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# Generalized convex sets

**Abstract.** In this paper we shall present a definition of convex set with respect to a two-parameter family of functions. It appears that these sets have some properties like convex sets in the usual sense.

For the convenience of the reader we first repeat two definitions and one theorem from [1].

Definition 1. A family $F$ of continuous real-valued functions $\varphi$, defined on an open interval $(a,b)$ is said to be a two-parameter family on $(a,b)$ if for any distinct points $x_1$, $x_2$ in $(a,b)$ and any numbers $y_1$, $y_2$ there exists exactly one $\varphi \in F$ satisfying

$$\varphi(x_i) = y_i, \quad i = 1,2.$$

Definition 2. Let $F$ be a two-parameter family on $(a,b)$. We say that a function $\psi$ continuous on $(a,b)$ is convex (concave) function with respect to the family $F$ if for any points $a < x_1 < x_2 < b$ the unique $\varphi \in F$ determined by

$$\varphi(x_i) = \psi(x_i), \quad i = 1,2 \tag{1}$$

satisfies the inequality

$$\psi(x) \underset{(\geq)}{\leq} \varphi(x), \quad x \in (x_1, x_2).$$

Theorem 1 (cf. [3]). *Let $F$ be a two-parameter family on $(a,b)$. Let*

$$a < x_1^n < x_2^n < b \ \ \textit{and} \ \ y_1^n,\ y_2^n \ \ \textit{be real numbers},$$

*for $n = 0, 1, 2, \dots$, such that*

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$$x_i^0 = \lim_{n\to\infty} x_i^n, \quad y_i^0 = \lim_{n\to\infty} y_i^n, \quad i = 1, 2.$$

*Let $\varphi_n$, where $n = 0, 1, 2, \ldots$, be the element of $F$ determined by the relations*

$$\varphi_n(x_i^n) = y_i^n, \quad i = 1, 2.$$

*Then $\varphi_n \to \varphi_0$ uniformly on every compact subinterval of $(a, b)$.*

Now we give a definition of convex set with respect to a two-parameter family. First we introduce some notation.

Let $F$ be a two-parameter family on $(a, b)$, $A, B \in (a, b) \times \mathbb{R}$, $A = (x_1, y_1)$, $B = (x_2, y_2)$. If $x_1 = x_2$, then

$$[A, B] := \{(x_1, y) : y_1 \le y \le y_2\}, \quad y_1 \le y_2,$$
$$[A, B] := \{(x_1, y) : y_2 \le y \le y_1\}, \quad y_1 > y_2.$$

If $x_1 \ne x_2$, then

$$[A, B] := \{(x, \varphi(x)) : x_1 \le x \le x_2\}, \quad x_1 < x_2,$$
$$[A, B] := \{(x, \varphi(x)) : x_2 \le x \le x_1\}, \quad x_1 > x_2,$$

where $\varphi \in F$ is determined by

$$\varphi(x_i) = y_i, \quad i = 1, 2. \tag{2}$$

DEFINITION 3. A set $D \subset (a, b) \times \mathbb{R}$ will be called convex with respect to a two-parameter family $F$ (or briefly $F$-convex) iff for any $A, B \in D$ we have

$$[A, B] \subset D.$$

If $F$ is the family of straight lines, then a set is $F$-convex iff it is convex in the usual sense.

Let $C(a, b)$ denote the set of all continuous function $\psi : (a, b) \to \mathbb{R}$. Set

$$D^\psi := \{(x, y) : x \in (a, b),\ y \ge \psi(x)\},$$
$$D_\psi := \{(x, y) : x \in (a, b),\ y \le \psi(x)\},$$

for $\psi \in C(a, b)$.

We give a generalization of the theorem: "A function $\psi$ is convex (concave) iff the set $D^\psi$ $(D_\psi)$ is convex."

THEOREM 2. *Let $F$ be a two-parameter family on $(a, b)$ and let $\psi \in C(a, b)$. Then*

*a) $\psi$ is convex function with respect to the family $F$ iff the set $D^\psi$ is $F$-convex;*

b) *$\psi$ is concave function with respect to the family $F$ iff the set $D_\psi$ is $F$-convex.*

*Proof.* We shall prove *a)*, the proof of *b)* is analogous. Let us assume that $\psi$ is convex function with respect to $F$ and let

$$A = (x_1, y_1), \quad B = (x_2, y_2), \quad A, B \in D^\psi.$$

Hence

$$y_i \geq \psi(x_i), \qquad i = 1, 2. \tag{3}$$

If $x_1 = x_2$, then obviously $[A, B] \subset D^\psi$. Let $x_1 \neq x_2$. Without loss of generality we may assume that $x_1 < x_2$. Then

$$[A, B] = \{(x, \varphi(x)) : x_1 \leq x \leq x_2\},$$

where $\varphi \in F$ is determined by (2). From (2) and (3) we deduce that

$$\varphi(x_i) \geq \psi(x_i), \qquad i = 1, 2. \tag{4}$$

Suppose, on the contrary, that $[A, B] \not\subset D^\psi$. This means that there exists a $c \in (x_1, x_2)$ such that

$$\varphi(c) < \psi(c). \tag{5}$$

Hence, by the continuity of $\varphi$ and $\psi$ and from (4), it follows that there exist $c_1 \in (x_1, c)$ and $c_2 \in (c, x_2)$ such that

$$\varphi(c_i) = \psi(c_i), \qquad i = 1, 2.$$

Since $\psi$ is convex with respect to $F$,

$$\varphi(x) \geq \psi(x), \qquad x \in (c_1, c_2).$$

In particular, $\varphi(c) \geq \psi(c)$, contrary to (5).

To prove the converse implication, we assume that the set $D^\psi$ is $F$-convex. Let $a < x_1 < x_2 < b$ and let $\varphi \in F$ be determined by (1). We have to show that

$$\psi(x) \leq \varphi(x), \qquad x \in (x_1, x_2).$$

Set

$$A := (x_1, \psi(x_1)), \quad B := (x_2, \psi(x_2)).$$

It is evident that $A, B \in D^\psi$. Since $D^\psi$ is $F$-convex, $[A, B] \subset D^\psi$. Hence

$$(x, \varphi(x)) \in D^\psi \quad \text{for} \quad x \in [x_1, x_2],$$

i.e. $\psi(x) \leq \varphi(x)$ for $x \in (x_1, x_2)$. This completes the proof.

If $\varphi \in F$, then $\varphi$ is convex and concave with respect to $F$. This way from Theorem 2 we get

REMARK 1. *If $\varphi \in F$, then the sets $D^\varphi, D_\varphi$ are $F$-convex.*

Now we give an example of a set, which is $F$-convex but is not convex in the usual sense and an example of a set, which is convex in the usual sense but is not $F$-convex.

EXAMPLE 1. Let

$$F := \{\alpha x + \beta - x^2 : \alpha, \beta \in \mathbb{R},\ x \in \mathbb{R}\} \quad \text{and} \quad \varphi(x) := -x^2,\ x \in \mathbb{R}.$$

Since $\varphi \in F$, the set $D^\varphi$ is $F$-convex (see Remark 1). It is evident that $D^\varphi$ is not convex in the usual sense.

EXAMPLE 2. Let

$$D := \{(x,y) : x^2 + y^2 \leq 1\} \quad \text{and} \quad F := \{\alpha x + \beta + 2x^2 - 2 : \alpha, \beta \in \mathbb{R};\ x \in \mathbb{R}\}.$$

It is obvious that $D$ is convex in the usual sense. We prove that $D$ is not $F$-convex. Consider points $A, B \in D$ such that $A = (-1,0)$, $B = (1,0)$. A simple computation shows that the unique $\varphi \in F$ satisfying

$$\varphi(-1) = 0, \quad \varphi(1) = 0$$

is $\varphi(x) = 2x^2 - 2$. Therefore

$$[A,B] = \{(x, 2x^2 - 2) : -1 \leq x \leq 1\}.$$

Since $(0,-2) \notin D$, $[A,B] \not\subset D$. This means that the set is not $F$-convex.

It is well known that if $D$ is a convex set, then $\operatorname{int} D$ and $\operatorname{cl} D$ are convex sets. We shall give a generalization of this theorem. First we prove a lemma needed in the sequel.

LEMMA 1. *Let $F$ be a two-parameter family on $(a,b)$, $a < x_1 < x_2 < b$, $y_1' < y_1$, $y_2' < y_2$. Assume that $\varphi_1, \varphi_2 \in F$ and*

$$\varphi_1(x_i) = y_i, \quad i = 1,2; \qquad \varphi_2(x_i) = y_i', \quad i = 1,2.$$

*Put*

$$P := \{(x,y) : x_1 \leq x \leq x_2,\ \varphi_2(x) \leq y \leq \varphi_1(x)\},$$

$$F_1 := \{\varphi \in F : y_1' \leq \varphi(x_1) \leq y_1,\ y_2' \leq \varphi(x_2) \leq y_2\}.$$

*Under the above assumptions*

$$P = \bigcup_{\varphi \in F_1} \bigcup_{x \in [x_1, x_2]} (x, \varphi(x)).$$

*Proof.* If $\varphi \in F_1$, then

$$\varphi(x_1) \leq y_1 = \varphi_1(x_1), \quad \varphi(x_2) \leq y_2 = \varphi_1(x_2).$$

Hence

$$\varphi(x) \leq \varphi_1(x), \quad x \in [x_1, x_2],$$

because $F$ is a two-parameter family. Similarly $\varphi(x) \geq \varphi_2(x)$ for $x \in [x_1, x_2]$. This means that

$$\bigcup_{\varphi \in F_1} \bigcup_{x \in [x_1, x_2]} (x, \varphi(x)) \subset P.$$

To prove the converse inclusion, it suffices to show that if $(x_0, y_0) \in P$, then

$$(x_0, y_0) \in \bigcup_{\varphi \in F_1} (x_0, \varphi(x_0)).$$

Fix $(x_0, y_0) \in P$, i.e.

$$x_1 \leq x_0 \leq x_2, \quad \varphi_2(x_2) \leq y_0 \leq \varphi_1(x_0).$$

If $y_0 = \varphi_2(x_0)$, then $(x_0, y_0) = (x_0, \varphi_2(x_0))$ and consequently

$$(x_0, y_0) \in \bigcup_{\varphi \in F_1} (x_0, \varphi(x_0)),$$

because $\varphi_2 \in F_1$.
The same conclusion can be drawn if $y_0 = \varphi_1(x_0)$.

Let $\varphi_2(x_0) < y_0 < \varphi_1(x_0)$ and let $\varphi_3 \in F$ be determined by

$$\varphi_3(x_1) = y_1', \quad \varphi_3(x_2) = y_2 .$$

It is seen at once that $\varphi_3 \in F_1$ and

$$\varphi_2(x) \leq \varphi_3(x) \leq \varphi_1(x), \quad x \in [x_1, x_2].$$

Three cases are possible

1. $y_0 = \varphi_3(x_0)$,
2. $y_0 > \varphi_3(x_0)$,
3. $y_0 < \varphi_3(x_0)$.

1. It is evident that

$$(x_0, y_0) \in \bigcup_{\varphi \in F_1} (x_0, \varphi(x_0)).$$

2. Let $\varphi_4 \in F$ be determined by

$$\varphi_4(x_0) = y_0 , \quad \varphi_4(x_2) = y_2 .$$

Since $\varphi_1(x_2) = \varphi_4(x_2) = \varphi_3(x_2)$ and $\varphi_3(x_0) < \varphi_4(x_0) < \varphi_1(x_0)$,

$$\varphi_3(x) < \varphi_4(x) < \varphi_1(x), \quad x \in (a, x_2).$$

In particular, $\varphi_3(x_1) < \varphi_4(x_1) < \varphi_1(x_1)$. But $\varphi_1(x_1) = y_1$ and $\varphi_3(x_1) = y_1'$. Therefore $y_1' < \varphi_4(x_1) < y_1$ , i.e. $\varphi_4 \in F_1$ . Hence it follows that

$$(x_0, y_0) \in \bigcup_{\varphi \in F_1} (x_0, \varphi(x_0)).$$

3. The proof is similar to the proof in case 2, then we omit it.

THEOREM 3. *Let $F$ be a two-parameter family on $(a,b)$ and let $D \subset (a,b) \times \mathbb{R}$. If $D$ is $F$-convex, then* $\operatorname{int} D$ *and* $\operatorname{cl} D$ *are $F$-convex.*

*Proof.* To prove the first part of the theorem, it suffices to show the following implication

$$A, B \in \operatorname{int} D \Rightarrow [A, B] \subset \operatorname{int} D.$$

Let $A, B \in \operatorname{int} D$ and $A = (x_1, y_1)$, $B = (x_2, y_2)$. Consider first the case $x_1 = x_2$. Since $A, B \in \operatorname{int} D$, there exists $r > 0$ such that

$$K(A, r) \subset D \quad \text{and} \quad K(B, r) \subset D,$$

where $K(C, r)$ is the open ball centered at $C$ and with the radius $r$. Therefore

$$\{(x, y_i) : x_1 - r < x < x_1 + r\} \subset D, \qquad i = 1, 2.$$

Hence

$$\{(x, y) : x_1 - r < x < x_1 + r,\ y_1 \leq y \leq y_2\} \subset D,$$

because $D$ is $F$-convex. It follows that for any $C \in [A, B]$ we have $K(C, r) \subset D$. Consequently

$$[A, B] \subset \operatorname{int} D.$$

Consider now the case where $x_1 \neq x_2$. Let for example $x_1 < x_2$. Then

$$[A, B] = \{(x, \varphi(x)) : x_1 \leq x \leq x_2\},$$

where $\varphi \in F$ is determined by (2). Analysis similar to that in the proor of the case where $x_1 = x_2$ shows that there exists an $r > 0$ such that

$$K(A, r) \subset D \quad \text{and} \quad K(B, r) \subset D.$$

Hence

$$E_1 := \left\{(x_1, y) : y_1 - \frac{r}{2} \leq y \leq y_1 + \frac{r}{2}\right\} \subset D,$$

$$E_2 := \left\{(x_2, y) : y_2 - \frac{r}{2} \leq y \leq y_2 + \frac{r}{2}\right\} \subset D.$$

Let $\varphi_1, \varphi_2 \in F$ be determined by the conditions

$$\varphi_1(x_i) = y_i + \tfrac{r}{2}, \quad i = 1, 2,$$

$$\varphi_2(x_i) = y_i - \tfrac{r}{2}, \quad i = 1, 2.$$

It follows from the definitions of $\varphi_1$, $\varphi_2$ and $\varphi$ that

$$\varphi_2(x) < \varphi(x) < \varphi_1(x), \quad x \in [x_1, x_2]. \tag{6}$$

Since $D$ is $F$-convex and $E_1, E_2 \subset D$, we have

$$\{(x,y) : x_1 \leq x \leq x_2,\ \varphi_2(x) \leq y \leq \varphi_1(x)\} \subset D, \tag{7}$$

by Lemma 1. Let

$$r_1 := \inf\{d((x,\varphi(x)),(\bar{x},\varphi_1(\bar{x}))) : x,\bar{x} \in [x_1,x_2]\},$$

$$r_2 := \inf\{d((x,\varphi(x)),(\bar{x},\varphi_2(\bar{x}))) : x,\bar{x} \in [x_1,x_2]\},$$

$$\bar{r} := \min(r_1,r_2),$$

where $d$ is the euclidean metric. By the continuity of $\varphi$, $\varphi_1$ and $\varphi_2$, and from (6) we get $r_1 > 0$, $r_2 > 0$. Therefore $\bar{r} > 0$. It follows from the definition of $\bar{r}$ and from (7) that for any $C \in [A,B]$ we have $K(C,\bar{r}) \subset D$. This means that $[A,B] \subset \operatorname{int} D$, which completes the proof of the first part of the theorem.

To prove the second part of the theorem, it suffices to show the following implication

$$A, B \in \operatorname{cl} D \Rightarrow [A,B] \subset \operatorname{cl} D.$$

Let $A, B \in \operatorname{cl} D$ and $A = (x_1,y_1)$, $B = (x_2,y_2)$. First we consider the case where $x_1 = x_2$. Since $A, B \in \operatorname{cl} D$, there exist sequences $\{A_n\}, \{B_n\} \subset D$ such that

$$A_n \to A \quad \text{and} \quad B_n \to B.$$

Hence $[A_n, B_n] \subset D$ for $n = 1, 2, \ldots$, because $D$ is $F$-convex. Fix $r > 0$ and let $n_0 \in N$ satisfies the conditions

$$d(A, A_{n_0}) < r, \quad d(B, B_{n_0}) < r.$$

It is easy to check that for any $C \in [A,B]$ we have

$$[A_{n_0}, B_{n_0}] \cap K(C,r) \neq \emptyset.$$

It follows from this that for any $C \in [A,B]$

$$D \cap K(C,r) \neq \emptyset.$$

Consequently

$$[A,B] \subset \operatorname{cl} D.$$

Consider now the case where $x_1 \neq x_2$. Let for example $x_1 < x_2$. Then

$$[A,B] = \{(x,\varphi(x)) : x_1 \leq x \leq x_2\},$$

where $\varphi \in F$ is determined by (2). As in the proof in the case where $x_1 = x_2$, there exist sequences $\{A_n\}, \{B_n\} \subset D$ such that

$$A_n \to A, \quad B_n \to B \ \text{ and } \ [A_n, B_n] \subset D \text{ for } n = 3, 4, \ldots.$$

Let $A_n = (x_n, y_n)$, $B_n = (x'_n, y'_n)$. Whithout loss of generality we may assume that $x_n < x'_n$ for $n = 3, 4, \ldots$. Obviously, $x_n \to x_1$, $y_n \to y_1$, $x'_n \to x_2$ and $y'_n \to y_2$. Let $\varphi_n \in F$ for $n = 3, 4, \ldots$, be determined by the conditions

$$\varphi_n(x_n) = y_n , \quad \varphi_n(x'_n) = y'_n.$$

Then

$$[A_n, B_n] = \{(x, \varphi_n(x)) : x_n \leq x \leq x'_n\} \subset D, \quad n = 3, 4, \ldots .$$

By Theorem 1 $\varphi_n \to \varphi$ uniformly on every compact subinterval of $(a, b)$. From this we deduce that

$$[A, B] \subset \operatorname{cl} D,$$

which proves the theorem.

An easy consequence of the definition of $F$-convex set is

LEMMA 2. *Let $F$ be a two-parameter family on $(a, b)$. The intersection of any family of $F$-convex subsets of $(a, b) \times \mathbb{R}$ is $F$-convex.*

From the above lemma and from Remark 1 it follows

REMARK 2. *If $\varphi_1, \varphi_2 \in F$, then the set*

$$D^{\varphi_1} \cap D_{\varphi_2}$$

*is $F$-convex.*

As in the case of the usual convexity, we may introduce the definition of the convex hull.

DEFINITION 4. Let $F$ be a two-parameter family on $(a, b)$ and $D \subset (a, b) \times \mathbb{R}$. The set

$$\operatorname{conv}_F D := \bigcap \{U \subset (a, b) \times \mathbb{R} : U \text{ is } F\text{-convex}, \ D \subset U\}$$

is called the convex hull of $D$ with respect to the family $F$.

From this definition and from Lemma 2 we get

THEOREM 4. *Let $F$ be a two-parameter family on $(a, b)$ and let $D, D_1$, $D_2 \subset (a, b) \times \mathbb{R}$. Then*

*1.* $D \subset \operatorname{conv}_F D$,

*2.* $\operatorname{conv}_F D$ *is the smallest $F$-convex set containing $D$,*

*3.* *$D$ is $F$-convex set iff* $D = \operatorname{conv}_F D$,

*4.* *if* $D_1 \subset D_2$, *then* $\operatorname{conv}_F D_1 \subset \operatorname{conv}_F D_2$.

One can prove that if $D$ is a closed (an open) set, then the set $\operatorname{conv}_F D$ is closed (open).

D. Brydak in [2] has proved the following

THEOREM 5. *Let $F$ be a two-parameter family of differentiable functions on $(a, b)$ such that for any $x_0 \in (a, b)$ and for any real numbers $y_0, y_1$ there exists exactly one element $\varphi \in F$ satisfying*

$$\varphi(x_0) = y_0, \quad \varphi'(x_0) = y_1.$$

*Suppose that a function $\psi$ is differentiable on $(a, b)$.*

*Under the above assumptions the following conditions are equivalent:*

(A) *the function $\psi$ is convex with respect to the family $F$;*

(B) *for any $x_0 \in (a, b)$*

$$\psi(x) \geq \varphi_{x_0}(x) \quad \text{for } x \in (a, b),$$

*where $\varphi_{x_0} \in F$ is determined by*

$$\varphi_{x_0}(x_0) = \psi(x_0), \quad \varphi'_{x_0}(x_0) = \psi'(x_0). \tag{8}$$

We are able to give a simpler proof of the implication $(B) \Rightarrow (A)$ than that given in [2].

We see at once (from (B)) that

$$D^{\psi} = \bigcap_{x_0 \in (a,b)} D^{\varphi_{x_0}},$$

where $\varphi_{x_0} \in F$ is determined by (8). By Remark 1, the set $D^{\varphi_{x_0}}$ is $F$-convex (for any $x_0 \in (a, b)$). It follows from Lemma 2 that the set $D^{\psi}$ is $F$-convex and consequently $\psi$ is convex with respect to the family $F$ (see Theorem 2).

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