ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE
Zeszyt 189 Prace Matematyczne XIV 1997

JÓZEF TABOR

Quasi-linear mappings

Abstract. Let E_1, E_2 be real normed spaces and let $\varepsilon \in [0,1)$. The paper deals with the system of inequalities

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \min\{\|f(x+y)\|, \|f(x)+f(y)\|\} \quad \text{for } x, y \in E_1,$$

$$\|f(\alpha x) - \alpha f(x)\| \le \varepsilon \min\{\|f(\alpha x)\|, \|\alpha f(x)\|\} \quad \text{for } x \in E_1, \alpha \in \mathbf{R},$$

where f maps E_1 into E_2.
We prove that some basic theorems concerning linear operators also hold for mappings satisfying these inequalities. In the next part of the paper we assume additionally that $E_2 = \mathbf{R}$ and f is continuous. Then we prove that there exists a continuous linear mapping $L : E_1 \to \mathbf{R}$ such that

$$| f(x) - L(x) | \le \varepsilon \min\{| f(x) |, | L(x) |\} \quad \text{for } x \in E_1 .$$

In the set of such linear mappings there exists a unique one, which is the best linear approximation of f.

1. Introduction

S. M. Ulam posed in [5] the following question: "When does for a nearly linear mapping f there exist a linear mapping which is near to f?" Let E_1 be a real vector space and E_2 a real normed space. D. H. Hyers [1] meant the term "nearly linear mapping" as a mapping $f : E_1 \to E_2$ satisfying for some $\varepsilon \ge 0$ the following inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \quad \text{for } x, y \in E_1 . \tag{1}$$

In the present-day terminology such a mapping is called *nearly additive* or *approximately additive*. By a linear mapping we mean a mapping $g : E_1 \to E_2$

AMS (1991) subject classification: Primary 39B72, Secondary 39A11, 15A04.

satisfying the following conditions

$$g(x+y) = g(x) + g(y) \quad \text{for } x, y \in E_1\,, \tag{2}$$

$$g(\alpha x) = \alpha g(x) \quad \text{for } x \in E_1,\ \alpha \in \mathbf{R}. \tag{3}$$

Following Hyers' idea we may say that a mapping $f : E_1 \to E_2$ is nearly linear if it is nearly additive and nearly homogeneous, i.e. if it satisfies (1) and

$$\|f(\alpha x) - \alpha f(x)\| \le \varepsilon \quad \text{for } x \in E_1,\ \alpha \in \mathbf{R}. \tag{4}$$

In fact such a mapping must be linear (cf. [4]). Inequalities (1) and (4) can be obtained by replacing in (2) and (3) equalities by "equalities up to ε". In other words, (1) and (4) mean that we deal with a linear mapping up to some error, namely, with an absolute error less than or equal to ε. But from the point of view of applications an absolute error is not so important as a relative error. Therefore it is reasonable to investigate additive up to small relative errors mappings, i.e. mappings $f : E_1 \to E_2$ satisfying for some $\varepsilon \in [0,1)$ the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \min\{\|f(x+y)\|, \|f(x)+f(y)\|\} \quad \text{for } x, y \in E_1\,. \tag{5}$$

Such a mapping f is called quasi-addititve (cf. [3]). A similar reasoning leads to the following generalization of equation (3)

$$\|f(\alpha x) - \alpha f(x)\| \le \varepsilon \min\{\|f(\alpha x)\|, |\,\alpha\,|\,\|f(x)\|\} \quad \text{for } x \in E_1,\ \alpha \in \mathbf{R}, \tag{6}$$

where $\varepsilon \in [0,1)$.
A mapping $f : E_1 \to E_2$ satisfying (5) and (6) will be called *ε-quasi-linear* (*quasi-linear* if the value of ε is inessential).

2. General properties

It appears that quasi-linear mappings have properties very similar to those of linear ones.

THEOREM 1. *Let E_1 be a real vector space, E_2 a real normed space and let $f : E_1 \to E_2$ be quasi-linear. Then*
(i) $\ker f$ *is a subspace of* E_1,
(ii) $f(x) = f(y) \Leftrightarrow x - y \in \ker f$ *for* $x, y \in E_1$.

Proof. (i). By (6) $f(0)=0$, i.e. $0 \in \ker f$. If $x, y \in \ker f$ then by (5) $x+y \in \ker f$. If $x \in \ker f$ then we obtain from (6) that $\alpha x \in \ker f$ for $\alpha \in \mathbf{R}$.
(ii). Suppose that $f(x)=f(y)$. By Proposition 2 from [3] f is odd. Making use of (5) and the oddness of f we obtain

$$\|f(x-y)\| = \|f(x-y)-f(x)-f(-y)\| \le \varepsilon\|f(x)+f(-y)\| = 0,$$

which means that $x-y \in \ker f$.
Now suppose that $x-y \in \ker f$. Then by (5)

$$\begin{aligned}\|f(x)-f(y)\| &= \|f(y+(x-y))-f(y)-f(x-y)\| \\ &\le \varepsilon \min\{\|f(y)\|, \|f(x-y)\|\} = 0,\end{aligned}$$

i.e. $f(x)=f(y)$.

THEOREM 2. *Let E_1, E_2 be real normed spaces and let $f : E_1 \to E_2$ be quasi-linear. Then the following conditions are equivalent:*
(i) *f is continuous,*
(ii) *there exists a $c \in \mathbf{R}$ such that*

$$\|f(x)\| \le c\|x\| \quad \textit{for } x \in E_1,$$

(iii) *f is bounded on the unit sphere $S = \{x \in E_1 : \|x\| = 1\}$.*

Proof. (i)⇒(ii). It is clear that condition (ii) is equivalent to the following one

$$\sup\left\{\frac{\|f(x)\|}{\|x\|} : x \in E_1, x \neq 0\right\} < \infty.$$

Suppose the contrary i.e.

$$\sup\left\{\frac{\|f(x)\|}{\|x\|} : x \in E_1, x \neq 0\right\} = \infty.$$

Then there exists a sequence $\{x_n\}$, $x_n \in E_1 \setminus \{0\}$ such that

$$\frac{\|f(x_n)\|}{\|x_n\|} \ge n \quad \text{for } n \in \mathbf{N}. \tag{7}$$

Let $z_n := \frac{1}{n\|x_n\|}x_n$. Obviously $z_n \to 0$. Since f is continuous and $f(0)=0$,

$$\lim_{n\to\infty} f(z_n) = 0. \tag{8}$$

By (6) we have

$$\left\|f(z_n) - \frac{1}{n\|x_n\|}f(x_n)\right\| \le \frac{\varepsilon}{n\|x_n\|}\|f(x_n)\| \quad \text{for } n \in \mathbf{N},$$

whence by (7) we get

$$\|f(z_n)\| \geq (1-\varepsilon)\frac{1}{n\|x_n\|}\|f(x_n)\| \geq 1-\varepsilon \quad \text{for } n \in N,$$

which contradicts (8).
(ii)⇒(iii). Obvious.
(iii)⇒(i). Let

$$\|f(x)\| \leq M \quad \text{for } x \in S.$$

For an arbitrary $x \in E_1$ such that $0 < \|x\| \leq 1$, we have by (6)

$$\|f(x)\| = \left\|f\left(\|x\|\frac{x}{\|x\|}\right)\right\| \leq (\varepsilon+1)\|x\|\left\|f\left(\frac{x}{\|x\|}\right)\right\| \leq (\varepsilon+1)M.$$

Hence f is locally bounded at zero. By Theorem 1 from [3] f is continuous.

For the next theorem we need the following simple lemma.

LEMMA 1. *Let E_1 be a real vector space and E_2 a real normed space. If a mapping $f: E_1 \to E_2$ satisfies* (5) *and* (6) *then*

$$\|f(\alpha_1 x_1 + \ldots + \alpha_n x_n)\| \leq (1+\varepsilon)^n(|\alpha_1|\,\|f(x_1)\| + \ldots + |\alpha_n|\,\|f(x_n)\|)$$
$$\textit{for } x_1, \ldots, x_n \in E_1,\ \alpha_1, \ldots, \alpha_n \in \mathbf{R},\ n \in \mathbf{N}.$$

Proof. Easy induction.

THEOREM 3. *Let E_1 and E_2 be real normed spaces and let* $\dim E_1 < \infty$. *Then every quasi-linear mapping $f: E_1 \to E_2$ is continuous.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of E_1. We have, by Lemma 1, for some $\varepsilon \in [0,1)$

$$\|f(\alpha_1 e_1 + \ldots + \alpha_n e_n)\| \leq (1+\varepsilon)^n(|\alpha_1|\,\|f(e_1)\| + \ldots + |\alpha_n|\,\|f(e_n)\|)$$
$$\text{for } \alpha_1, \ldots, \alpha_n \in \mathbf{R}.$$

Hence f is locally bounded at zero. By Theorem 1 from [3] f is continuous.

LEMMA 2. *Let E_1 be a real vector space, E_2 a normed space and let $f: E_1 \to E_2$ satisfy* (5) *for some $\varepsilon \in [0,1)$. Then*

$$\|f(\alpha x) - \alpha f(x)\| \leq \frac{2\varepsilon}{1-\varepsilon}\min\{|\alpha|\,\|f(x)\|, \|f(\alpha x)\|\} \quad \textit{for } x \in E_1,\ \alpha \in \mathbf{Q}. \tag{9}$$

Proof. Obviously (9) can be written as a conjuction of

$$\|f(\alpha x) - \alpha f(x)\| \leq \frac{2\varepsilon}{1-\varepsilon}|\alpha|\,\|f(x)\| \quad \text{for } x \in E_1, \alpha \in \mathbf{Q} \tag{10}$$

and

$$\|f(\alpha x) - \alpha f(x)\| \leq \frac{2\varepsilon}{1-\varepsilon}\|f(\alpha x)\| \quad \text{for } x \in E_1,\ \alpha \in \mathbf{Q}. \tag{11}$$

By Lemma 1 from [3] we have

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon \min\{\|f(x)\|, \|f(y)\|\} \quad \text{for } x,y \in E_1\,. \tag{12}$$

We prove inductively that

$$\|f(nx)-nf(x)\| \le n\varepsilon\|f(x)\| \quad \text{for } x \in E_1,\ n \in \mathbf{N}. \tag{13}$$

For $n=1$ it is obvious. Suppose that this inequality holds for some $n \in \mathbf{N}$. Then by (12) for $x \in E_1$ we obtain

$$\begin{aligned}\|f((n+1)x)-(n+1)f(x)\| &\le \|f((n+1)x)-f(nx)-f(x)\| \\ &\quad +\|f(nx)-nf(x)\| \le \varepsilon\|f(x)\| + n\varepsilon\|f(x)\| \\ &= (n+1)\varepsilon\|f(x)\|.\end{aligned}$$

Thus (13) is proved. Since by Proposition 2 from [3] f is odd, we have by (13)

$$\|f(nx)-nf(x)\| \le \varepsilon \mid n \mid \|f(x)\| \quad \text{for } x \in E_1,\ n \in \mathbf{Z}. \tag{14}$$

For $x \in E_1$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$, we get by (14)

$$\left\|\frac{m}{n}f(nx)-mf(x)\right\| \le \varepsilon \mid m \mid \|f(x)\|$$

and

$$\|f(mx)-mf(x)\| \le \varepsilon \mid m \mid \|f(x)\|.$$

Adding these inequalities side by side, applying the triangle inequality and replacing x by $\frac{x}{n}$ we obtain

$$\left\|f\left(\frac{m}{n}x\right)-\frac{m}{n}f(x)\right\| \le 2\varepsilon \mid m \mid \left\|f\left(\frac{x}{n}\right)\right\| \quad \text{for } x \in E_1,\ m \in \mathbf{Z},\ n \in \mathbf{N}. \tag{15}$$

From (14) we get

$$\|f(x)\| \le \frac{1}{\mid n \mid (1-\varepsilon)}\|f(nx)\| \quad \text{for } x \in E_1,\ n \in \mathbf{Z}. \tag{16}$$

Replacing in this inequality x by $\frac{x}{n}$, we obtain

$$\left\|f\left(\frac{x}{n}\right)\right\| \le \frac{1}{\mid n \mid (1-\varepsilon)}\|f(x)\| \quad \text{for } x \in E_1,\ n \in \mathbf{Z},$$

which together with (15) yields

$$\left\|f\left(\frac{m}{n}x\right)-\frac{m}{n}f(x)\right\| \le \frac{2\varepsilon}{1-\varepsilon}\left|\frac{m}{n}\right| \|f(x)\| \quad \text{for } x \in E_1,\ m \in \mathbf{Z},\ n \in N.$$

This means that (10) is valid.

For $x \in E_1$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$, we have by (15) and (16)

$$\left\| f\left(\frac{m}{n}x\right) - \frac{m}{n}f(x)\right\| \leq 2\varepsilon \mid m \mid \left\| f\left(\frac{x}{n}\right)\right\| \leq \frac{2\varepsilon}{1-\varepsilon}\left\| f\left(\frac{m}{n}x\right)\right\|.$$

THEOREM 4. *Let E_1, E_2 be real normed spaces, and let $\varepsilon \in [0,1)$. Let $f : E_1 \to E_2$ be an ε-quasi-additive mapping such that for each $x \in E_1$ the mapping $\mathbf{R} \ni \alpha \to f(\alpha x)$ is continuous. Then f is $\frac{2\varepsilon}{1-\varepsilon}$-quasi-linear.*

Proof. Let $x \in E_1$, $\alpha \in \mathbf{R}$, and let $\{\alpha_n\}$, $\alpha_n \in \mathbf{Q}$ be a sequence such that $\lim_{n\to\infty} \alpha_n = \alpha$. By Lemma 2 we have

$$\|f(\alpha_n x) - \alpha_n f(x)\| \leq \frac{2\varepsilon}{1-\varepsilon} \min\{\mid \alpha_n \mid \|f(x)\|, \|f(\alpha_n x)\|\} \quad \text{for } n \in \mathbf{N}.$$

Letting $n \to \infty$ we obtain

$$\|f(\alpha x) - \alpha f(x)\| \leq \frac{2\varepsilon}{1-\varepsilon} \min\{\mid \alpha \mid \|f(x)\|, \|f(\alpha x)\|\}.$$

3. Ulam's problem

We turn again to Ulam's question. If we understand the term "nearly linear mapping" as quasi-linear one, it may be formulated as follows.

Let E_1, E_2 be normed spaces and let $\varepsilon \in [0,1)$. Does there exist an $\varepsilon_1 \in [0,1)$ such that for each ε-quasi-linear mapping $f : E_1 \to E_2$ there exists a linear mapping $L : E_1 \to E_2$ satisfying the following condition

$$\|f(x) - L(x)\| \leq \varepsilon_1 \min\{\|f(x)\|, \|L(x)\|\} \quad \text{for } x \in E_1\,? \tag{17}$$

We will answer this question affirmatively for $E_2 = \mathbf{R}$ and f being continuous. We will also prove that in this case there exists the best linear approximation of f. The meaning of the term "the best linear approximation" will be specified later on.

We start with the following lemma.

LEMMA 3. *Let E_1 be a real normed space and let $f : E_1 \to \mathbf{R}$ be quasi-linear and continuous. Then there exists a subspace A of E_1 such that* $\dim A \leq 1$ *and*

$$E_1 = \ker f \oplus A. \tag{18}$$

Proof. By Theorem 1(i) E_1 can be written in the form (18). We need only to prove that $\dim A \leq 1$. Let $g := f \mid A$. Then $\ker g = \{0\}$, and hence $A \setminus \{0\}$ is the disjoint union of the open sets $g^{-1}((0,\infty))$ and $g^{-1}((-\infty,0))$. If $\dim A > 1$ then, since g is odd, neither of these sets is void so $A \setminus \{0\}$ is not connected – a contradiction.

THEOREM 5. *Let E_1 be a real normed space, let $\varepsilon \in [0,1)$, and let $f : E_1 \to \mathbf{R}$ be ε-quasi-linear and continuous. Then there exist uniquely determined: an $\varepsilon_f^0 \in [0,\varepsilon)$ and a continuous linear mapping $L_f : E_1 \to \mathbf{R}$ such that*

$$|f(x) - L_f(x)| \leq \varepsilon_f^0 \min\{|f(x)|, |L_f(x)|\} \quad \textit{for } x \in E_1 \tag{19}$$

and there is no smaller ε_1 satisfying (19) *with some linear $L : E_1 \to \mathbf{R}$.*

Proof. Let D_f denote the set of all linear mappings $L : E_1 \to \mathbf{R}$ satisfying with a certain $\varepsilon_1 \geq 0$ the following condition

$$|f(x) - L(x)| \leq \varepsilon_1 \min\{|f(x)|, L(x)|\} \quad \text{for } x \in E_1 . \tag{20}$$

Since f is continuous, every $L \in D_f$ is locally bounded and hence continuous. It is clear that

$$\ker f = \ker L \quad \text{for } L \in D_f . \tag{21}$$

Hence if $f = 0$ then $L_f = 0$ and $\varepsilon_f^0 = 0$. From now on we assume that $f \neq 0$. We begin with the case $E_1 = \mathbf{R}$. Then by Lemma 3 from [2] f is strictly monotonic. Let

$$\begin{aligned} c_f &:= \inf\left\{\left|\frac{f(x)}{x}\right| : x \in \mathbf{R} \setminus \{0\}\right\}, \\ d_f &:= \sup\left\{\left|\frac{f(x)}{x}\right| : x \in \mathbf{R} \setminus \{0\}\right\}. \end{aligned} \tag{22}$$

It follows from (6) that

$$\frac{1}{1+\varepsilon}|\alpha||f(1)| \leq c_f \leq d_f \leq (1+\varepsilon)|\alpha||f(1)| \quad \text{for } \alpha \in \mathbf{R}.$$

Hence we obtain

$$0 < \frac{1}{1+\varepsilon}|f(1)| \leq c_f \leq d_f \leq (1+\varepsilon)|f(1)|. \tag{23}$$

For any $a \in \mathbf{R}$ let

$$L_a(x) := ax \quad \text{for } x \in \mathbf{R}.$$

We shall prove that $D_f = \{L_a : a \neq 0\}$. If $L_a \in D_f$ then, by (21), $L_a \neq 0$, i.e. $a \neq 0$. On the other hand, if $a \neq 0$ then, by (22) and (23), we have for $x \in \mathbf{R}$

$$|f(x) - ax| \leq (d_f + |a|)|x| = \frac{d_f + |a|}{|a|}|ax|,$$

and

$$|f(x) - ax| \leq (d_f + |a|)|x| = \frac{d_f + |a|}{c}c_f|x| \leq \frac{d_f + |a|}{c_f}f(x).$$

It means that L_a satisfies (20) with $\varepsilon_1 = \max\left\{\frac{d_f+|a|}{|a|}, \frac{d_f+|a|}{c_f}\right\}$, i.e. $L_a \in D_f$. Hence $D_f = \{L_a : a \neq 0\}$.

We define a function $\varepsilon_f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ by putting

$$\varepsilon_f(a) := \sup\left\{ \frac{|f(x) - ax|}{\min\{|f(x)|, |ax|\}} : x \neq 0 \right\}.$$

Clearly $\varepsilon_f(a)$ is the smallest ε_1 satisfying (20) with $L = L_a$. Therefore it is needed to prove that the function $\varepsilon_f(\cdot)$ reaches the minimum. We have

$$\begin{aligned} \varepsilon_f(a) &= \sup\left\{ \max\left\{ \frac{|f(x) - ax|}{|f(x)|}, \frac{|f(x) - ax|}{|ax|} \right\} : x \neq 0 \right\} \\ &= \max\left\{ \sup\left\{ \frac{|f(x) - ax|}{|f(x)|} : x \neq 0, \right\}, \sup\left\{ \frac{|f(x) - ax|}{|ax|} : x \neq 0 \right\} \right\}. \end{aligned}$$

Since f is odd, we obtain further

$$\begin{aligned} \varepsilon_f(a) &= \max\left\{ \sup\left\{ \frac{|f(x) - ax|}{|f(x)|} : x > 0 \right\}, \sup\left\{ \frac{|f(x) - ax|}{|ax|} : x > 0 \right\} \right\} \\ &= \max\left\{ \sup\left\{ \left|1 - a\frac{x}{f(x)}\right| : x > 0 \right\}, \sup\left\{ \left|\frac{1}{a}\frac{f(x)}{x} - 1\right| : x > 0 \right\} \right\}. \end{aligned}$$

For any $a \in \mathbf{R} \setminus \{0\}$ we put

$$\begin{aligned} G_f(a) &:= \sup\left\{ \left|1 - a\frac{x}{f(x)}\right| : x > 0 \right\}, \\ H_f(a) &:= \sup\left\{ \left|\frac{1}{a}\frac{f(x)}{x} - 1\right| : x > 0 \right\}. \end{aligned} \tag{24}$$

Then

$$\varepsilon_f(a) = \max\{G_f(a), H_f(a)\} \quad \text{for } a \in \mathbf{R} \setminus \{0\}. \tag{25}$$

Suppose that f is strictly increasing. Then

$$\begin{aligned} c_f &= \inf\left\{ \frac{f(x)}{x} : x > 0 \right\}, \\ d_f &= \sup\left\{ \frac{f(x)}{x} : x > 0 \right\}. \end{aligned} \tag{26}$$

We are going to show that

$$\varepsilon_f(a) > \varepsilon_f(c_f) \quad \text{for } a \in (\mathbf{R} \setminus \{0\}) \setminus [c_f, d_f]. \tag{27}$$

In view of (24) and (25), $\varepsilon_f(a) < \varepsilon_f(-a)$ for $a > 0$. Making use of (26) we obtain

$$\begin{aligned} G_f(a) &= \max\left\{\left|1-\frac{a}{c_f}\right|,\left|1-\frac{a}{d_f}\right|\right\},\\ H_f(a) &= \max\left\{\left|\frac{c_f}{a}-1\right|,\left|\frac{d_f}{a}-1\right|\right\} \end{aligned} \tag{28}$$

for $a > 0$.
Hence we have

$$H_f(a) = \frac{d_f}{a} - 1 > \frac{d_f}{c_f} - 1 = \varepsilon_f(c_f) \quad \text{for } a \in (0, c_f),$$

and

$$G_f(a) = \frac{a}{c_f} - 1 > \frac{d_f}{c_f} - 1 = \varepsilon_f(c_f) \quad \text{for } a \in (d_f, \infty).$$

This means that (27) is valid.
Let $a \in [c_f, d_f]$. By (25) and (28) we have

$$\begin{aligned} \varepsilon_f(a) &= \max\left\{\left|1-\frac{a}{c_f}\right|,\left|1-\frac{a}{d_f}\right|,\left|\frac{c_f}{a}-1\right|,\left|\frac{d_f}{a}-1\right|\right\}\\ &= \max\left\{\frac{a}{c_f}-1,\frac{d_f}{a}-1\right\}. \end{aligned}$$

Applying this equality one can calculate easily that $\varepsilon_f(\cdot)$ has a strict minimum at $a_f := \sqrt{c_f d_f}$. Let $\varepsilon_f^0 := \varepsilon_f(a_f) = \sqrt{\frac{d_f}{c_f}} - 1$. Since $f(1) > 0$, as $f(0) = 0$ and f is strictly increasing, we obtain from (23)

$$\varepsilon_f^0 = \sqrt{\frac{d_f}{c_f}} - 1 \le \sqrt{(1+\varepsilon)^2} - 1 = \varepsilon.$$

We define a linear mapping $L_f : \mathbf{R} \to \mathbf{R}$ by putting

$$L_f(x) = a_f \cdot x = \sqrt{c_f d_f}\, x \quad \text{for } x \in \mathbf{R}.$$

It is clear that ε_f^0 and L_f satisfy the assertion of the theorem.

Now suppose that f is strictly decreasing. Let $g(x) := -f(x)$. Then, by the preceding part of the proof, $\varepsilon_g^0 := \sqrt{\frac{d_g}{c_g}} - 1$ and L_g defined by the formula

$$L_g(x) = \sqrt{c_g d_g}\, x \quad \text{for } x \in \mathbf{R}$$

satisfy the theorem for g in place of f. But $c_g = c_f$, $d_g = d_f$, and hence $\varepsilon_g^0 = \sqrt{\frac{d_g}{c_g}} - 1 = \sqrt{\frac{d_f}{c_f}} - 1$. Since f and L_g are odd we conclude that $\varepsilon_f^0 := \sqrt{\frac{d_f}{c_f}} - 1$

and $L_f := -L_g$ satisfy the assertion for the function f. It is clear that there are no other constants and linear mappings with this property. We have proved that, in the case where $E_1 = \mathbf{R}$ and $f \neq 0$, $\varepsilon_f^0 := \sqrt{\frac{d_f}{c_f}} - 1$ and L_f defined by the formula

$$L_f(x) = \operatorname{sgn} f(1)\sqrt{c_f d_f}\, x \quad \text{for } x \in \mathbf{R}$$

is the unique pair satisfying the conclusion of our theorem.

Now we consider the general case where E_1 is a normed space. Since $f \neq 0$, there exists an $e_1 \in E_1$ such that $f(e_1) \neq 0$. By Lemma 3

$$E_1 = \ker f \oplus \operatorname{Lin}(e_1).$$

Let

$$f^*(t) := f(te_1) \quad \text{for } t \in \mathbf{R}.$$

For each linear mapping $L : E_1 \to \mathbf{R}$ we define a corresponding mapping $L^* : \mathbf{R} \to \mathbf{R}$ by putting

$$L^*(t) := L(te_1) \quad \text{for } t \in \mathbf{R}.$$

Making use of Theorem 1 (ii) and (21) we obtain

$$f(x_1 + te_1) = f(te_1) = f^*(t) \quad \text{for } x_1 \in \ker f,\ t \in \mathbf{R},$$

$$L(x_1 + te_1) = L(te_1) = L^*(t) \quad \text{for } x_1 \in \ker f,\ t \in \mathbf{R}.$$

By the last equality and Theorem 1 (i), each linear mapping of $\mathbf{R}$ into $\mathbf{R}$ determines uniquely a corresponding linear mapping of E_1 into $\mathbf{R}$. In this way we have reduced the problem of finding ε_f^0 and L_f^0 to the real case. We have

$$\varepsilon_f^0 = \sqrt{\frac{d_f^*}{c_f^*}} - 1,$$

$$L_f^0(x_1 + te_1) = \operatorname{sgn} f^*(1)\sqrt{c_{f^*} d_{f^*}}\, t \quad \text{for } x_1 \in \ker f,\ t \in \mathbf{R}.$$

In order to complete the proof it is sufficient to show that ε_f^0 and L_f^0 are determined uniquely, i.e. that they do not depend on the choice of e_1. Let $e_2 \notin \ker f$, and let

$$f_1^*(t) := f(te_2) \quad \text{for } t \in \mathbf{R},$$

$$L_1(x_1 + te_2) := \operatorname{sgn} f_1^*(1)\sqrt{c_{f_1^*} d_{f_1^*}}\, t \quad \text{for } x_1 \in \ker f,\ t \in \mathbf{R}.$$

The element e_2 can be uniquely represented in the form

$$e_2 = x_0 + t_0 e_1,$$

where $x_0 \in \ker f$ and $t_0 \in \mathbf{R} \setminus \{0\}$.

Applying Theorem 1 (ii) and the definitions of f^* and f_1^*, we obtain

$$f_1^*(t) = f(te_2) = f(tx_0 + tt_0e_1) = f(tt_0e_1) = f^*(tt_0) \quad \text{for } t \in \mathbf{R}.$$

Hence we have

$$c_{f_1^*} = \inf\left\{\frac{|f_1^*(t)|}{|t|} : t \neq 0\right\} = \inf\left\{\frac{|f^*(tt_0)|}{|t|} : t \neq 0\right\} = |t_0|c_f^*,$$

$$d_{f_1^*} = \sup\left\{\frac{|f_1^*(t)|}{|t|} : t \neq 0\right\} = \sup\left\{\frac{|f^*(tt_0)|}{|t|} : t \neq 0\right\} = |t_0|d_f^*.$$

Thus

$$\sqrt{\frac{d_{f_1^*}}{c_{f_1^*}}} - 1 = \sqrt{\frac{d_{f^*}}{c_{f^*}}} - 1 = \varepsilon_f^0.$$

We also have

$$\operatorname{sgn} f_1^*(1) = \operatorname{sgn} f^*(t_0) = \operatorname{sgn} t_0 \operatorname{sgn} f^*(1).$$

Finally, for $x_1 \in \ker f$, $t \in \mathbf{R}$, we obtain

$$\begin{aligned} L_1(x_1 + te_1) &= L_1(x_1 - \frac{t}{t_0}x_0 + \frac{t}{t_0}e_2) = \operatorname{sgn} f_1^*(1)\sqrt{c_{f_1^*}^0 d_{f_1^*}}\frac{t}{t_0} \\ &= \operatorname{sgn} t_0 \operatorname{sgn} f^*(1)\sqrt{c_{f^*} d_{f^*}}|t_0|\frac{t}{t_0} \\ &= \operatorname{sgn} f^*(1)\sqrt{c_{f^*} d_{f^*}}t = L_f^0(x_1 + te_1), \end{aligned}$$

which means that $L_1 = L_f^0$.

From Theorem 5 we obtain directly the following

COROLLARY. *Let E_1 be a real normed space and let $\varepsilon \in [0, 1)$. Then for each ε-quasi-linear continuous mapping $f : E_1 \to \mathbf{R}$ there exists a continuous linear mapping $L : E_1 \to \mathbf{R}$ such that*

$$|f(x) - L(x)| \leq \varepsilon \min\{|f(x)|, |L(x)|\} \quad \textit{for } x \in E_1.$$

The question whether or not Theorem 5 and Corollary hold without continuity of f is an open problem. The same concerns the question whether or not $\mathbf{R}$ can be replaced by a normed space.

References

[1] Hyers D. H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222-224.

[2] Tabor J., *On functions behaving like additive functions*, Aequationes Math. **35** (1988), 164-185.

[3] Tabor J., *Quasi-additive functions*, Aequationes Math. **39** (1990), 179-197.

[4] Tabor J., *On approximately linear mappings*, Stability of Mappings of Hyers-Ulam Type, Hadronic Press, 1994, 157-163.

[5] Ulam S. M., *Problems in Modern Mathematics*, Chapter VI, Science Editions, Wiley, New York, 1966.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
Poland

Manuscript received March 22, 1995