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STANISŁAW SIUDUT

On the convergence of singular integrals of vector-valued functions

Abstract. The pointwise convergence of abstract singular convolution integrals $K(\cdot, v) * f(x - \cdot)$ depending on two parameters x, v ($x \in \mathbb{R}$, v belongs to a metric space E) is studied. The convergence of the parameters is restricted to some subsets of $\mathbb{R} \times E$ (i.e. the Fatou's type of the convergence is discussed). Parameter x tends to a generalized D or L-point of the Banach space-valued function f, whereas v tends to an accumulation point of E. Some applications of the obtained results to the norm-convergence of real-valued singular integrals and partial differential equations are presented.

Introduction

Singular integrals of the convolution type occur in approximation theory, as solutions of partial differential equations and in the theory of Fourier series. For example, the sequence of arithmetic means of partial sums of the Fourier series of a function f may be written as a singular integral with the Fejér kernel. The singular integral having as approximate identity the Abel-Poisson kernel is a solution of Dirichlet's boundary value problem for the unit disc. The solution of this problem for the half-plane is a singular integral of Cauchy-Poisson (it is an integral on the line group). The results on singular integrals on the line group are essential in the summability of Fourier inversion integrals. For further informations on singular integrals see [1], p. 25-161.

In this paper we study the pointwise convergence of some class of singular integrals of vector-valued functions. Roughly speaking, the nontangential limits of these integrals are considered (contrary to [1], theorems 1.4.2-1.4.7 and 3.2.1-3.2.5, where the parameter x is fixed, the parameter ρ is variable only). An early and important paper on singular integrals depending on two parameters is that of Taberski [5] (1962).

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Some preliminary notation and definitions are presented in Section 1. In Section 2 three theorems are formulated.

Theorem 2 from [2] is extended on several directions, for example on vector-valued functions and on non-periodic case (see Theorems 1, 2, parts (c), (d), (c′), (d′)).

The parts (b), (b′) of Theorems 1, 2 are essential in the proof of Theorem 3, whereas the parts (a), (a′) are necessary in Section 4. These parts are formulated for the convenience of the reader (cp. Theorems 1-4, [4]).

Statements (c), (d), (c′), (d′), Theorem 3 and the results given in Section 4 are the essential new results.

In Section 3 we prove Theorem 3 and we sketch proofs of Theorems 1, 2. Section 4 is concerned with applications of the theorems and some comments.

1. Notations and definitions

In the sequel we denote by

l – a positive real number,

G – a Banach space with the norm $\| \ \|$,

$L_{2l}(G) = \{f | f : \mathbb{R} \to G$ is $2l$-periodic and Bochner integrable on $[-l, l]\}$,

$S(G) \ \ = \{f | f : \mathbb{R} \to G$ is strongly measurable$\}$,

$$L^p(G) = \begin{cases} \{f \in S(G) | \int_{-\infty}^{\infty} \|f(t)\|^p dt < \infty\} & \text{if } 1 \le p < \infty, \\ \{f \in S(G) | f \text{ is essentially bounded}\} & \text{if } p = \infty, \end{cases}$$

$L^p = L^p(\mathbb{R}),\ L = L^1,\ L_{2l} = L_{2l}(\mathbb{R})$,

E – a metric space with the metric ϱ,

v_0 – an accumulation point of E (in the case $E \subset \mathbb{R}$ we admit also $v_0 = \infty$),

v – a parameter belonging to E,

K, K^* – real-valued functions defined on $\mathbb{R} \times E$, even, bounded and measurable with respect to the first variable for every fixed the second one – moreover, K^* is assumed to be an absolute majorant of K.

For $x_0 \in \mathbb{R}$, $C \ge 0$, $\beta > 0$, $\delta > 0$, $0 < \varepsilon < 1$, $f \in L_{2l}(G)$ or $f \in L^p(G)$ $(1 \le p \le \infty)$, we define the sets

$Z_{C,\beta} = Z_{C,\beta}(K) = \{(x, v) \in \mathbb{R} \times E |\ |x - x_0|^\beta K(0, v) \le C\}$,

$Z^*_{C,\beta} = Z_{C,\beta}(K^*)$,

$$Z_{C,\delta,\varepsilon} = Z_{C,\delta,\varepsilon}(K)$$
$$= \left\{(x,v) \in \mathbb{R} \times E \mid \left(\int_{-\delta}^{\delta} \|f(t+x) - f(t+x_0)\| dt\right)^{1-\varepsilon} K(0,v) \le C\right\},$$
$$Z^*_{C,\delta,\varepsilon} = Z_{C,\delta,\varepsilon}(K^*).$$

Consider the expression

$$\psi_x(u,d) = f(x+u) + f(x-u) - 2d$$

in which $x, u \in \mathbb{R}$, $d \in G$, $f \in L_{2l}(G)$ or $f \in L^p(G)$, $1 \le p \le \infty$.

DEFINITION. The number x is a D_s-point of f [L_s-point of f] if there is a vector $d \in G$ such that

$$\lim_{h\to 0} \int_0^h \psi_x(u,d)du = 0 \quad \left[\lim_{h\to 0} \int_0^h ||\psi_x(u,d)||du = 0\right],$$

resp.

The definitions of D_*, L_*, D, L-points of f with the explanation of notations $f_+(x)$, $f_-(x)$, $f(x)$ one can found in [4], p. 171, 172.
$A(f)$ is the set of all A-points of f, where A is standing for D_s, L_s, D_*, ... etc.

2. Statement of results

Consider a function $f \in L_{2l}(G)$ and the numbers β, C, δ, ε, such that $0 < \beta < 1$, $C \ge 0$, $0 < \delta < l$, $0 < \varepsilon < 1$. We shall study the pointwise convergence of singular integrals

$$U(x,v) = U_w(x,v) = \int_{-w}^{w} K(x-t,v) f(t)dt,$$

where $w = l$ in the periodic case or $w = \infty$ in the non-periodic case.

THEOREM 1. *Suppose that for every $v \in E$ the function $K(\cdot,v)$ is $2l$-periodic, non-negative, and that*
$K(\cdot,v)$ is non-increasing on $[0,l]$ for each $v \in E$,
$\lim_{v\to v_0} K(u,v) = 0$ for every $u \in (0,l)$,

$$\lim_{v\to v_0} \int_{-l}^{l} K(t,v)dt = 1.$$

Then:

(a) *if $x_0 \in D_*(f)$, then $U(x,v) \to \frac{1}{2}\{f_-(x_0) + f_+(x_0)\}$ as $(x,v) \to (x_0,v_0)$ in $Z_{C,\beta}$,*

(b) *if $x_0 \in D(f)$, then $U(x,v) \to f(x_0)$ as $(x,v) \to (x_0,v_0)$ in $Z_{C,1}$,*

(c) *if* $x_0 \in D_s(f)$, *then* $U(x,v) \to f(x_0)$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z_{C,\delta,\varepsilon}$,

(d) *if* $x_0 \in D_s(f)$ *and in addition there is* $\alpha > 0$ *such that* $\int_{-\delta}^{\delta} \|f(t+x) - f(t+x_0)\|dt = O(|x-x_0|^\alpha)$ *as* $x \to x_0$, *then* $U(x,v) \to f(x_0)$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z_{C,\gamma}$, *where* γ *is fixed and* $0 < \gamma < \alpha$.

Consider a function $f \in L^p(G)$ $(1 \le p \le \infty)$ and the numbers β, C, δ, ε, where $0 < \beta < 1$, $C \ge 0$, $\delta > 0$, $0 < \varepsilon < 1$.

THEOREM 2. *If*

$$\lim_{v \to v_0} \int_{-\infty}^{\infty} K(t,v)dt = 1,$$

$K^*(\cdot,v)$ *is non-increasing on* $[0,\infty)$ *for all* $v \in E$,

$$\lim_{v \to v_0} \int_{u}^{\infty} K^*(t,v)dt = 0 \quad \text{for every} \quad u > 0,$$

there is $B > 0$ *such that*

$$\int_{-\infty}^{\infty} K^*(t,v)dt \le B \quad \text{for all} \quad v \in E,$$

then:

(a′) *if* $x_0 \in L_*(f)$, *then* $U(x,v) \to \frac{1}{2}\{f_-(x_0) + f_+(x_0)\}$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z^*_{C,\beta}$,

(b′) *if* $x_0 \in L(f)$, *then* $U(x,v) \to f(x_0)$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z^*_{C,1}$,

(c′) *if* $x_0 \in L_s(f)$, *then* $U(x,v) \to f(x_0)$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z^*_{C,\delta,\varepsilon}$,

(d′) *if* $x_0 \in L_s(f)$ *and in addition there is* $\alpha > 0$ *such that* $\int_{-\delta}^{\delta} \|f(t+x) - f(t+x_0)\|dt = O(|x-x_0|^\alpha)$ *as* $x \to x_0$, *then* $U(x,v) \to f(x_0)$ *as* $(x,v) \to (x_0,v_0)$ *in* $Z^*_{C,\gamma}$, *where* γ *is fixed and* $0 < \gamma < \alpha$.

THEOREM 3. *If the assumptions of one of the preceding theorems are satisfied, then* $U(x,v)$ *has a limit as* $(x,v) \to (x_0,v_0)$ *if and only if the function* f *is essentially continuous at* x_0 *(i.e. there is a continuous at* x_0 *function* g, $g = f$ *a.e.).*

REMARK. *If* $f \in L_{2l}(G)$ *and* K, K^* *satisfy the assumptions of Th. 2 from* [4], *p. 172, then we have* (a′), (b′), (c′), (d′). *If* $f \in L^p(G)$, $1 \le p \le \infty$, *and* K *satisfies the assumptions of Th. 3 in* [4], *p. 173, then the statements* (a), (b), (c), (d) *hold.*

3. Outlines of proofs

Since the suitable generalizations of Natanson's lemma are shown in [3], then the proofs of parts (b), (b′) run similarly to the proofs of Theorems 3.1, 3.2 of Taberski's paper [5]. He has considered the periodic case only – however for example , in the non-periodic case, the assumptions on K^* lead to

$$\lim_{v\to v_0} |K^*(u/2,v)| = 0, \quad \lim_{v\to v_0} \int_{\mathbf{R}\setminus(-u/2,u/2)} |K^*(t,v)|^q dt = 0, \quad (q \geq 1, u > 0)$$

therefore applying this and the Hölder inequality we obtain

$$\begin{aligned}
\lim_{v\to v_0} \int_{\mathbf{R}\setminus(x_0-u,x_0+u)} & K(s-x,v)\{f(s)-f(x_0)\}ds \\
&= \lim_{v\to v_0} \left[\int_{\mathbf{R}\setminus(x_0-u,x_0+u)} K(s-x,v)f(s)ds \right. \\
&\quad \left. - \left(\int_{\mathbf{R}\setminus(x_0-u,x_0+u)} K(s-x,v)ds\right) f(x_0)\right] \\
&= 0 \quad \text{if } |x-x_0| \leq u/2.
\end{aligned}$$

The rest of the proof of Th. 2 (b′) is the same as in the paper by R. Taberski. Namely, it is sufficient to use the lemmas of Natanson's type to the expression:

$$\int_{x_0-u}^{x_0+u} K^*(s-x,v)\|f(s)-f(x_0)\|ds.$$

The proofs of the parts (a), (a′) are given in [4]. The proofs of parts (c), (c′), (d), (d′) runs similarly as in the periodic and real-valued case, see Theorem 2 and Corollary of [2].

We complete this section by indication of the main alterations in the proof of Th. 2, [2], which are essential for the proof of (c) when $f \in L^p(G)$.

Consider first the integral of the type J_1 (cp. [2], p. 138), which occurs in the proof of (c) when $f \in L^p(G)$. In this integral the expression

$$\int_{\mathbf{R}\setminus I_\delta} ||f(t-x)+f(t+x_0)||K(t,v)dt := E_v$$

one can estimate by

$$\int_{\mathbf{R}\setminus I_\delta} K^q(t,v)dt \cdot 2||f||_p \quad \text{if } p > 1 \quad (q \text{ is conjugated to } p),$$

or

$$K(\delta,v)\cdot 2||f||_1 \quad \text{if } p = 1.$$

For these estimations if follows that

$$\lim_{v \to v_0} E_v = 0.$$

The "new" integral of the type Y_2 (cp. [2], p.138) one can estimate by

$$\begin{aligned} ||Y_2|| \leq & \int_\delta^\infty ||f(x_0+t)+f(x_0-t)||K(t,v)dt \\ & +2||f(x_0)|| \int_\delta^\infty K(t,v)dt \\ := & A_v + B_v, \quad \text{say}. \end{aligned}$$

Similarly to E_v, $\lim_{v \to v_0} A_v = 0$. Evidently, $\lim_{v \to v_0} B_v = 0$. Therefore $\lim_{v \to v_0} Y_2 = 0$.

Finally, in the "new" integral of the type Y_1 (cp. [2], p. 138) we must use the Natanson's lemmas from [3] instead of [5].

Proof of Theorem 3. ($\Rightarrow$) Suppose that

$$\lim_{(x,v)\to(x_0,v_0)} U(x,v) = b. \tag{5}$$

Consider the function g equivalent to f, given by

$$g(x) = \begin{cases} f(x) & \text{if } x \in L(f), \\ b & \text{if } x \notin L(f). \end{cases}$$

Assume that g is discontinuous at x_0. Then there are $\varepsilon > 0$ and a sequence $\{x_n\}$, $x_n \in L(f)$ $(n = 1, 2, \ldots)$, such that $\lim_{n\to\infty} x_n = x_0$ and

$$||f(x_n) - b|| \geq 2\varepsilon, \; n = 1, 2, \ldots \,. \tag{6}$$

Fix x_n, for the moment. Since $x_n \in L(f)$, then (from (b)-parts) there is v_n for which

$$||U(x_n, v_n) - f(x_n)|| < \varepsilon. \tag{7}$$

Clearly, one may assume that $\lim_{n\to\infty} v_n = v_0$.

This above and (6), (7) yield $\lim_{n\to\infty}(x_n, v_n) = (x_0, v_0)$ and $||U(x_n, v_n) - b|| \geq \varepsilon$, contrary to (5).

For ($\Leftarrow$) see [5], p. 173, 1.1.

4. Examples and applications

A. Let $1 \leq p < \infty$, $G = L_0^p$, where $L_0^p = L^p/N$,

$$N = \left\{ u \in L^p \,\middle|\, ||u||_p = \left(\int_{-\infty}^\infty ||u(t)||^p dt\right)^{\frac{1}{p}} = 0 \right\}.$$

Let $H : \mathbb{R} \to \mathbb{R}$ be a measurable function such that

$$\lim_{x\to x_0-} H(x) = n_-, \quad \lim_{x\to x_0+} H(x) = n_+. \tag{8}$$

Fix g belonging to L_0^p and define the function f by

$$f : \mathbb{R} \to L_0^p \quad \text{and} \quad f(t) = g(\cdot + H(t)), \; t \in \mathbb{R}.$$

Observe that $f \in L^\infty(L_0^p)$, because $\operatorname{ess\,sup}_{t\in\mathbb{R}} \|f(t)\| = \|g\|_p < \infty$ and $f \in S(L^p)$ from measurability of H.

Furthermore, $x_0 \in D_*(f)$ and $x_0 \in L_*(f)$ in view of (8) and Hölder-Minkowski inequality ([1], p. 3). Clearly, $f_-(x_0) = g(\cdot + n_-)$, $f_+(x_0) = g(\cdot + n_+)$; thus, applying Th. 3 or Th. 4 we obtain the following

COROLLARY. *Suppose K satisfies the assumptions of Theorem 3. If p, g, H are as above, $0 < \beta < 1$, $C > 0$, then*

$$\lim_{\substack{(x,v)\to(x_0,v_0)\\ (x,v)\in Z_{C,\beta}}} \left\| \int_{-\infty}^{\infty} K(t-x,v) g(\cdot + H(t))dt - \frac{1}{2}\{g(\cdot + n_-) + g(\cdot + n_+)\} \right\|_p = 0. \tag{9}$$

If K, K^* satisfy the assumptions of Theorem 4, then (9) *is true with $Z^*_{C,\beta}$ instead of $Z_{C,\beta}$. In particular (for $x = x_0 = 0$, $H(t) = t$)*

$$\lim_{v\to v_0} \left\| \int_{-\infty}^{\infty} K(t,v) g(\cdot + t)dt - g(\cdot) \right\|_p = 0.$$

A similar corollary may be stated in the periodic case. Thus, the convergence of real-valued singular integrals in the L^p-norm is a consequence of the pointwise convergence of the adequate vector-valued ones.

B. Let $1 \le p < \infty$ and $G = X_{2\pi}$, where $X_{2\pi}$ denotes one of the spaces $C_{2\pi}$ or $L^p_{2\pi}$ with the sup or $L^p_{2\pi}$ norm, respectively ([1], p. 9). This norm is denoted by $\| \; \|_{X_{2\pi}}$.

Let $\theta_3(x,t)$ be Jacobi's theta-function which is given for $x \in \mathbb{R}$, $t > 0$ by

$$\begin{aligned}\theta_3(x,t) &= 1 + 2\sum_{k=1}^{\infty} e^{-tk^2} \cos kx \\ &= \prod_{k=1}^{\infty} (1 - e^{-2kt})(1 + 2l^{-(2k-1)t} \cos x + e^{-2(2k-1)t})\end{aligned}$$

([1], p. 61).

Let $E = (0, \infty)$, $v = t$, $v_0 = 0$. Then the function $K(x,v) = \frac{\theta_3(x,v)}{2\pi}$ satisfies the assumptions of Theorem 1. Indeed, in view of the second equality in the definition of $\theta_3(x,t)$, $K(\cdot, v)$ is non-increasing on $[0,\pi]$ for all $v \in E$. The other assumptions on K follows from Problem 10 p. 61 and Theorem 1.3.7 p. 58 of [1] (see also p. 63 ibid.).

Applying Theorem 1 in a similar manner as in **A** we get

$$\lim_{\substack{(x,t)\to(x_0,0+)\\(x,t)\in Z_{C,\beta}(\theta_3)}} \Big\| \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta_3(u-x,t)g(\cdot + H(u))du - \frac{1}{2}\{g(\cdot + n_-) + g(\cdot + n_+)\}\Big\|_{X_{2\pi}} = 0, \tag{10}$$

where H, n_-, n_+ are the same as in **A**, g belongs to $X_{2\pi}$.

The special singular integral of Weierstrass

$$W_t(g;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta_3(u-x,t)g(u)du, \quad g \in X_{2\pi}$$

is the solution of Fourier's problem in the ring with the initial condition $u(x,0) = g(x)$, $-\pi \le x \le \pi$ ([1] p. 281-283). Thus, (10) is a complement of properties (i)-(v) (ibid. p. 281) of the solution. The property (iii) is a particular case of (10) (take $x = x_0 = 0$, $H(u) = u$).

By identity (3.1.36) of [1],

$$(\pi/t)^{1/2} \le \theta_3(0,t) \le 2(\pi/t)^{1/2} \quad \text{for} \quad 0 < t \le T_0,$$

where T_0 is positive and small enough. Therefore the set $Z_{C,\beta}(\theta_3)$ in (10) may be replaced by

$$A_C = \{(x,t) \in \mathbb{R} \times (0,\infty) \mid |x - x_0|^{\beta} t^{-1/2} \le C\}.$$

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Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
Poland

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