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General solution of a linear functional equation of even order

Abstract. In this paper we show that the functional equation

$$A\varphi(g(x)) = \sum_{i=-n}^{n} A_i \varphi(h^i(x))$$

has in an open interval I a solution φ depending on an arbitrary function. An analogous result we obtain for continuous solutions.

In this note the functional equation

$$A\varphi(g(x)) = \sum_{i=-n}^{n} A_i \varphi(h^i(x)), \tag{1}$$

where unknown function φ maps an open interval $I = (a, b)$ into a linear space X over a field K, $n \in \mathbf{N}$, $A, A_i \in K$, $i \in \{-n, \ldots, n\}$ is considered. One or both endpoints of I may be infinite. Assume that functions h, g fulfil the following conditions:

(H) $h : I \mapsto I$ is an increasing bijection such that $h(x) > x$ for all $x \in I$,

(G) $g : I \mapsto I$ is an increasing bijection such that for some $x_0 \in I$ $g(x) < x$ for $x > x_0$, $g(x) > x$ for $x < x_0$.

For $I = K = X = \mathbf{R}$, $g(x) = qx$, $q \in [0, 1]$, $h(x) = x + 1$, $A_{-1} \neq 0 \neq A_1$ the general solution of (1) gave A. Grząślewicz in [3] (the same result has been obtained by W. Főrg-Rob, see [2]). The aim of this paper is to generalize Grząślewicz's result.

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By f^n we denote here the n-th iterate of the function $f : I \mapsto I$, that is

$$f^0 := \mathrm{id}_{|I}, \quad f^1 := f, \quad f^{n+1} := f \circ f^n, \quad f^{-n} := (f^n)^{-1}, \quad n = 1, 2, \dots .$$

It is easy to check that

$$\begin{gathered}\{h^n(x_0),\ n \in \mathbf{Z}\} \text{ is a strictly increasing sequence,} \\ \lim_{n\to\infty} h^n(x_0) = b, \quad \lim_{n\to\infty} h^{-n}(x_0) = a.\end{gathered} \tag{2}$$

Put

$$I_k := [h^k(x_0),\ h^{k+1}(x_0)] \ \text{ for } \ k \in \mathbf{Z}. \tag{3}$$

Now, note that relations (2) and (3) imply

$$I = \bigcup_{k\in\mathbf{Z}} I_k.$$

Define a function $E : I \mapsto \mathbf{Z}$ by the formula

$$E(x) := k \ \text{ for } \ x \in I_k \setminus \{h^{k+1}(x_0)\}, \ \ k \in \mathbf{Z}. \tag{4}$$

First we formulate two lemmas

LEMMA 1. *The functional equation* (1) *is equivalent to each of the following functional equations:*

$$A_n\varphi(x) = A\varphi(g(h^{-n}(x))) - \sum_{i=-n}^{n-1} A_i\varphi(h^{i-n}(x)), \tag{5}$$

$$A_{-n}\varphi(x) = A\varphi(g(h^n(x))) - \sum_{i=-n+1}^{n} A_i\varphi(h^{i+n}(x)). \tag{6}$$

Proof. It is sufficient to put $h^{-n}(x)$ or $h^n(x)$ in place of x, respectively.

LEMMA 2. *If* $x \in I_k$, *then*

$$0 \le E(g(h^{-n}(x))) < k + 1 - n \ \ \text{for} \ \ k \ge n \tag{7}$$

and

$$k + n - 1 < E(g(h^n(x))) \le 0 \ \ \text{for} \ \ k \le -n - 1. \tag{8}$$

Proof. By (3), (H), (G) we get

$$g(h^{k-n}(x_0)) \le g(h^{-n}(x)) \le g(h^{k+1-n}(x_0)).$$

Let $k \ge n$, then $x_0 \le h^{k-n}(x_0) < h^{k-n+1}(x_0)$ and by (G) we obtain

$$x_0 = g(x_0) \le g(h^{k-n}(x_0)) \le g(h^{-n}(x)) \le g(h^{k-n+1}(x_0)) < h^{k-n+1}(x_0),$$

which together with (4) gives (7). We apply the same arguments for $k \le -n - 1$ to obtain (8). This completes the proof.

Now, we can prove the following

THEOREM 1. *Let* $n \in \mathbf{N}$, $A, A_i \in K$, $i \in \{-n, \dots, n\}$, $A_{-n} \neq 0 \neq A_n$ *and let functions* h, g *fulfil* (H) *and* (G). *Then equation* (1) *has in* I *a solution* φ *depending on an arbitrary function. More precisely, for an arbitrary function*

$$\psi : \bigcup_{i=-n}^{n-1} I_i \mapsto X$$

fulfilling the condition

$$A\psi(x_0) = \sum_{i=-n}^{n} A_i \psi(h^i(x_0)) \tag{9}$$

there exists exactly one function $\varphi : I \mapsto X$ *satisfying equation* (1) *and such that*

$$\varphi(x) = \psi(x) \quad for \quad x \in \bigcup_{i=-n}^{n-1} I_i. \tag{10}$$

This function is given by

$$\varphi = \bigcup_{k \in \mathbf{Z}} \varphi_k, \tag{11}$$

where the functions $\varphi_k : I_k \mapsto X$ *are defined as follows:*

$$\varphi_k(x) = \psi(x) \quad if \quad k \in \{-n, \dots, n-1\}, \quad x \in I_k, \tag{12}$$

$$\varphi_k(x) = A_n^{-1}[A\varphi_{E(g(h^{-n}(x)))}(g(h^{-n}(x))) - \sum_{i=-n}^{n-1} A_i \varphi_{k+i-n}(h^{i-n}(x))] \quad if \quad k \geq n, \quad x \in I_k, \tag{13}$$

$$\varphi_k(x) = A_{-n}^{-1}[A\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n+1}^{n} A_i \varphi_{k+i+n}(h^{i+n}(x))] \quad if \quad k \leq -n-1, \quad x \in I_k. \tag{14}$$

Moreover, every solution $\varphi : I \mapsto X$ *of equation* (1) *is given by* (11), *where the functions* φ_k *are given by* (12), (13), (14) *and*

$$\psi : \bigcup_{i=-n}^{n-1} I_i \mapsto X$$

is a function satisfying condition (9).

Proof. Let $A, A_i \in K$ for $i \in \{-n, \dots, n\}$. $A_{-n} \neq 0 \neq A_n$ and let the functions h, g fulfil (H) and (G), respectively. Assume first that the function φ satisfies equation (1) and put $\varphi_k := \varphi_{|I_k}$ for all $k \in \mathbf{Z}$,

$$\psi := \bigcup_{k=-n}^{n-1} \varphi_k.$$

Clearly, conditions (11), (12) are satisfied. Substituting x_0 for x in (1) and using (G) we get (9). If $x \in I_k$ for $k \geq n$, then from (3), (H) and (7)

$$h^{i-n}(x) \in I_{k+i-n} \quad \text{for } \ i \in \{-n, \ldots, n-1\} \ \text{ and } \ g(h^{-n}(x)) \in I_j,$$

where $j = E(g(h^{-n}(x)))$ and $0 \leq j < k+1-n$, whence by Lemma 1 we get (13). Similarly, we obtain (14). This completes the first part of the proof.

Assume now that

$$\psi : \bigcup_{k=-n}^{n-1} I_k \mapsto X$$

is an arbitrary function fulfilling condition (9). Define a sequence of functions

$$\{\varphi_k \ , \ k \in \mathbf{Z}\},$$

by formulas (12), (13) and (14), respectively. Clearly, the functions φ_k are well defined on I_k for $k \in \{-n, \ldots, n-1\}$. Let $x \in I_n$. Then by Lemma 2

$$g(h^{-n}(x)) \in I_0$$

and in view of (3) and (H) we get

$$h^{i-n}(x) \in I_i \quad \text{for } \ i \in \{-n, \ldots, n-1\},$$

which implies that the function φ_n is defined on I_n by φ_i for $-n \leq i \leq n-1$. Let now $k > n$ and the functions φ_i be well defined on I_i for $-n \leq i < k$. Let us note that (13) define the function φ_k on I_k by φ_i for $-n+1 \leq i < k$. In fact, if $i \in \{-n, \ldots, n-1\}$ and $x \in I_k$, then $h^{i-n}(x) \in I_{k+i-n}$ and $-n < k+i-n \leq k-1$, since $n < k$. Further by Lemma 2 $0 \leq E(g(h^{-n}(x))) < k+1-n \leq k$, so $g(h^{-n}(x)) \in I_j$, where $j = E(g(h^{-n}(x)))$ and $0 \leq j < k$. Thus φ_k is defined by the previous functions. Consequently, by induction the functions φ_k are well defined on I_k for $k \geq n$. The same conclusion can be drawn for $k \leq -n-1$. Thus the functions φ_k for every $k \in \mathbf{Z}$ are well defined. Now we shall prove that for every $k \in \mathbf{Z}$

$$\varphi_{k+1}(y) = \varphi_k(y) \ \text{ for } \ y \in I_{k+1} \cap I_k,$$

that is, the condition

$$\varphi_{k+1}(h^{k+1}(x_0)) = \varphi_k(h^{k+1}(x_0)) \tag{15}$$

for $k \in \mathbf{Z}$ holds. By (12) we get

$$\varphi_{k+1}(h^{k+1}(x_0)) = \varphi_k(h^{k+1}(x_0)) \ \text{ for } \ k \in \{-n, \ldots, n-2\}. \tag{16}$$

Now we show that condition (15) holds for $k = n-1$ and $k = -n-1$. In virtue of (9) and (12) we see that

$$A\varphi_0(x_0) = \sum_{i=-n}^{-1} A_i\varphi_i(h^i(x_0)) + \sum_{i=0}^{n} A_i\varphi_{i-1}(h^i(x_0)) \tag{17}$$

whence by (13), (G), (4) and (16) we obtain

$$\begin{aligned}
\varphi_n(h^n(x_0)) &= A_n^{-1}[A\varphi_0(x_0) - \sum_{i=-n}^{n-1} A_i\varphi_i(h^i(x_0))] \\
&= A_n^{-1}[\sum_{i=-n}^{-1} A_i\varphi_i(h^i(x_0)) + \sum_{i=0}^{n} A_i\varphi_{i-1}(h^i(x_0)) \\
&\quad - \sum_{i=-n}^{-1} A_i\varphi_i(h^i(x_0)) - \sum_{i=0}^{n-1} A_i\varphi_i(h^i(x_0))] \\
&= A_n^{-1}[\sum_{i=0}^{n} A_i\varphi_{i-1}(h^i(x_0)) - \sum_{i=0}^{n-1} A_i\varphi_{i-1}(h^i(x_0))] \\
&= \varphi_{n-1}(h^n(x_0)).
\end{aligned}$$

Using (14), (G), (4), (17), (16) we get the following equalities

$$\begin{aligned}
\varphi_{-n-1}(h^{-n}(x_0)) &= A_{-n}^{-1}[A\varphi_0(x_0) - \sum_{i=-n+1}^{n} A_i\varphi_{i-1}(h^i(x_0))] \\
&= A_{-n}^{-1}[\sum_{i=-n}^{-1} A_i\varphi_i(h^i(x_0)) + \sum_{i=0}^{n} A_i\varphi_{i-1}(h^i(x_0)) \\
&\quad - \sum_{i=-n+1}^{-1} A_i\varphi_{i-1}(h^i(x_0)) - \sum_{i=0}^{n} A_i\varphi_{i-1}(h^i(x_0))] \\
&= A_{-n}^{-1}[\sum_{i=-n}^{-1} A_i\varphi_i(h^i(x_0)) - \sum_{i=-n+1}^{-1} A_i\varphi_i(h^i(x_0))] \\
&= \varphi_{-n}(h^{-n}(x_0)).
\end{aligned}$$

Hence and by (16) we have (15) for $k \in \{-n-1, \dots, n-1\}$. Now, fix $k \geq n$ and assume

$$\varphi_{i+1}(h^{i+1}(x_0)) = \varphi_i(h^{i+1}(x_0)) \quad \text{for} \quad -n-1 \leq i < k. \tag{18}$$

We shall show that

$$\varphi_{k+1}(h^{k+1}(x_0)) = \varphi_k(h^{k+1}(x_0)). \tag{19}$$

By (13) and (18) we have

$$\begin{aligned}\varphi_{k+1}(h^{k+1}(x_0)) &= A_n^{-1}[A\varphi_{E(g(h^{k+1-n}(x_0)))}(g(h^{k+1-n}(x_0))) \\ &\quad - \sum_{i=-n}^{n-1} A_i\varphi_{k+1+i-n}(h^{k+1+i-n}(x_0))] \\ &= A_n^{-1}[A\varphi_{E(g(h^{k+1-n}(x_0)))}(g(h^{k+1-n}(x_0))) \\ &\quad - \sum_{i=-n}^{n-1} A_i\varphi_{k+i-n}(h^{k+1+i-n}(x_0))] \\ &= \varphi_k(h^{k+1}(x_0)),\end{aligned}$$

i. e. (19). Consequently, by induction, condition (15) holds for every $k \geq -n-1$.
Now, let $k \leq -n-2$ and assume that

$$\varphi_{i+1}(h^{i+1}(x_0)) = \varphi_i(h^{i+1}(x_0)) \text{ for } i > k. \tag{20}$$

We shall prove that condition (20) holds for $i = k$, i. e. condition (19). By (14) and (20) we get

$$\begin{aligned}\varphi_k(h^{k+1}(x_0)) &= A_{-n}^{-1}[A\varphi_{E(g(h^{n+1+k}(x_0)))}(g(h^{n+1+k}(x_0))) \\ &\quad - \sum_{i=-n+1}^{n} A_i\varphi_{k+i+n}(h^{k+1+i+n}(x_0))] \\ &= A_{-n}^{-1}[A\varphi_{E(g(h^{n+1+k}(x_0)))}(g(h^{n+1+k}(x_0))) \\ &\quad - \sum_{i=-n+1}^{n} A_i\varphi_{k+1+i+n}(h^{k+1+i+n}(x_0))] \\ &= \varphi_{k+1}(h^{k+1}(x_0)),\end{aligned}$$

since $k+i+n > k$ for $-n+1 \leq i \leq n$. Thus we have (19) and by induction we infer that (15) holds for $k \leq -n-2$, which consequently gives (15) for $k \in \mathbf{Z}$. Since for every $k \in \mathbf{Z}$ φ_k maps I_k into X, $I = \bigcup_{k\in\mathbf{Z}} I_k$, $I_k \cap I_{k+1} = \{h^{k+1}(x_0)\}$ and condition (15) holds we may define the function $\varphi : I \mapsto X$ by formula (11).

Now, we shall show that φ satisfies equation (1). To this aim, we prove that

$$\varphi_k(x) = A_{-n}^{-1}[A\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n+1}^{n} A_i\varphi_{k+i+n}(h^{i+n}(x))] \quad \text{for } k \in \mathbf{Z},\ x \in I_k. \tag{21}$$

Let $k \geq -n$, $x \in I_k$. Then $k+2n \geq n$, $h^{2n}(x) \in I_{k+2n}$ and by (13) we obtain

$$A_{-n}^{-1}[A\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n+1}^{n} A_i\varphi_{k+i+n}(h^{i+n}(x))]$$

$$= A_{-n}^{-1}[A\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n+1}^{n-1} A_i\varphi_{k+i+n}(h^{i+n}(x))$$

$$-A_n\varphi_{k+2n}(h^{2n}(x))]$$

$$= A_{-n}^{-1}[A\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n+1}^{n-1} A_i\varphi_{k+i+n}(h^{i+n}(x))$$

$$-A_n(A_n^{-1}\{\varphi_{E(g(h^n(x)))}(g(h^n(x))) - \sum_{i=-n}^{n-1} A_i\varphi_{k+i+n}(h^{i+n}(x))\})]$$

$$= \varphi_k(x),$$

which together with (14) gives (21). By (21), (11) we immediately deduce that the function φ satisfies (6) and consequently, by Lemma 1, φ satisfies equation (1). The uniqueness of φ follows immediately from induction and the above proof. This completes the proof.

Since I_k, $k \in \mathbf{Z}$ are closed intervals, the proof of Th. 1 and induction give the following

COROLLARY 1. *If X is a linear topological space over the field K of real or complex numbers and ψ is a continuous function, then the unique extension of ψ to a solution of equation* (1) *is continuous.*

In the case A=0 we get

COROLLARY 2. *Let $n \in \mathbf{N}$, $A_i \in K$, $i \in \{-n, \dots, n\}$, $A_{-n} \neq 0 \neq A_n$ and let h fulfils* (H). *Then equation*

$$\sum_{i=-n}^{n} A_i\varphi(h^i(x)) = 0 \tag{22}$$

has in I a solution φ depending on an arbitrary function. More precisely, for $x_0 \in I$ and an arbitrary function

$$\psi : \bigcup_{i=-n}^{n-1} I_i \mapsto X$$

fulfilling the condition

$$\sum_{i=-n}^{n} A_i\psi(h^i(x_0)) = 0 \tag{23}$$

there exists exactly one function $\varphi : I \mapsto X$ *satisfying equation* (22) *and such that condition* (10) *holds. This function is given by formula* (11), *where the functions* $\varphi_k : I_k \mapsto X$ *are defined as follows:*

$$\varphi_k(x) = \psi(x) \qquad if \;\; k \in \{-n, \ldots, n-1\}, \quad x \in I_k, \tag{24}$$

$$\varphi_k(x) = -A_n^{-1} \sum_{i=-n}^{n-1} A_i \varphi_{k+i-n}(h^{i-n}(x)) \qquad if \;\; k \geq n, \quad x \in I_k, \tag{25}$$

$$\varphi_k(x) = -A_{-n}^{-1} \sum_{i=-n+1}^{n} A_i \varphi_{k+i+n}(h^{i+n}(x)) \qquad if \;\; k \leq -n-1, \quad x \in I_k. \tag{26}$$

Moreover, every solution $\varphi : I \mapsto X$ *of equation* (22) *is given by* (11), *where the functions* φ_k *are given by* (24), (25), (26) *and* $x_0 \in I$,

$$\psi : \bigcup_{i=-n}^{n-1} I_i \mapsto X$$

is a function satisfying condition (23).

If X *is a linear topological space over the field* K *of real or complex numbers and* ψ *is a continuous function, then the unique extension of* ψ *to a solution of equation* (22) *is continuous.*

The paper [1] deals with the case $I = K = X = \mathbf{R}$, $n = 1$, $A_{-1} = 0$.

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