0$, $g_1(x) < -x$ for $x < 0$.

Let us note that $0 < h_1(0)$ and $h^{-1}(0) < 0$.
Hence it follows from (G) and (H_1) that there exists exactly one $Q_1 > 0$ such that

$$g(Q) < h_1(Q) \quad \text{for } Q < Q_1,$$
$$g(Q_1) = h_1(Q_1),$$
$$h_1(Q) < g(Q) \quad \text{for } Q > Q_1.$$

Similarly it follows from (G_1) and (H) that there exists exactly one $Q_2 > 0$ such that

$$h^{-1}(Q) < g_1(Q) \quad \text{for } Q < Q_2,$$
$$h^{-1}(Q_2) = g_1(Q_2),$$
$$g_1(Q) < h^{-1}(Q) \quad \text{for } Q > Q_2.$$

Define

$$Q_0 := \min\{Q_1, Q_2\}.$$

Then for $Q \leq Q_0$,

$$g(Q) \leq h_1(Q) = h(-Q) \quad \text{and} \quad h^{-1}(Q) \leq g_1(Q) = g(-Q)$$

and for $Q > Q_0$,

$$h(-Q) = h_1(Q) < g(Q) \quad \text{or} \quad g(-Q) = g_1(Q) < h^{-1}(Q).$$

This ends the proof.

REMARK 3. *Theorem 2 implies quoted at the beginning of this paper Baron's result.*

Indeed, in the case $a(x) = b(x) = \frac{1}{4q}$, $c(x) = \frac{1}{2q}$, $h(x) = x + 1$, $g(x) = qx$ equations (3) and (1) are equivalent. Let $q \in (0, \frac{1}{2})$ and $Q = \lambda(q)$, where $\lambda : (0, \frac{1}{2}) \mapsto \mathbf{R}$ is an arbitrary function. It is easy to verify that conditions (H), (G), (5), (6) are satisfied by functions h, g and c, respectively and (27) and (28) hold true. Moreover, Remark 2 gives now $Q_0 = \frac{1}{1+q}$. Consequently, from Th. 2, we conclude that if $0 < \lambda(q) \leq \frac{1}{1+q}$ and $\lambda(\frac{1}{4}) < \frac{1}{2}$, then the zero function is the only solution $\varphi : \mathbf{R} \mapsto \mathbf{R}$ of equation (1) bounded in a neighbourhood of the origin and such that condition (8) is satisfied. If we put $\lambda(q) = \frac{q}{1-q}$ then of course $\lambda(\frac{1}{4}) = \frac{1}{3} < \frac{1}{2}$, and $0 < \lambda(q) \leq \frac{1}{1+q}$ iff $q \in (0, \sqrt{2}-1]$. Thus we have Baron's Theorem.

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