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The stability of a generalized quadratic functional equation

Abstract. In the paper we consider the stability of the functional equation

$$f(x+y)+f(x-y)+\|f(x+y)-f(x-y)\|u=2f(x)+2f(y),$$

where $f: G \to X$, X is a normed space, $u \in X$ such that $\|u\| \geq 1$ and $\|u\| \neq 3$ is fixed and G is an abelian group, with suitable assumptions on X and G.

In [1] A. Chaljub-Simon and P. Volkmann proved that if G is an abelian group and $f: G \to \mathbb{R}$ then the general solution of the functional equation

$$\max\{f(x+y), f(x-y)\} = f(x)+f(y) \quad \text{for } x, y \in G \tag{1}$$

is of the form $f(x) = |a(x)|$ for $x \in G$, where $a: G \to \mathbb{R}$ is an additive function. J. Tabor considered in [3] the following generalization of the above equation

$$f(x+y)+f(x-y)+\|f(x+y)-f(x-y)\|u=2f(x)+2f(y) \quad \text{for } x, y \in G, \tag{2}$$

where $f: G \to X$, G is a group, X is a normed space and $u \in X$, $\|u\| = 1$. Then, under the assumption that there exists a subspace X_1 of X such that $X = \operatorname{Lin}(u) \oplus X_1$ and $\|a+b\| \geq \max(\|a\|, \|b\|)$ for $a \in \operatorname{Lin}(u)$ and $b \in X_1$, the general solution of equation (1) is given by the formula $f(x) = f_u(x)u$ for $x \in G$, where $f_u: G \to \mathbb{R}$ satisfies the equation

$$f(x+y)+f(x-y)+|f(x+y)-f(x-y)|\,\|u\|=2f(x)+2f(y) \quad \text{for } x, y \in G \tag{3}$$

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Let us notice that (3) with $\|u\| = 1$ is equivalent to (1). Z. Moszner and B. Pilecka-Oborska observed in [2] that using the similar reasoning we get the same characterization of f in the case $\|u\| > 1$. They also gave the general solution of the equation

$$f(x+y) + f(x-y) + |f(x+y) - f(x-y)|c = 2f(x) + 2f(y) \quad \text{for } x, y \in G,$$

where $f : G \to \mathbb{R}$, G is an abelian group and c is a fixed element of the set $\mathbb{R} \setminus \{1\}$.

In this paper we consider the stability of equation (2). We prove the following theorem.

THEOREM 1. *Let X be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let a $u \in X$ such that $\|u\| \geq 1$ and $\|u\| \neq 3$ be fixed and let G be an abelian group (additionally, in the case $\|u\| > 3$, uniquely divisible by 2). We assume that there exists a subspace X_1 of X such that*

(i) $X = \mathrm{Lin}(u) \oplus X_1$

and

(ii) $\|a + b\| \geq \max\{\|a\|, \|b\|\}$ *for* $a \in \mathrm{Lin}(u), b \in X_1$.

If $\varepsilon \geq 0$ and $f : G \to X$ satisfies the inequality

$$\|f(x+y) + f(x-y) + \|f(x+y) - f(x-y)\|u - 2f(x) - 2f(y)\| \leq \varepsilon \quad \text{for } x, y \in G \tag{4}$$

then there exists a unique solution $T : G \to X$ of (2) such that

$$\|T(x) - f(x)\| \leq k\varepsilon \quad \text{for } x \in G, \tag{5}$$

where

$$k := \begin{cases} \dfrac{1}{2}\left(\dfrac{4\|u\| + 3}{|\,\|u\| - 3|} + 1\right) & \text{if } \mathbb{K} = \mathbb{R}, \\[2ex] \dfrac{1}{2}\left(\sqrt{\left(\dfrac{5\|u\| + 3}{\|u\| - 3}\right)^2 + 1} + 1\right) & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Proof. Let $f : G \to X$ satisfy (4). Due to (i) we have uniquely determined mappings $f_u : G \to \mathbb{K}$, $g_u : G \to X_1$ such that $f(x) = f_u(x)u + g_u(x)$ for $x \in G$. In the case $\mathbb{K} = \mathbb{C}$ we denote by f_1, f_2 the real and the imaginary part of f_u, respectively. Putting in (4) $x = y$ and applying (ii) we obtain

$$\|f_u(2x) + f_u(0) + \|f(2x) - f(0)\| - 4f_u(x)\|\,\|u\| \leq \varepsilon \quad \text{for } x \in G \tag{6}$$

and

$$\|g_u(2x) + g_u(0) - 4g_u(x)\| \leq \varepsilon \quad \text{for } x \in G. \tag{7}$$

Then we get from (6)

$$|f_1(2x) + f_1(0) + \|f(2x) - f(0)\| - 4f_1(x)|\,\|u\| \leq \varepsilon \quad \text{for } x \in G \tag{8}$$

and

$$|f_2(2x) + f_2(0) - 4f_2(x)|\|u\| \leq \varepsilon \quad \text{for } x \in G. \tag{9}$$

Inserting into (4), (7), (8) and (9) $x = y = 0$ we have

$$\|f(0)\| \leq \frac{\varepsilon}{2}, \ \|g_u(0)\| \leq \frac{\varepsilon}{2}, \ |f_1(0)|\,\|u\| \leq \frac{\varepsilon}{2} \text{ and } |f_2(0)|\,\|u\| \leq \frac{\varepsilon}{2}. \tag{10}$$

Moreover, by (ii) we get

$$|f_1(x) - f_1(y)|\|u\| \leq \|f(x) - f(y)\| \quad \text{for } x, y \in G. \tag{11}$$

Now we prove that

$$|f_2(x)|\,\|u\| \leq \frac{\varepsilon}{2} \quad \text{for } x \in G \tag{12}$$

and

$$\|g_u(x)\| \leq \frac{\varepsilon}{2} \quad \text{for } x \in G. \tag{13}$$

By (8) and (10) we obtain

$$|f_1(2x) + \|f(2x) - f(0)\| - 4f_1(x)|\,\|u\| \leq \frac{3}{2}\varepsilon \quad \text{for } x \in G. \tag{14}$$

Condition (11) and $\|u\| \geq 1$ yield

$$|f_1(2x) - f_1(0)|\,\|u\| \leq \|f(2x) - f(0)\| \leq \|f(2x) - f(0)\|\,\|u\| \quad \text{for } x \in G,$$

and hence in view of (10)

$$|f_1(2x)|\,\|u\| \leq \|f(2x) - f(0)\|\,\|u\| + \frac{\varepsilon}{2} \quad \text{for } x \in G.$$

Using the above inequality we get for $x \in G$

$$(f_1(2x) + |f_1(2x)| - 4f_1(x))\|u\| \leq f_1(2x)\|u\| + \|f(2x) - f(0)\|\,\|u\| + \frac{\varepsilon}{2} - 4f_1(x)\|u\|,$$

which together with (14) implies

$$(f_1(2x) + |f_1(2x)| - 4f_1(x))\|u\| \leq 2\varepsilon \quad \text{for } x \in G. \tag{15}$$

In the same way, in view of (11), $\|u\| \geq 1$, (10) and (14), we obtain

$$(f_1(2x) + |f_2(2x)| - 4f_1(x))\|u\| \leq 2\varepsilon \quad \text{for } x \in G \tag{16}$$

and

$$f_1(2x)\|u\| + \|g_u(2x)\| - 4f_1(x)\|u\| \leq 2\varepsilon \quad \text{for } x \in G. \tag{17}$$

By (15) we have

$$-4f_1(x)\|u\| \leq 2\varepsilon \quad \text{for } x \in G,$$

and further

$$0 \leq f_1(x)\|u\| + \frac{\varepsilon}{2} \quad \text{for } x \in G. \tag{18}$$

Next, (9) and (10) imply that

$$4|f_2(x)|\,\|u\| - |f_2(2x)|\,\|u\| \leq |f_2(2x) - 4f_2(x)|\,\|u\| \leq \frac{3}{2}\varepsilon \quad \text{for } x \in G$$

i.e.

$$4|f_2(x)|\,\|u\| \leq \frac{3}{2}\varepsilon + |f_2(2x)|\,\|u\| \quad \text{for } x \in G. \tag{19}$$

We get from (18), (19) and (16)

$$\begin{aligned} 4|f_2(x)|\,\|u\| - 4f_1(x)\|u\| &\leq \frac{\varepsilon}{2} + f_1(2x)\|u\| + 4|f_2(x)|\,\|u\| - 4f_1(x)\|u\| \\ &\leq \frac{\varepsilon}{2} + f_1(2x)\|u\| + \frac{3}{2}\varepsilon + |f_2(2x)|\,\|u\| \\ &\quad -4f_1(x)\|u\| \leq 4\varepsilon \quad \text{for } x \in G, \end{aligned}$$

and hence

$$|f_2(x)|\,\|u\| \leq \varepsilon + f_1(x)\|u\| \quad \text{for } x \in G. \tag{20}$$

Making use of (19) and (16) we obtain

$$\begin{aligned} f_1(2x)\|u\| + 4|f_2(x)|\,\|u\| &\leq f_1(2x)\|u\| + \frac{3}{2}\varepsilon + |f_2(2x)|\,\|u\| \\ &\leq 4f_1(x)\|u\| + 2\varepsilon + \frac{3}{2}\varepsilon \quad \text{for } x \in G, \end{aligned}$$

and further

$$\frac{1}{4}f_1(2x)\|u\| + |f_2(x)|\,\|u\| \leq f_1(x)\|u\| + \frac{7}{8}\varepsilon \quad \text{for } x \in G. \tag{21}$$

We will show that for every $n \in \mathbb{N}$

$$f_1(x)\|u\| + \frac{n+1}{2}\varepsilon \geq n|f_2(x)|\,\|u\| \quad \text{for } x \in G. \tag{22}$$

By (20) this inequality holds for $n = 1$. Assume now (22) to hold for an $n \in \mathbb{N}$ and fix an $x \in X$. From (21), (22) and (19) we get

$$\begin{aligned} f_1(x)\|u\| &\geq \frac{1}{4} f_1(2x)\|u\| + |f_2(x)|\,\|u\| - \frac{7}{8}\varepsilon \\ &\geq \frac{n}{4}|f_2(2x)|\,\|u\| - \frac{n+1}{8}\varepsilon + |f_2(x)|\,\|u\| - \frac{7}{8}\varepsilon \\ &\geq n|f_2(x)|\,\|u\| - \frac{3n}{8}\varepsilon - \frac{n+1}{8}\varepsilon + |f_2(x)|\,\|u\| - \frac{7}{8}\varepsilon \\ &= (n+1)|f_2(x)|\,\|u\| - \frac{n+2}{2}\varepsilon, \end{aligned}$$

which completes the inductive proof of (22).
By (22) we have

$$|f_2(x)|\,\|u\| \leq \frac{1}{n}|f_1(x)|\,\|u\| + \frac{n+1}{2n}\varepsilon \quad \text{for } x \in G.$$

Letting n increase to ∞ we obtain immediately (12).
In the similar way we prove (13). Namely, (7) and (10) yield

$$4\|g_u(x)\| \leq \frac{3}{2}\varepsilon + \|g_u(2x)\| \quad \text{for } x \in G, \tag{23}$$

whence by (18) and (17) we have

$$\begin{aligned} 4\|g_u(x)\| - 4f_1(x)\|u\| &\leq \frac{\varepsilon}{2} + f_1(2x)\|u\| + 4\|g_u(x)\| - 4f_1(x)\|u\| \\ &\leq \frac{\varepsilon}{2} + f_1(2x)\|u\| + \frac{3}{2}\varepsilon \\ &\quad + \|g_u(2x)\| - 4f_1(x)\|u\| \leq 4\varepsilon \quad \text{for } x \in G \end{aligned}$$

and, as a consequence,

$$\|g_u(x)\| \leq \varepsilon + f_1(x)\|u\| \quad \text{for } x \in G. \tag{24}$$

From (23) and (17) we get

$$\begin{aligned} f_1(2x)\|u\| + 4\|g_u(x)\| &\leq f_1(2x)\|u\| + \frac{3}{2}\varepsilon + \|g_u(2x)\| \\ &\leq 4f_1(x)\|u\| + 2\varepsilon + \frac{3}{2}\varepsilon \quad \text{for } x \in G, \end{aligned}$$

and hence

$$\frac{1}{4}f_1(2x)\|u\| + \|g_u(x)\| \leq f_1(x)\|u\| + \frac{7}{8}\varepsilon \quad \text{for } x \in G. \tag{25}$$

With the aid of (24), (25) and (23) we easily obtain

$$\frac{1}{n}f_1(x)\|u\| + \frac{n+1}{2n}\varepsilon \geq \|g_u(x)\| \quad \text{for } x \in G$$

(it is sufficient to substitute $|f_2(\cdot)|\,\|u\|$ for $\|g_u(\cdot)\|$ in the proof of (22)), which implies immediately (13).
Now, in view of (11), (10) and (14) we have

$$\begin{aligned}(1+\|u\|)f_1(2x)-4f_1(x) &\leq f_1(2x)+|f_1(2x)|\,\|u\|-4f_1(x)\\ &\leq f_1(2x)+|f_1(2x)-f_1(0)|\,\|u\|\\ &\quad +|f_1(0)|\,\|u\|-4f_1(x)\\ &\leq f_1(2x)+\|f(2x)-f(0)\|+|f_1(0)|\,\|u\|-4f_1(x)\\ &\leq \frac{\varepsilon}{2}+\frac{3}{2\|u\|}\varepsilon \quad \text{for } x\in G.\end{aligned}$$

Therefore

$$(1+\|u\|)f_1(2x)-4f_1(x)\leq \varepsilon\left(\frac{1}{2}+\frac{3}{2\|u\|}\right) \quad \text{for } x\in G. \tag{26}$$

By (18) it is evident that

$$|f_1(x)|\,\|u\|\leq f_1(x)\|u\|+\varepsilon \quad \text{for } x\in G. \tag{27}$$

Conditions (12) and (13) yield

$$\|f(2x)\|\leq |f_u(2x)|\,\|u\|+\|g_u(2x)\|\leq |f_1(2x)|\,\|u\|+c\cdot\varepsilon \quad \text{for } x\in G,$$

where $c:=\begin{cases}\frac{1}{2} & \text{if } \mathbb{K}=\mathbb{R},\\ 1 & \text{if } \mathbb{K}=\mathbb{C},\end{cases}$
whence using (10) we get

$$\|f(2x)-f(0)\|\leq |f_1(2x)|\,\|u\|+\varepsilon\left(c+\frac{1}{2}\right) \quad \text{for } x\in G.$$

By (27), the above inequality and (14) we have

$$\begin{aligned}(1+\|u\|)f_1(2x)-4f_1(x) &\geq f_1(2x)+|f_1(2x)|\,\|u\|-\varepsilon-4f_1(x)\\ &\geq f_1(2x)+\|f(2x)-f(0)\|\\ &\quad -\left(c+\frac{1}{2}\right)\varepsilon-\varepsilon-4f_1(x)\\ &\geq -\varepsilon\left(c+\frac{1}{2}+1+\frac{3}{2\|u\|}\right) \quad \text{for } x\in G,\end{aligned}$$

which together with (26) gives

$$|(1+\|u\|)f_1(2x)-4f_1(x)|\leq \left(c+\frac{3}{2}+\frac{3}{2\|u\|}\right)\varepsilon \quad \text{for } x\in G,$$

and consequently,

$$\left|\frac{1+\|u\|}{4}f_1(2x)-f_1(x)\right|\le\frac{1}{4}\left(c+\frac{3}{2}+\frac{3}{2\|u\|}\right)\varepsilon\quad\text{for } x\in G \tag{28}$$

or, assuming that G is uniquely divisible by 2 and replacing x by $\frac{x}{2}$,

$$\left|\frac{4}{1+\|u\|}f_1\left(\frac{x}{2}\right)-f_1(x)\right|\le\frac{1}{1+\|u\|}\left(c+\frac{3}{2}+\frac{3}{2\|u\|}\right)\varepsilon\quad\text{for } x\in G. \tag{29}$$

Put

$$a:=\begin{cases}2 & \text{if } 1\le\|u\|<3,\\ \frac{1}{2} & \text{if } \|u\|>3,\end{cases}\qquad b:=\begin{cases}\frac{1+\|u\|}{4} & \text{if } 1\le\|u\|<3,\\ \frac{4}{1+\|u\|} & \text{if } \|u\|>3,\end{cases}$$

and

$$p:=\begin{cases}\frac{1}{4}\left(c+\frac{3}{2}+\frac{3}{2\|u\|}\right) & \text{if } 1\le\|u\|<3,\\ \frac{1}{1+\|u\|}\left(c+\frac{3}{2}+\frac{3}{2\|u\|}\right) & \text{if } \|u\|>3.\end{cases}$$

We show that for every $n\in\mathbb{N}$

$$|b^n f_1(a^n x)-f_1(x)|\le p\varepsilon\sum_{i=0}^{n-1}b^i\quad\text{for } x\in G. \tag{30}$$

Namely, from (28) or (29) we get (30) for $n=1$. Assume now that (30) holds for an $n\in\mathbb{N}$. Then

$$\begin{aligned}|b^{n+1}f_1(a^{n+1}x)-f_1(x)|&\le|b\cdot b^n f_1(a^n\cdot ax)-bf_1(ax)|+|bf_1(ax)-f_1(x)|\\ &\le p\varepsilon\sum_{i=0}^{n-1}b^{i+1}+p\varepsilon=p\varepsilon\sum_{i=0}^{n}b^i\quad\text{for } x\in G,\end{aligned}$$

which gives (30) for $(n+1)$.
Now, fix an $x\in G$ and choose arbitrary $m,n\in\mathbb{N}$ such that $m>n$. Then, by (30)

$$\begin{aligned}|b^m f_1(a^m x)-b^n f_1(a^n x)|&=b^n|b^{m-n}f_1(a^{m-n}a^n x)-f_1(a^n x)|\\ &\le p\varepsilon b^n\sum_{i=0}^{m-n-1}b^i=p\varepsilon\sum_{i=n}^{m-1}b^i,\end{aligned}$$

which, seeing that $0<b<1$, means that the sequence $\{b^n f_1(a^n x):n\in\mathbb{N}\}$ satisfies Cauchy's condition, and therefore it is convergent for each $x\in G$.

We define mappings $A : G \to \mathbb{R}$ and $T : G \to X$ by the formulas

$$A(x) := \lim_{n\to\infty} b^n f_1(a^n x), \quad x \in G,$$
$$T(x) := A(x)u, \quad x \in G.$$

From (30) we get at once

$$|A(x) - f_1(x)| \le \frac{p\varepsilon}{1-b} \quad \text{for } x \in G. \tag{31}$$

Now we prove that T fulfils the equation (2). By (12), (13) and (11) we obtain for $x, y \in G$

$$\begin{aligned} \|f(x+y) - f(x-y)\| - 2\varepsilon &\le |f_u(x+y) - f_u(x-y)|\,\|u\| \\ &\quad + \|g_u(x+y) - g_u(x-y)\| - 2\varepsilon \\ &\le |f_1(x+y) - f_1(x-y)|\,\|u\| \\ &\le \|f(x+y) - f(x-y)\|. \end{aligned}$$

Using (ii), (4) and the above inequalities we get for $x, y \in G$

$$\begin{aligned} -\frac{\varepsilon}{\|u\|} - 2\varepsilon &\le f_1(x+y) + f_1(x-y) + \|f(x+y) - f(x-y)\| \\ &\quad -2f_1(x) - 2f_1(y) - 2\varepsilon \le f_1(x+y) + f_1(x-y) \\ &\quad +|f_1(x+y) - f_1(x-y)|\,\|u\| - 2f_1(x) - 2f_1(y) \\ &\le f_1(x+y) + f_1(x-y) \\ &\quad +\|f(x+y) - f(x-y)\| - 2f_1(x) - 2f_1(y) \le \frac{\varepsilon}{\|u\|}. \end{aligned}$$

Hence

$$|f_1(x+y) + f_1(x-y) + |f_1(x+y) - f_1(x-y)|\,\|u\| - 2f_1(x) - 2f_1(y)| \le 3\varepsilon$$
$$\text{for } x, y \in G.$$

Thus

$$\begin{aligned} &\|b^n f_1(a^n x + a^n y)u + b^n f_1(a^n x - a^n y)u \\ &+\|b^n f_1(a^n x + a^n y)u - b^n f_1(a^n x - a^n y)u\|u \\ &-2b^n f_1(a^n x)u - 2b^n f_1(a^n y)u\| \le 3b^n\|u\|\varepsilon \end{aligned}$$

for $x, y \in G$, which, letting $n \to \infty$, implies that T fulfils (2).
To prove (5) notice that

$$\|T(x) - f(x)\| = \|A(x)u - (f_u(x)u + g_u(x))\|$$
$$\leq |A(x) - f_u(x)|\,\|u\| + \|g_u(x)\| \quad \text{for } x \in G.$$

Therefore, in the case $\mathbb{K} = \mathbb{R}$, we have from (31) and (13)

$$\|T(x) - f(x)\| \leq |A(x) - f_1(x)|\,\|u\| + \|g_u(x)\| \leq \frac{p\varepsilon}{1-b}\|u\| + \frac{\varepsilon}{2}$$
$$= \left(\frac{2 + \frac{3}{2\|u\|}}{|\,\|u\| - 3|}\|u\| + \frac{1}{2} \right)\varepsilon$$
$$= \frac{1}{2}\left(\frac{4\|u\| + 3}{|\,\|u\| - 3|} + 1 \right)\varepsilon \quad \text{for } x \in G.$$

In the case $\mathbb{K} = \mathbb{C}$ we obtain by (31), (12) and (13)

$$\|T(x) - f(x)\| \leq \sqrt{[(A(x) - f_1(x))\|u\|]^2 + (|f_2(x)|\,\|u\|)^2} + \|g_u(x)\|$$
$$\leq \sqrt{\left[\frac{\left(\frac{5}{2} + \frac{3}{2\|u\|}\right)\|u\|}{|\,\|u\| - 3|}\varepsilon \right]^2 + \left(\frac{\varepsilon}{2}\right)^2} + \frac{\varepsilon}{2}$$
$$= \frac{1}{2}\left(\sqrt{\left(\frac{5\|u\| + 3}{\|u\| - 3}\right)^2 + 1} + 1 \right)\varepsilon \quad \text{for } x \in G.$$

It results from Theorem 3 in [2] that the only one solution of equation (2) for $\|u\| > 1$ and $\|u\| \neq 3$ is the function $f \equiv 0$. Therefore, to prove the uniqueness, it is sufficient to consider the case $\|u\| = 1$. To this end assume the existence of two solutions $T_1, T_2 : G \to X$ of equation (2) and two constants $\Theta_1, \Theta_2 \geq 0$ such that

$$\|T_i(x) - f(x)\| \leq \Theta_i \quad \text{for } x \in G,\ i = 1, 2.$$

By Theorem 2 in [3] there exist additive functions $a_1, a_2 : G \to \mathbb{R}$ such that $T_i(x) = |a_i(x)|u$ for $x \in G$, $i = 1, 2$. Hence in view of the last inequality we get

$$|\,|a_1(x)| - |a_2(x)|\,|\,\|u\| \leq \Theta_1 + \Theta_2 \quad \text{for } x \in G,$$

and further

$$2^n|\,|a_1(x)| - |a_2(x)|\,|\,\|u\| = |\,|a_1(2^n x)| - |a_2(2^n x)|\,|\,\|u\| \leq \Theta_1 + \Theta_2$$
$$\text{for } x \in G \text{ and } n \in \mathbb{N}.$$

Since the right-hand of this inequality is constant, it becomes apparent that $|a_1(x)| = |a_2(x)|$ for $x \in G$ and as a consequence $T_1 = T_2$.

From Theorem 2 in [3] and Theorem 3 in [2] we obtain directly the following corollary.

COROLLARY. *Let X be a normed space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, let a $u \in X$ such that $\|u\| \geq 1$ and $\|u\| \neq 3$ be fixed and let G be an abelian group (additionally, in the case $\|u\| > 3$, uniquely divisible by 2). We assume that there exists a subspace X_1 of X such that* (i) *and* (ii) *hold. If $\varepsilon \geq 0$ and $f : G \to X$ satisfies the inequality* (4) *then, in the case $\|u\| = 1$, there exists an additive function $a : G \to \mathbb{R}$ such that $\|f(x) - |a(x)|u\| \leq k\varepsilon$ for $x \in G$, or, in the case $\|u\| > 1$, $\|f(x)\| \leq k\varepsilon$ for $x \in G$, where the constant k is defined as in Theorem 1.*

We have also the following Theorem.

THEOREM 2. *Let G be a group, let X be a normed space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, let a $u \in X$ such that $u \neq 0$ be fixed and let $T : G \to X$ be a solution of* (2). *If $\eta \geq 0$ and $f : G \to X$ satisfies the inequality*

$$\|T(x) - f(x)\| \leq \eta \quad \textit{for } x \in G, \tag{32}$$

then f fulfils (4) *with $\varepsilon = \bar{k}\eta$, where $\bar{k} := (6 + 2\|u\|)$.*

Proof. By the fact that T fulfils (2) and use (32) we obtain for $x, y \in G$

$$\begin{aligned}
&\|f(x+y) + f(x-y) + \|f(x+y) - f(x-y)\|u - 2f(x) - 2f(y)\| \\
&\quad = \|f(x+y) - T(x+y) + f(x-y) - T(x-y) + \|f(x+y) - f(x-y)\|u \\
&\qquad - \|T(x+y) - T(x-y)\|u - 2(f(x) - T(x)) - 2(f(y) - T(y))\| \\
&\quad \leq 6\eta + \| \, \|f(x+y) - f(x-y)\|u - \|T(x+y) - T(x-y)\|u\| \\
&\quad = 6\eta + | \, \|f(x+y) - f(x-y)\| - \|T(x+y) - T(x-y)\| \, | \, \|u\| \\
&\quad \leq 6\eta + \|f(x+y) - T(x+y)\| \, \|u\| + \|f(x-y) - T(x-y)\| \, \|u\| \\
&\quad \leq 6\eta + 2\eta\|u\|.
\end{aligned}$$

The following problems remain open.

PROBLEM 1. Solve the problem of stability of the equation (2) in the case $\|u\| = 3$ (Let us notice that the case $\|u\| = 3$ is also non-typical for the solutions of equation (2) (see Theorem 3 in [2])).

PROBLEM 2. Do the above results hold without the assumption that G is uniquely divisible by 2 in the case $\|u\| > 3$?

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