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Remarks on real-valued quasi-isometries

Abstract. In this paper we consider the class of solutions of the functional inequality

$$||f(x) - f(y)| - |x - y|| \le \varepsilon \min\{|f(x) - f(y)|, |x - y|\}, \ x, y \in \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the unknown function and $\varepsilon \in [0, 1)$. The two main tasks of the paper are to compare the above class of quasi-isometries with the class of quasi-additive functions and to answer the question of stability.

Introduction

In the previous volume of this series Z. Hajto and J. Tabor [2] introduced a class of approximate isometries which they called *quasi-isometries.* The method applied to define such a class was based on the J. Tabor's idea of defining approximate homomorphisms (*quasi-additive* mappings) presented in [6] and [7]. This class is different from the one defined much earlier by D. H. Hyers and S. M. Ulam in [3]. In the present paper we are going to make further investigations on quasi-isometries restricting ourselves to the case of the real line as the domain and the target space. The main tasks are the following. At first, we try to state a stability result of some kind. We prove that a quasi-isometry can be approximated by an isometry. It could be treated as a complementary problem to the one of Hyers-Ulam stability of isometries posed in [3]. Secondly, we would like to raise the question as to whether or not a quasi-isometric mapping has to be quasi-affine (analogously to the Mazur-Ulam theorem for isometries in [4]).

Let X and Y be normed spaces. For $\varepsilon \ge 0$ a function $f : X \to Y$ is called *ε-isometry* iff (cf. [3])

$$| \|f(x) - f(y)\| - \|x - y\| | \le \varepsilon \quad \text{for } x, y \in X. \tag{1}$$

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For $\varepsilon \in [0,1)$ a function $f : X \to Y$ is called *ε-quasi-isometry* iff (cf. [2])

$$|\,\|f(x)-f(y)\|-\|x-y\|\,| \le \varepsilon \min\{\|f(x)-f(y)\|, \|x-y\|\} \quad \text{for } x,y \in X \quad (2)$$

or, equivalently, if f is a solution of a system of functional inequalities

$$|\,\|f(x)-f(y)\|-\|x-y\|\,| \le \varepsilon\|f(x)-f(y)\| \quad \text{for } x,y \in X \quad (3)$$

and

$$|\,\|f(x)-f(y)\|-\|x-y\|\,| \le \varepsilon\|x-y\| \quad \text{for } x,y \in X. \quad (4)$$

With $\varepsilon = 0$ both (1) and (2) become the definition of *isometry*. Notice that if f is an ε-quasi-isometry, then $f+c$ (with any $c \in Y$) is an ε-quasi-isometry as well. That is why we may assume, without loss of generality, that $f(0) = 0$. The following lemma can be compiled from Proposition 1 and Proposition 2 in [2].

LEMMA 1. *For X, Y – normed spaces, $f : X \to Y$ and $\varepsilon \in [0,1)$ the following conditions are equivalent*

1. f is an ε-quasi-isometry;

2.

$$\frac{1}{1+\varepsilon}\|x-y\| \le \|f(x)-f(y)\| \le (1+\varepsilon)\|x-y\| \quad \textit{for } x,y \in X; \quad (5)$$

3.

$$\frac{1}{1+\varepsilon}\|x-y\| \le \|f^{-1}(x)-f^{-1}(y)\| \le (1+\varepsilon)\|x-y\| \quad \textit{for } x,y \in f(X). \quad (6)$$

Condition (5) implies injectivity of f and since then (6) makes sense. Moreover, (5) implies that f is Lipschitzian.
A simple example of an ε-quasi-isometry which is neither an isometry nor an approximate isometry (in the sense of Hyers-Ulam) can be given for the Euclidean plane. Let $X = Y = \mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows

$$f(x) = (\alpha x_1, \beta x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2$$

where α and β are constants taken from $\left[-(1+\varepsilon), -\frac{1}{1+\varepsilon}\right] \cup \left[\frac{1}{1+\varepsilon}, 1+\varepsilon\right]$. Obviously f satisfies (5), i.e., f is an ε-quasi-isometry. On the other hand, taking $\alpha = 1+\varepsilon$ and $x_k = (k,0) \in \mathbb{R}^2$, $k > 0$, one gets $|\,\|f(x_k)-f(0)\|-\|x_k-0\|\,| = \varepsilon k$, which means that f is not a δ-isometry for any δ.

In considerations that follows we restrict ourselves to the case $X = Y = \mathbb{R}$.

LEMMA 2. *An ε-quasi-isometry $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitzian homeomorphism.*

Proof. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) and $f(0) = 0$. As a continuous and injective mapping f is strictly monotonic. Thus either

$$(a) \qquad \frac{f(x)}{x} > 0 \quad \text{for } x \in \mathbb{R} \setminus \{0\}$$

or

$$(b) \qquad \frac{f(x)}{x} < 0 \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

Putting 0 in place of y in (5) we get

$$\frac{1}{1+\varepsilon} \leq \frac{f(x)}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\} \quad \text{in the case (a)}$$

or

$$\frac{f(x)}{x} \leq -\frac{1}{1+\varepsilon} \quad \text{for } x \in \mathbb{R} \setminus \{0\} \quad \text{in the case (b).}$$

It yields that $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$ in the case (a) and $\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$ in the case (b), whence f is bounded neither above nor below and consequently, as an injective and continuous mapping, has to be bijective. Because of (5) and (6) f is Lipschitzian homeomorphism.

Stability

The problem of the stability of isometries was posed by D. H. Hyers and S. M. Ulam in [3] and exhaustively solved by M. Omladič and P. Šemrl [5]. In the case of real Banach spaces X and Y for a surjective ε-*isometry* $f : X \to Y$ satisfying $f(0) = 0$ there exists a unique surjective linear isometry $I : X \to Y$ such that $\|f(x) - I(x)\| \leq 2\varepsilon$ holds true for each $x \in X$. The constant 2ε on the right hand side of the last inequality is optimal.
A question arises: is it possible to prove the stability of isometric mappings with respect to the class of quasi-isometries? A positive answer to this question, but only in the case considered in this paper, follows.

For $f : \mathbb{R} \to \mathbb{R}$ let $\operatorname{Gr} f$ stand for the graph of f. We define two subsets of the plane $\mathbb{R}^2$:

$$\mathcal{K}_1 := \left\{(x, y) \in \mathbb{R}^2 : \ \frac{1}{1+\varepsilon} \leq \frac{y}{x} \leq (1+\varepsilon)\right\} \cup \{(0,0)\}$$

and

$$\mathcal{K}_2 := \left\{(x, y) \in \mathbb{R}^2 : \ -(1+\varepsilon) \leq \frac{y}{x} \leq -\frac{1}{1+\varepsilon}\right\} \cup \{(0,0)\}.$$

Now we prove the main result of this section.

THEOREM 1. *If $f : \mathbb{R} \to \mathbb{R}$ is an ε-quasi-isometry (with $\varepsilon \in [0,1)$), then there exists a unique isometry $i : \mathbb{R} \to \mathbb{R}$ satisfying the inequality*

$$|f(x) - i(x)| \le \varepsilon \min\{|f(x) - f(0)|, |i(x) - i(0)|\} \quad \textit{for } x \in \mathbb{R}. \tag{7}$$

Proof. We start with the case $f(0) = 0$. Putting in (5) 0 in place of y and using the monotonicity of f (cf. the proof of Lemma 2) we get that $\operatorname{Gr} f \subset \mathcal{K}_1$ or $\operatorname{Gr} f \subset \mathcal{K}_2$. Suppose that the first case holds. For $i = \mathrm{id}_{\mathbb{R}}$ we have then

$$\frac{1}{1+\varepsilon}x - x \le f(x) - i(x) \le (1+\varepsilon)x - x \quad \text{for } x \ge 0$$

and

$$(1+\varepsilon)x - x \le f(x) - i(x) \le \frac{1}{1+\varepsilon}x - x \quad \text{for } x < 0.$$

Combining these inequalities we get

$$|f(x) - i(x)| \le \varepsilon |i(x)| \quad \text{for } x \in \mathbb{R}.$$

On the other hand, we have

$$\frac{1}{1+\varepsilon}x \le f(x) \le (1+\varepsilon)x \le \frac{1}{1-\varepsilon}x \quad \text{for } x \ge 0$$

and

$$\frac{1}{1-\varepsilon}x \le (1+\varepsilon)x \le f(x) \le \frac{1}{1+\varepsilon}x \quad \text{for } x < 0,$$

which yields

$$-\varepsilon|f(x)| = -\varepsilon f(x) \le f(x) - x \le \varepsilon f(x) = \varepsilon |f(x)| \quad \text{for } x \ge 0$$

and

$$-\varepsilon|f(x)| = \varepsilon f(x) \le f(x) - x \le -\varepsilon f(x) = \varepsilon |f(x)| \quad \text{for } x < 0,$$

whence

$$|f(x) - i(x)| \le \varepsilon |f(x)| \quad \text{for } x \in \mathbb{R}.$$

Thus we have obtained

$$|f(x) - i(x)| \le \varepsilon \min\{|f(x)|, |i(x)|\} \quad \text{for } x \in \mathbb{R},$$

i.e., we have proven that there exists an isometry $\mathbb{R} \to \mathbb{R}$ satisfying (7). In order to prove that such an isometry is unique notice that (7) implies $i(0) = 0$. The only isometries satisfying this condition are $i = \mathrm{id}_{\mathbb{R}}$ and $i = -\mathrm{id}_{\mathbb{R}}$. Supposing that $i = -\mathrm{id}_{\mathbb{R}}$ we would have $|f(x) + x| \le \varepsilon |x|$ and this would imply $\operatorname{Gr} f \subset \mathcal{K}_2$, a contradiction.

In the case $\mathrm{Gr}\, f \subset \mathcal{K}_2$ we can take the function $-f$ which is an ε-quasi-isometry and $\mathrm{Gr}\,(-f) \subset \mathcal{K}_1$. Applying for the function $(-f)$ what we have proved above we obtain that (7) is satisfied with the only isometry $i = -\mathrm{id}_{\mathbb{R}}$. If $f(0) = c$, one can apply the above result for function $\bar{f}(x) = f(x) - c$ and obtain (7) in that case easily.

Quasi-additivity

As it was said at the beginning of the paper the definition of quasi-isometry (inequality (2)) shows close connections with the definition of quasi-additive mappings. Recall that function $f : X \to Y$ is called *ε-quasi-additive* (with $\varepsilon \in [0,1)$) iff (cf. [6], [7])

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \min\{\|f(x+y)\|, \|f(x) + f(y)\|\}$$

holds true for each $x, y \in X$. It is well known, that under some additional assumptions on a function or on the target space an isometry $I : X \to Y$ satisfying $I(0) = 0$ is additive (cf. [4], [1]). A similar result for approximate isometries and approximate additive mappings can be derived from the main result of [5]. Namely, it is true that if X and Y are real Banach spaces, $\varepsilon \geq 0$ and $f : X \to Y$ is a surjective ε-isometry, then f is 6ε-additive up to a constant. Indeed, there exists a surjective and linear isometry $i : X \to Y$ such that for $f_0 = f - f(0)$ there is $\|f_0(x) - i(x)\| \leq 2\varepsilon$ for $x \in X$ (cf. [5]). Thus for $x, y \in X$

$$\|f_0(x+y) - f_0(x) - f_0(y)\| = \|f_0(x+y) - i(x+y) - f_0(x) + i(x) - f_0(y) + i(y)\| \leq 6\varepsilon.$$

It seems natural to conjecture that a result of the same type holds as far as quasi-isometries and quasi-additive mappings are concerned. In what follows we make such an attempt in the case of real-valued quasi-isometries.

THEOREM 2. *If $f : \mathbb{R} \to \mathbb{R}$ is an odd ε-quasi-isometry (with $\varepsilon \in [0,1)$), then f is a continuous $(2+\varepsilon)\varepsilon$-quasi-additive mapping.*

Proof. Let

$$D_1 f(x_0) := \limsup_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D_2 f(x_0) := \limsup_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D_3 f(x_0) := \liminf_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

$$D_4 f(x_0) := \liminf_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

be the Dini derivatives at a point x_0. J. Tabor proved ([6], Theorem 4) that for an odd and continuous function $f : \mathbb{R} \to \mathbb{R}$ we have

(i) If there exist $A, B \in \mathbb{R}$, $0 < A \leq B$ and $i \in \{1,2,3,4\}$ such that $A \leq D_i f(x) \leq B$ for $x \in \mathbb{R}$, then

$$|f(x+y) - f(x) - f(y)| \leq \left(\frac{B}{A} - 1\right) \min\{|f(x+y)|, |f(x) + f(y)|\} \quad \textit{for } x, y \in \mathbb{R}.$$

(ii) If there exist $C, D \in \mathbb{R}$, $D \leq C < 0$ and $i \in \{1,2,3,4\}$ such that $D \leq D_i f(x) \leq C$ for $x \in \mathbb{R}$, then

$$|f(x+y) - f(x) - f(y)| \leq \left(\frac{D}{C} - 1\right) \min\{|f(x+y)|, |f(x) + f(y)|\} \quad \textit{for } x, y \in \mathbb{R}.$$

Putting in (5) $x_0 + h$ in place of x and x_0 in place of y we get

$$\frac{1}{1+\varepsilon} \leq \left|\frac{f(x_0+h) - f(x_0)}{h}\right| \leq 1 + \varepsilon \quad \text{for } x_0 \in \mathbb{R},\ h \in \mathbb{R} \setminus \{0\},$$

whence

$$\frac{1}{1+\varepsilon} \leq |D_i f(x)| \leq 1 + \varepsilon \quad \text{for } x \in \mathbb{R},\ i = 1,2,3,4.$$

The monotonicity and continuity of f yields that all the Dini derivatives have the same sign, whence

$$0 < \frac{1}{1+\varepsilon} \leq D_i f(x) \leq 1 + \varepsilon, \quad x \in \mathbb{R},\ i = 1,2,3,4$$

or

$$-(1+\varepsilon) \leq D_i f(x) \leq -\frac{1}{1+\varepsilon} < 0, \quad x \in \mathbb{R},\ i = 1,2,3,4.$$

In both cases the assumptions of Tabor's theorem are satisfied and we get that

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon' \min\{|f(x+y)|, |f(x) + f(y)|\}$$

holds true for $x, y \in \mathbb{R}$ with $\varepsilon' = (1+\varepsilon)^2 - 1$.

The assumption that f is odd is essential. Indeed, we used this assumption in the proof of Theorem 2 as it was necessary to apply Tabor's theorem. It is easily seen that the definition of quasi-additive function yields that such a function has to be odd. The question is whether this property can be derived from (2). Putting in (5) 0 in place of y and $-x$ in place of x one obtains

$$\frac{1}{1+\varepsilon} \le \frac{\|f(x)\|}{\|x\|} \le 1+\varepsilon \quad \text{for } x \in X \setminus \{0\}$$

and

$$\frac{1}{1+\varepsilon} \le \frac{\|x\|}{\|f(-x)\|} \le 1+\varepsilon \quad \text{for } x \in X \setminus \{0\}.$$

Multiplying the above inequalities one gets

$$\frac{1}{(1+\varepsilon)^2} \le \frac{\|f(x)\|}{\|f(-x)\|} \le (1+\varepsilon)^2 \quad \text{for } x \in X \setminus \{0\}.$$

If $\varepsilon = 0$, then we have $\|f(x)\| = \|f(-x)\|$ which, in the case $Y = \mathbb{R}$, implies that f is odd. Indeed, $|f(x)| = |f(-x)|$ implies $f(-x) = f(x)$ or $f(-x) = -f(x)$. Suppose that $f(-x) = f(x)$. Putting $y = -x$ in (2) we would have $2|x| \le 0$. Thus $f(x) = -f(x)$ for $x \in \mathbb{R}$. For $\varepsilon > 0$ the oddness of f does not result from (2). It is easily seen when considering the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (1+\varepsilon)x, & x \ge 0, \\ \dfrac{1}{1+\varepsilon}x, & x < 0, \end{cases}$$

which satisfies (2) but is not odd.

One can replace, in the above considerations, the target space by a one-dimensional normed space Y which implies that the domain is also one-dimensional. As it is not much more general we considered the $\mathbb{R} \to \mathbb{R}$ case only. And since we have restricted ourselves merely to the real case, we only might hope that in the general case the results of the paper are still valid.

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