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On some subsemigroups of the group L_s^1

Abstract. In this paper we generalize the results concerning determination of some form subsemigroups of the group L_s^1. We show that for $s \geq 4+2i$ there are no subsemigroups $Z^i_{s,s}(f)$ and $T^i_{i,s}(g)$. We determine all subsemigroups of mentioned form for $s \leq 3+2i$. Moreover we use obtained results to determine the subsemigroups $P^{s,i}_i(h)$ and $P^{s,i}_{1,s}(h_1, h_s)$.

0. Denote by $\mathbf{Z}$ the set of all integral numbers and by $\mathbf{R}$ – the set of all real numbers. Let

$$\mathbf{R}_0 := \mathbf{R} \setminus \{0\}, \qquad |k,l| := \{n \in \mathbf{Z} : k \leq n \leq l\} \qquad \text{for } k,l \in \mathbf{Z}.$$

We adhere to the convention that

$$0^0 = 1,$$

$$\sum_{k=m}^{n} a_k = 0 \quad \text{and} \quad |m,n| = \emptyset \quad \text{for} \quad m > n\,.$$

DEFINITION 1. Let s be a natural number. A set

$$Z_s := \{\bar{x}_s := (x_1, \ldots, x_s) \in \mathbf{R}^s : \; x_1 \neq 0\}$$

with the operation

$$\bar{x}_s \cdot \bar{y}_s = \bar{z}_s \tag{0.1}$$

if and only if

$$z_n = \sum_{k=1}^{n} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} y_j^{u_j} \qquad \text{for } n \in |1,s|, \tag{0.2}$$

where

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$$U_{n,k} := \left\{ \bar{u}_n := (u_1, \ldots, u_n) \in |0, k|^n : \sum_{i=1}^{n} u_i = k \wedge \sum_{i=1}^{n} i u_i = n \right\}, \tag{0.3}$$

$$A_{\bar{u}_n} := \frac{n!}{\prod_{i=1}^{n} (u_i!(i!)^{u_i})}, \tag{0.4}$$

is a group, which is denoted by L_s^1 (see [3]).

In this paper we will consider the subsemigroups of the group L_s^1 such that the last parameter or the first one and the last one are the functions of the remaining ones. Such subsemigroups for $s \le 6$ have been determined in [3], [4], [7]-[11].

At first we will give some properties of the sets $U_{n,k}$.

(i) $U_{n,1} = \{(0, \ldots, 0, 1)\}$.

(ii) $U_{n,n} = \{(n, 0, \ldots, 0)\}$.

(iii) If $\bar{u}_n \in U_{n,k}$ $(2 \le k \le n)$, then $u_j = 0$ for all $j \in |n-k+2, n|$.

Let $n \ge 3$ and $k \in |2, n-1|$.

(iv) If $\bar{u}_n \in U_{n,k}$, then there exists $j \in |2, n-k+1|$ such that $u_j \ge 1$.

(v) If $\bar{u}_n \in U_{n,k}$ and $u_j = 0$ for all $j \in |2, n-k|$, then $u_1 = k-1$ and $u_{n-k+1} = 1$.

(vi) If $u_1 = k-1$, $u_{n-k+1} = 1$ and $u_j = 0$ for all $j \in |2, n| \setminus \{n-k+1\}$, then $\bar{u}_n \in U_{n,k}$.

(vii) If $4 \le 2k \le n$, $\bar{u}_n \in U_{n,k}$ and $u_1 = i \le k-2$, then there exists $j \in \left|2, \left[\frac{n}{k-i}\right]\right|$ such that $u_j \ge 1$.

($[x]$ denotes the integral part of x).

Proof. Properties (i)-(iv) have been proved in [2].
Let $n \ge 3$, $k \in |2, n-1|$, $\bar{u}_n \in U_{n,k}$ and $u_j = 0$ for all $j \in |2, n-k|$.
By (iii) we have $u_j = 0$ for all $j \in |n-k+2, n|$. Thus from (0.3) we get

$$\begin{cases} u_1 + u_{n-k+1} = k \\ u_1 + (n-k+1)u_{n-k+1} = n \end{cases}$$

so $u_1 = k-1$ and $u_{n-k+1} = 1$.
From (0.3) we obtain (vi), immediately.
Let $4 \le 2k \le n$, $\bar{u}_n \in U_{n,k}$ and $u_1 = i \le k-2$.

Suppose that $u_j = 0$ for all $j \in \left|2, \left[\frac{n}{k-i}\right]\right|$. From (0.3) we get

$$n = \sum_{j=1}^{n} ju_j = i + \sum_{j=\left[\frac{n}{k-i}\right]+1}^{n} ju_j \geq i + \left(\left[\tfrac{n}{k-i}\right] + 1\right) \sum_{j=\left[\frac{n}{k-i}\right]+1}^{n} u_j$$
$$= i + \left(\left[\frac{n}{k-i}\right] + 1\right)(k-i) = \left[\frac{n}{k-i}\right](k-i) + k > n.$$

This demonstrated (vii).

From (i), (ii) and (0.4) we have

Corollary 1. *$A_{\bar{u}_n} = 1$ for all $\bar{u}_n \in U_{n,1} \cup U_{n,n}$.*

We will characterize some properties of expressions (0.2). In particular cases we get

$$z_1 = x_1 y_1,$$
$$z_2 = x_1 y_2 + x_2 y_1^2,$$
$$z_3 = x_1 y_3 + 3x_2 y_1 y_2 + x_3 y_1^3.$$

Lemma 1. *For every $n \geq 2$ we have*

$$z_n = x_1 y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} + x_n y_1^n. \tag{0.5}$$

Proof. From (i), (ii) and Corollary 1 we get

$$z_n = x_1 y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u} \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} y_j^{u_j} + x_n y_1^n. \tag{0.6}$$

For $n = 2$, from (0.6), on the ground of accepted agreement, (0.5) follows, whereas for $n \geq 3$, $k \in |2, n-1|$ and $\bar{u}_n \in U_{n,k}$, by (iii) we obtain

$$u_j = 0 \text{ for all } j \in |n-k+2, n| \supset \{n\}$$

and then

$$\prod_{j=1}^{n} y_j^{u_j} = \prod_{j=1}^{n-1} y_j^{u_j}.$$

Consequently from (0.6) we get (0.5).

Next we prove

Lemma 2. *Let p, q be natural numbers such that $1 \leq p \leq q$ and let $r = p + q + 1$. If $x_j = 0$ for all $j \in |2, q|$ and $y_j = 0$ for all $j \in |2, p|$, then z_n specified by (0.2) will be of the form*

1) $z_1 = x_1 y_1$,
2) $z_n = 0$ *for* $n \in |2, p|$,
3) $z_n = x_1 y_n$ *for* $n \in |p+1, q|$,
4) $z_n = x_1 y_n + x_n y_1^n$ *for* $n \in |q+1, p+q|$,
5) $z_r = x_1 y_r + \binom{r}{p+1} x_{q+1} y_1^q y_{p+1} + x_r y_1^r$.

Proof. By the assumption, for $n \in |2, q+1|$, we have

$$\sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} = 0. \tag{0.7}$$

Thus from (0.5) we get 2), 3) and 4) for $n = q+1$.
Let us fix $n \in |q+2, p+q|$. Then

$$\sum_{k=2}^{q} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} = 0. \tag{0.8}$$

If $k \in |q+1, n-1|$, then by (iv), for every $\bar{u}_n \in U_{n,k}$, there exists

$$j \in |2, n-k+1| \subset |2, n-(q+1)+1| \subset |2, p| \text{ such that } u_j \geq 1.$$

By the assumption ($y_j = 0$ for all $j \in |2, p|$) we obtain

$$\sum_{k=q+1}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} = 0.$$

Thus from (0.8) and (0.5) it follows that 4) holds for $n \in |q+2, p+q|$.
Now let $n = r$. For every $k \in |q+2, r-1|$ and $\bar{u}_r \in U_{r,k}$, by (iv) there exists

$$j \in |2, r-k+1| \subset |2, r-(q+2)+1| \subset |2, p| \text{ such that } u_j \geq 1,$$

hence

$$\sum_{k=q+2}^{r-1} x_k \sum_{\bar{u}_r \in U_{r,k}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = 0. \tag{0.9}$$

Since (0.8) holds too, so

$$\sum_{k=2}^{r-1} x_k \sum_{\bar{u}_r \in U_{r,k}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = x_{q+1} \sum_{\bar{u}_r \in U_{r,q+1}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j}.$$

Let us notice that if for some $\bar{u}_r \in U_{r,q+1}$ there exists $j \in |2, p|$ such that $u_j \geq 1$, then we have

$$\prod_{j=1}^{r-1} y_j^{u_j} = 0.$$

Moreover, if for an $\bar{u}_r \in U_{r,q+1}$ $u_j = 0$ for all $j \in |2,p|$, then by (v) we have $u_1 = q$, $u_{p+1} = 1$ and next from (0.4) we get

$$x_{q+1} \sum_{\bar{u}_r \in U_{r,q+1}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = \frac{r!}{(p+1)!q!} x_{q+1} y_1^q y_{p+1} .$$

Thus from (0.5) we obtain 5).

LEMMA 3. *Let p, q be natural numbers such that $1 \leq p < q$ and let $r = p + q + 1$. If $x_j = 0$ for all $j \in |2,p|$ and $y_j = 0$ for all $j \in |2,q|$, then*

1) $z_1 = x_1 y_1$,
2) $z_n = 0$ *for* $n \in |2,p|$,
3) $z_n = x_n y_1^n$ *for* $n \in |p+1,q|$,
4) $z_n = x_1 y_n + x_n y_1^n$ *for* $n \in |q+1,p+q|$,
5) $z_r = x_1 y_r + \binom{r}{p} x_{p+1} y_1^p y_{q+1} + x_r y_1^r$.

Proof. By the assumption, (0.7) holds for $n \in |2,p+1|$, and from (0.5) we get 2) and 3) for $n = p+1$.
Now let $n \in |p+2, p+q|$. Then

$$\sum_{k=2}^{p} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} = 0. \tag{0.10}$$

If $k \in |p+1, n-1|$, then by (iv), for every $\bar{u}_n \in U_{n,k}$, there exists

$$j \in |2, n-k+1| \subset |2, n-(p+1)+1| \subset |2,q| \text{ such that } u_j \geq 1.$$

and

$$\sum_{k=p+1}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} = 0.$$

From (0.10) and (0.5) we get that 3) for $n \in |p+2,q|$ and 4) hold.
Let $n = r$. For $k \in |p+2, r-1|$ and every $\bar{u}_r \in U_{r,k}$, by (iv) there exists

$$j \in |2, r-k+1| \subset |2, r-(p+2)+1| \subset |2,q| \text{ such that } u_j \geq 1,$$

therefore

$$\sum_{k=p+2}^{r-1} x_k \sum_{\bar{u}_r \in U_{r,k}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = 0.$$

Since (0.10) for $n = r$ holds too, so

$$\sum_{k=2}^{r-1} x_k \sum_{\bar{u}_r \in U_{r,k}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = x_{p+1} \sum_{\bar{u}_r \in U_{r,k}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} . \tag{0.11}$$

Let us notice that for every $\bar{u}_r \in U_{r,p+1}$ we have $u_1 \le p$.
Indeed, if $u_1 = p+1$, then we will get $u_j = 0$ for all $j \in |2, r|$, so

$$p + q + 1 = r = \sum_{i=1}^{r} i u_i = u_1 = p + 1$$

and we obtain a contradiction with the assumption $q > 1$.
From (vi) we get

if $u_1 = p$, $u_{q+1} = 1$ and $u_j = 0$ for all $j \in |2, r| \setminus \{q+1\}$, then $\bar{u}_r \in U_{r,p+1}$.

Hence from (0.4), on the right side of (0.11) we get

$$\frac{r!}{p!(q+1)!} x_{p+1} y_1^p y_{q+1} .$$

On the other hand, if for an $\bar{u}_r \in U_{r,p+1}$ $u_1 = i \le p-1$ holds, then we obtain

$$4 \le 2(p+1) \le r = p + q + 1 \quad \text{and} \quad 2 \le \left[\frac{r}{p+1-i}\right] \le q.$$

Thus by (vii) there exists $j \in \left|2, \left[\frac{r}{p+1-i}\right]\right| \subset |2, q|$ such that $u_j \ge 1$.
Therefore

$$\prod_{j=1}^{r-1} y_j^{u_j} = 0.$$

Consequently

$$x_{p+1} \sum_{\bar{u}_r \in U_{r,p+1}} A_{\bar{u}_r} \prod_{j=1}^{r-1} y_j^{u_j} = \frac{r!}{p!(q+1)!} x_{p+1} y_1^p y_{q+1} ,$$

which with (0.11) and (0.5) completes the proof.

REMARK 1. For any fixed natural numbers p and q Lemmas 2 and 3 determine some properties of the operation (0.1) in the group L_s^1 for $2 \le s \le p + q + 1$.

1. Denote by

$$Z_s^i := \{\bar{x}_s \in Z_s : \ \forall\, j \in |2, 1+i| \ x_j = 0\} ,$$
$$\tilde{Z}_s^i := \left\{\tilde{x}_s := (x_1, x_{i+2}, \ldots, x_s) : \ \bar{x}_s \in Z_s^i\right\} ,$$
$$T_s^i := \left\{\bar{x}_s \in Z_s^i : \ x_1 = 1\right\} ,$$
$$\tilde{T}_s^i := \left\{\hat{x}_s := (x_{i+2}, \ldots, x_s) : \ \bar{x}_s \in T_s^i\right\} .$$

One can prove that for any fixed non-negative integral number i the sets Z_s^i and T_s^i are closed with respect to the operation (0.1) in the group L_s^1.

Let $f : \tilde{Z}^i_{s-1} \to \mathbf{R}$, $g : \bar{T}^i_{s-1} \to \mathbf{R}$. Consider sets

$$Z^i_{s,s}(f) := \left\{(\bar{x}_{s-1}, f(\tilde{x}_{s-1})) : \ \bar{x}_{s-1} \in Z^i_{s-1}\right\} \quad \text{for} \quad s \geq 2+i, \tag{1.1}$$

$$T^i_{s,s}(g) := \left\{(\bar{x}_{s-1}, g(\hat{x}_{s-1})) : \ \bar{x}_{s-1} \in T^i_{s-1}\right\} \quad \text{for} \quad s \geq 3+i. \tag{1.2}$$

The subsemigroups $Z^0_{s,s}(f)$ of the group L^1_s for $s \in |2,6|$ have been determined in [3], [4], [9] and [11]. S. Midura has proved in [3] that for the groups L^1_2 and L^1_3 such subsemigroups belongs to the families

$$\left(Z^0_{2,2}\left(x_1 \to p(x_1^2 - x_1)\right)\right)_{p\in\mathrm{R}},$$

$$\left(Z^0_{3,3}\left(\bar{x}_2 \to \frac{3}{2}\frac{x_2^2}{x_1} + p(x_1^3 - x_1)\right)\right)_{p\in\mathrm{R}},$$

respectively, whereas in [4], [9], [11] it has been shown that for $i = 0$ there do not exist any subsemigroups of the form (1.1) of the groups L^1_4, L^1_5 and L^1_6. Furthermore it is known (see [2], [3]) that for $s \geq 4$ there does not exist any subsemigroup $Z^0_{s,s}(f)$ of the group L^1_s.
The subsemigroups $Z^1_{s,s}(f)$ for $s \in |4,6|$ have been considered in [4], [9], [11]. It has been proved that subsemigroups of the form (1.1) for $i = 1$ are the sets from the families

$$\left(Z^1_{4,4}\left(\bar{x}_3 \to p(x_1^4 - x_1)\right)\right)_{p\in\mathrm{R}} \qquad \text{for the group } L^1_4,$$

$$\left(Z^1_{5,5}\left(\bar{x}_4 \to 5\frac{x_3^2}{x_1} + p(x_1^5 - x_1)\right)\right)_{p\in\mathrm{R}} \qquad \text{for the group } L^1_5.$$

S. Midura has proved in [9] that there does not exist any subsemigroup $Z^1_{6,6}(f)$ of the group L^1_6.
In this part we will prove that for $s \geq 4+2i$ there do not exist any subsemigroups $Z^i_{s,s}(f)$ and $T^i_{s,s}(g)$. We will show it by proving that suitable functional equations have not any solutions.

The sets $Z^i_{s,s}(f)$ and $T^i_{s,s}(g)$ are subsemigroups of the group L^1_s if and only if the functions f and g are solutions of the equations:

$$f(\bar{z}_{s-1}) = x_1 f(\bar{y}_{s-1}) + \sum_{k=2}^{s-1} x_k \sum_{\bar{u}_s \in U_{s,k}} A_{\bar{u}_s} \prod_{j=1}^{s-1} y_j^{u_j} + y_1^s f(\bar{x}_{s-1}) \quad \text{for} \quad \bar{x}_{s-1}, \bar{y}_{s-1} \in Z^i_{s-1}, \tag{1.3}$$

$$g(\hat{z}_{s-1}) = g(\hat{y}_{s-1}) + \sum_{k=2}^{s-1} x_k \sum_{\bar{u}_s \in U_{s,k}} A_{\bar{u}_s} \prod_{j=1}^{s-1} y_j^{u_j} + g(\hat{x}_{s-1}) \quad \text{for} \quad \bar{x}_{s-1}, \bar{y}_{s-1} \in T^i_{s-1}, \tag{1.4}$$

where $\tilde{z}_{s-1}$ and $\hat{z}_{s-1}$ are defined by

$$z_n = \sum_{k=1}^{n} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} y_j^{u_j}. \tag{1.5}$$

We are going to prove

THEOREM 1. *For $s \geq 4+2i$ the equation* (1.4) *has not any solutions in the class of fonctions $g : \bar{T}_{s-1}^i \to \mathbf{R}$.*

Proof. Suppose that the function $g : \bar{T}_{s-1}^i \to \mathbf{R}$ is a solution of the equation (1.4) for $s \geq 4+2i$.
Put in (1.4) $x_j = 0$ for all $j \in |i+2, s-i-2|$. By Lemma 2 ($p = i+1$, $q = s-i-2$) we get

$$g(\hat{z}_{s-1}) = g(\hat{x}_{s-1}) + \binom{s}{i+2} x_{s-i-1} y_{i+2} + g(\hat{y}_{s-1}) \quad \text{for} \quad \bar{x}_{s-1} \in T_{s-1}^{s-i-3},\ \bar{y}_{s-1} \in T_{s-1}^{i}, \tag{1.6}$$

where

$$z_n = y_n \quad \text{for } n \in |i+2, s-i-2|,$$
$$z_n = x_n + y_n \quad \text{for } n \in |s-i-1, s-1|.$$

Set in (1.4) $y_j = 0$ for all $j \in |i+2, s-i-2|$ now. By Lemma 3 we obtain

$$g(\hat{z}_{s-1}) = g(\hat{x}_{s-1}) + \binom{s}{i+1} x_{i+2} y_{s-i-1} + g(\hat{y}_{s-1}) \quad \text{for} \quad \bar{x}_{s-1} \in T_{s-1}^{i},\ \bar{y}_{s-1} \in T_{s-1}^{s-i-3}, \tag{1.7}$$

where

$$z_n = x_n \quad \text{for } n \in |i+2, s-i-2|,$$
$$z_n = x_n + y_n \quad \text{for } n \in |s-i-1, s-1|.$$

If we put in (1.6) $\bar{v}_{s-1} := \bar{y}_{s-1}$, $\bar{w}_{s-1} := \bar{x}_{s-1}$, then we obtain

$$g(\hat{z}_{s-1}) = g(\hat{w}_{s-1}) + \binom{s}{i+2} w_{s-i-1} v_{i+2} + g(\hat{v}_{s-1}) \quad \text{for} \quad \bar{w}_{s-1} \in T_{s-1}^{s-i-3},\ \bar{v}_{s-1} \in T_{s-1}^{i}, \tag{1.8}$$

where

$$z_n = v_n \qquad \text{for } n \in |i+2, s-i-2|,$$
$$z_n = w_n + v_n \quad \text{for } n \in |s-i-1, s-1|. \tag{1.9}$$

If we set in (1.7) $\bar{v}_{s-1} := \bar{x}_{s-1}$, $\bar{w}_{s-1} := \bar{y}_{s-1}$, then we get

$$g(\hat{z}_{s-1}) = g(\hat{v}_{s-1}) + \binom{s}{i+1} w_{s-i-1}v_{i+2} + g(\hat{w}_{s-1}) \quad \text{for} \quad \bar{w}_{s-1} \in T_{s-1}^{s-i-3},\ \bar{v}_{s-1} \in T_{s-1}^{i}, \tag{1.10}$$

where $\hat{z}_{s-1}$ is defined by (1.9).
Compare (1.8) and (1.10). We obtain

$$\binom{s}{i+2} w_{s-i-1}v_{i+2} = \binom{s}{1+1} w_{s-i-1}v_{i+2} \quad \text{for all} \quad v_{i+2}, w_{s-i-1} \in \mathbf{R}.$$

Thus

$$s = 3 + 2i,$$

and we have a contradiction with the assumption $s \geq 4 + 2i$, which completes the proof.

From above theorem it results

THEOREM 2. *For $s \geq 4 + 2i$ there does not exist any subsemigroup $T_{s,s}^i(g)$ of the group L_s^1.*

Consider the equation (1.3) now. Fix $x_1 = y_1 = 1$. If we denote

$$g(\hat{x}_{s-1}) := f(1, \hat{x}_{s-1}) \quad \text{for} \quad \hat{x}_{s-1} \in \tilde{T}_{s-1}^{i},$$

then by Theorem 1 we will get

THEOREM 3.

(i) For $s \geq 4 + 2i$ the equation (1.3) *has not any solutions in the class of functions $f : \tilde{Z}_{s-1}^{i} \to \mathbf{R}$.*

(ii) For $s \geq 4+2i$ there does not exist any subsemigroup $Z_{s,s}^i(f)$ of the group L_s^1.

2. In this part we determine all the existing subsemigroups $Z_{s,s}^i(f)$ and $T_{s,s}^i(g)$. At first we present solutions of some functional equations.

LEMMA 4 ([3], Theorem 1). *Let $n > 1$ be a natural number. The general solution $\varphi : \mathbf{R}_0 \to \mathbf{R}$ of the equation*

$$\varphi(xy) = x\varphi(y) + y^n\varphi(x)$$

is the family of functions

$$\varphi(x) = a(x^n - x),$$

where a is an arbitrary real number.

LEMMA 5 ([11], Lemma 1). *Let t be a real number, n - a natural number. The general solution $\varphi : \mathbf{R}_0 \times \mathbf{R} \to \mathbf{R}$ of the equation*

$$\varphi(x_1y_1, x_1y_2 + x_2y_1^{n+1}) = x_1\varphi(y_1, y_2) + tx_2y_1^ny_2 + y_1^{2n+1}\varphi(x_1, x_2)$$

is given by

$$\varphi(x_1, x_2) = \frac{t}{2}\frac{x_2^2}{x_1} + a\left(x_1^{2n+1} - x_1\right),$$

where a is an arbitrary real constant.

In [11] it has been proved

LEMMA 6. *Let t be a real number. The general solution* $\varphi : \mathbf{R}\to\mathbf{R}$ *of the equation*

$$\varphi(x + y) = \varphi(x) + txy + \varphi(y)$$

is the family of functions

$$\varphi(x) = \psi(x) + \frac{t}{2}x^2,$$

where $\psi : \mathbf{R}\to\mathbf{R}$ *is an arbitrary additive function.*

LEMMA 7 ([8] Lemma 1). *Let* $b \geq 2$, $c \geq 2$, $b \neq c$ *be integers. The general solution* $\varphi : \mathbf{R}_0\times\mathbf{R}\to\mathbf{R}$ *of the equation*

$$\varphi(x_1y_1, x_1y_2 + x_2y_1^b) = x_1\varphi(y_1, y_2) + y_1^c\varphi(x_1, x_2)$$

is given by

$$\varphi(x_1, x_2) = a(x_1^c - x_1),$$

where a is an arbitrary real constant.

LEMMA 8 (see [1], Proposition 1, p. 35). *The general solution* $\varphi : \mathbf{R}^n\to\mathbf{R}$ *of the equation*

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

is given by

$$\varphi(x) = \sum_{k=1}^{n} \psi_k(x_k),$$

where $x = (x_1, \ldots, x_n)$ *and* $\psi_k : \mathbf{R}\to\mathbf{R}$ *for* $k \in |1, n|$ *are arbitrary additive functions.*

In the first part of our paper we have proved that for $s \geq 4 + 2i$ there do not exist any subsemigroups $Z^i_{s,s}(f)$ and $T^i_{s,s}(g)$. In order to determine the subsemigroups of the form (1.1), (1.2), we will consider the equations (1.3) and (1.4) for $s \leq 3 + 2i$.

Consider the equation (1.3) in two cases:

1) $s \in |i + 2, 2i + 2|$,

2) $s = 3 + 2i$.

1) For $s \in |i + 2, 2i + 2|$ we have (Lemma 2, $p = q = i + 1$)

$$f(\tilde{z}_{s-1}) = x_1 f(\tilde{y}_{s-1}) + y_1^s f(\tilde{x}_{s-1}) \quad \text{for} \quad \tilde{x}_{s-1}, \tilde{y}_{s-1} \in \tilde{Z}^i_{s-1}, \tag{2.1}$$

where

$$z_1 = x_1 y_1 ,$$
$$z_n = x_1 y_n + x_n y_1^n \ \text{ for } n \in |i+2, s-1|.$$

We will prove

THEOREM 4. *The general solution $f : \tilde{Z}^i_{s-1} \to \mathbf{R}$ of the equation (2.1) is the family of functions*

$$f(\tilde{x}_{s-1}) = a\,(x_1^s - x_1)\,,$$

where a is an arbitrary real constant.

Proof. Suppose that a function $f : \tilde{Z}^i_{s-1} \to \mathbf{R}$ is a solution of (2.1). In a case CARD$|i+2, s-1| \le 1$ we get the statement by Lemma 4 or Lemma 7. Let CARD$|i+2, s-1| \ge 2$. Put in (2.1) $x_j = y_j = 0$ for all $j \in |i+2, s-2|$. We have

$$f(x_1 y_1, 0, \ldots, 0, x_1 y_{s-1} + x_{s-1} y_1^{s-1}) = x_1 f(y_1, 0, \ldots, 0, y_{s-1}) + y_1^s f(x_1, 0, \ldots, 0, x_{s-1})$$

and by Lemma 7

$$f(x_1, 0, \ldots, 0, x_{s-1}) = a(x_1^s - x_1). \tag{2.2}$$

If we set in (2.1) $y_1 = 1$, $y_j = 0$ for $j \in |i+2, s-2|$, then by (2.2) we obtain

$$f(x_1, x_{i+2}, \ldots, x_{s-2}, x_1 y_{s-1} + x_{s-1}) = f(\tilde{x}_{s-1}) \quad \text{for} \quad \tilde{x}_{s-1} \in \tilde{Z}^i_{s-1},\ y_{s-1} \in \mathbf{R}$$

and so

$$f(\tilde{x}_{s-1}) = f(\tilde{x}_{s-2}, 0) \qquad \text{for} \quad \tilde{x}_{s-1} \in \tilde{Z}^i_{s-1}\,.$$

Suppose that for some $k \in |i+2, s-2|$

$$f(\tilde{x}_{s-1}) = f(\tilde{x}_k, 0, \ldots, 0) \qquad \text{for} \quad \tilde{x}_{s-1} \in \tilde{Z}^i_{s-1}\,.$$

From (2.1) we get

$$f(\tilde{z}_k, 0, \ldots, 0) = x_1 f(\tilde{y}_k, 0, \ldots, 0) + y_1^s f(\tilde{x}_k, 0, \ldots, 0) \quad \text{for } \tilde{x}_k, \tilde{y}_k \in \tilde{Z}^i_k\,.$$

Similarly like above, we can show that

$$f(\tilde{x}_k, 0, \ldots, 0) = f(\tilde{x}_{k-1}, 0, \ldots, 0)$$

and so

$$f(\tilde{x}_{s-1}) = f(\tilde{x}_{k-1}, 0, \ldots, 0).$$

Consequently

$$f(\tilde{x}_{s-1}) = f(x_1, 0, \ldots, 0)$$

and from (2.1), by Lemma 4, we get

$$f(\tilde{x}_{s-1}) = a\,(x_1^s - x_1)\,. \tag{2.3}$$

It is easy to see that every function of the form (2.3), where a is an arbitrary real number, is the solution of the equation (2.1).

2) Assume that $s = 3 + 2i$. From (1.3), by Lemma 2, we will obtain

$$f(\tilde{z}_{s-1}) = x_1 f(\tilde{y}_{s-1}) + \binom{s}{i+2} x_{i+2} y_1^{i+1} y_{i+2} + y_1^s f(\tilde{x}_{s-1}) \qquad \text{for } \tilde{x}_{s-1},\, \tilde{y}_{s-1} \in \tilde{Z}^1_{s-1}\,, \tag{2.4}$$

where

$$\begin{aligned} z_1 &= x_1 y_1, \\ z_n &= x_1 y_n + x_n y_1^n \quad \text{for } n \in |i+2, s-1|. \end{aligned}$$

THEOREM 5. *The general solution* $f : \tilde{Z}^i_{s-1} \to \mathbf{R}$ *of the equation* (2.4) *is the family of functions*

$$f(\tilde{x}_{s-1}) = \frac{1}{2}\binom{s}{i+2}\frac{x_{i+2}^2}{x_1} + a\,(x_1^s - x_1)\,, \tag{2.5}$$

where a is an arbitrary real constant.

Proof. Let $f : \tilde{Z}^i_{s-1} \to \mathbf{R}$ be a solution of the equation (2.4). If $i = 0$, then we get the statement from Lemma 5, directly.

Now let $i \geq 1$. Similarly like in the proof of Theorem 4 we can prove that

$$f(\tilde{x}_{s-1}) = f(x_1, x_{i+2}, 0, \ldots, 0) \quad \text{for } \tilde{x}_{s-1} \in \tilde{Z}^i_{s-1}\,.$$

Hence from (2.4), by Lemma 5 we get (2.5). One can verify that every function of the form (2.5), where a is an arbitrary real constant, is a solution of the equation (2.4).

From Theorems 4 and 5 we have

THEOREM 6. *Let us fix a non-negative integral number i.*

(i) The only subsemigroups of the form (1.1) *of L_s^1 for $s \in |i+2, 2i+2|$ are the sets from the families*

$$\left(Z^i_{s,s}\,(\tilde{x}_{s-1} \to a\,(x_1^s - x_1))\right)_{a \in \mathbf{R}}\,.$$

(ii) The only subsemigroups of the form (1.1) *of L_s^1 for $s = 3 + 2i$ are the sets from the families*

$$\left(Z^i_{s,s}\left(\tilde{x}_{s-1} \to \frac{1}{2}\binom{s}{i+2}\frac{x_{i+2}^2}{x_1} + a\,(x_1^s - x_1)\right)\right)_{a \in \mathbf{R}}\,.$$

Now consider the equation (1.4). For $s \in |i+3, 2i+2|$, by Lemma 2, we get

$$g(\hat{z}_{s-1}) = g(\hat{x}_{s-1}) + g(\hat{y}_{s-1}) \quad \text{for } \hat{x}_{s-1}, \hat{y}_{s-1} \in \bar{T}^i_{s-1}, \tag{2.6}$$

where

$$z_n = x_n + y_n \quad \text{for } n \in |i+2, s-1|.$$

From Lemma 8 we get

THEOREM 7. *The general solution $g : \bar{T}^i_{s-1} \to \mathbf{R}$ of the equation (2.6) is given by*

$$g(\hat{x}_{s-1}) = \sum_{k=i+2}^{s-1} \psi_k(x_k),$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

Let $s = 3 + 2i$. From (1.4), by Lemma 2, we obtain

$$g(\hat{z}_{s-1}) = g(\hat{x}_{s-1}) + \binom{s}{i+2} x_{i+2} y_{i+2} + g(\hat{y}_{s-1}) \quad \text{for } \hat{x}_{s-1}, \hat{y}_{s-1} \in \bar{T}^i_{s-1}, \tag{2.7}$$

where

$$z_n = x_n + y_n \quad \text{for } n \in |i+2, s-1|.$$

We will prove

THEOREM 8. *The general solution $g : \bar{T}^i_{s-1} \to \mathbf{R}$ of the equation (2.7) is given by*

$$g(\hat{x}_{s-1}) = \frac{1}{2}\binom{s}{i+2} x_{i+2}^2 + \sum_{k=i+2}^{s-1} \psi_k(x_k),$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

Proof. Let $g : \bar{T}^i_{s-1} \to \mathbf{R}$ be a solution of the equation (2.7). If $i = 0$, then the statement results from Lemma 6. Now let $i \geq 1$. Putting in (2.7) $x_{i+2} = y_{i+2} = 0$ we obtain

$$g(0, x_{i+3} + y_{i+3}, \ldots, x_{s-1} + y_{s-1}) = g(0, x_{i+3}, \ldots, x_{s-1}) + g(0, y_{i+3}, \ldots, y_{s-1})$$

and by Lemma 8 we get

$$g(0, x_{i+3}, \ldots, x_{s-1}) = \sum_{k=i+3}^{s-1} \psi_k(x_k), \tag{2.8}$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.
Set in (2.7) $x_j = y_j = 0$ for all $j \in |i+3, s-1|$. We have

$$g(x_{i+2}+y_{i+2},0,\dots,0) = g(x_{i+2},0,\dots,0)+\binom{s}{i+2} x_{i+2}y_{i+2}+g(y_{i+2},0,\dots,0).$$

From Lemma 6 we obtain

$$g(x_{i+2},0,\dots,0) = \frac{1}{2}\binom{s}{i+2} x_{i+2}^2 + \psi_{i+2}(x_{i+2}), \tag{2.9}$$

where $\psi_{i+2} : \mathbf{R}\to\mathbf{R}$ is an arbitrary additive function.

If we put in (2.7) $y_{i+2} = 0$ and $x_j = 0$ for all $j \in |i+3, s-1|$, then by (2.9) we get

$$g(x_{i+2}, y_{i+3},\dots,y_{s-1}) = \frac{1}{2}\binom{s}{i+2} x_{i+2}^2 + \psi_{i+2}(x_{i+2}) + \sum_{k=i+3}^{s-1} \psi_k(y_k). \tag{2.10}$$

It is easy to see that every function given by (2.10), where $\psi_k : \mathbf{R}\to\mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions, satisfies (2.7).

THEOREM 9. *Let us fix a non-negative integral number i.*

(i) The only subsemigroups of the form (1.2) *of L_s^1 for $s \in |i+3, 2i+2|$ are the sets*

$$T^i_{s,s}\left(\hat{x}_{s-1}\to \sum_{k=i+2}^{s-1} \psi_k(x_k)\right),$$

where $\psi_k : \mathbf{R}\to\mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

(ii) The only subsemigroups of the form (1.2) *of L_s^1 for $s = 3+2i$ are the sets*

$$T_{s,s}\left(\hat{x}_{s-1}\to \frac{1}{2}\binom{s}{i+2} x_{i+2}^2 + \sum_{k=i+2}^{s-1} \psi_k(x_k)\right),$$

where $\psi_k : \mathbf{R}\to\mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

3. Denote

$$P_s^i := \left\{\check{x}_s := (x_2,\dots,x_s) \in \mathbf{R}^{s-1} : \ \forall j \in |2, i+1| \ x_j = 0\right\},$$

$$\tilde{P}_s^i := \left\{\hat{x}_s := (x_{i+2},\dots,x_s) : \ \check{x}_s \in P_s^i\right\}.$$

Let $h : \tilde{P}_s^i \to \mathbf{R}_0$, $h_1 : \tilde{P}_{s-1}^i \to \mathbf{R}_0$, $h_s : \tilde{P}_{s-1}^i \to \mathbf{R}$. Consider sets

$$P_1^{s,i}(h) := \left\{(h(\hat{x}_s), \check{x}_s) : \ \check{x}_s \in P_s^i\right\} \quad \text{for} \quad s \geq i+2, \tag{3.1}$$

$$P_{1,s}^{s,i}(h_1, h_s) := \{(h_1(\hat{x}_{s-1}), \check{x}_{s-1}, h_s(\hat{x}_{s-1})) : \ \check{x}_{s-1} \in P_{s-1}^i\} \quad \text{for} \quad s \geq i+3. \tag{3.2}$$

Subsemigroups $P_1^{s,i}(h)$ and $P_{1,s}^{s,i}(h_1, h_s)$ of the group L_s^1 for $s \leq 6$, on some conditions, have been considered in [5], [7], [9], [10].

The set $P_1^{s,i}(h)$ with the operation (0.1) is a subsemigroup of L_s^1 if and only if the function $h : \tilde{P}_s^i \to \mathbf{R}_0$ is a solution of the equation

$$h(\hat{z}_s) = h(\hat{x}_s)h(\hat{y}_s) \qquad \text{for} \quad \breve{x}_s, \breve{y}_s \in P_s^i\,, \tag{3.3}$$

where

$$z_n = h(\hat{x}_s)y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} h(\hat{y}_s)^{u_1} \prod_{j=2}^{n-1} y_j^{u_j} + x_n h(\hat{y}_s)^n$$
$$\text{for} \quad n \in |i+2, s|.$$

Assume that

$$\text{the functions} \;\; x_j \to (0, \ldots, 0, x_j, 0, \ldots, 0) \;\; \text{for} \;\; j \in |i+2, s| \;\; \text{are continuous.} \tag{3.4}$$

We prove

THEOREM 10. *A unique solution $h : \tilde{P}_s^i \to \mathbf{R}_0$ of the equation* (3.3), *on condition* (3.4), *is the constant function $h \equiv 1$.*

Proof. We will use the following

LEMMA 9 ([12]). *A unique continuous solution $\varphi : \mathbf{R} \to \mathbf{R}_0$ of the equation*

$$\varphi\left(x\varphi(y)^k + y\varphi(x)^l\right) = \varphi(x)\varphi(y),$$

where k and l are natural numbers such that $k \neq l$, is the constant function $\varphi \equiv 1$.

If $s = i + 2$, then we have the statement from Lemma 9.
Now let $s \geq i + 3$. Putting in (3.3) $x_j = y_j = 0$ for all $j \in |i+2, s-1|$ we obtain

$$h\left(0, \ldots, 0, y_s h(0, \ldots, 0, x_s) + x_s h(0, \ldots, 0, y_s)^s\right) = h(0, \ldots, 0, x_s)h(0, \ldots, 0, y_s)$$

and by Lemma 9

$$h(0, \ldots, 0, x_s) = 1 \qquad \text{for} \quad x_s \in \mathbf{R}. \tag{3.5}$$

If we set in (3.3) $x_j = 0$ for all $j \in |i+2, s-1|$, then by (3.5) we get

$$h(y_{i+2}, \ldots, y_{s-1}, x_s h(\hat{y}_s) + y_s) = h(\hat{y}_s) \qquad \text{for} \quad x_s \in \mathbf{R},\; \hat{y}_s \in \tilde{P}_s^i$$

and for $x_s = -y_s\left[h(\hat{y}_s)\right]^{-1}$ we have

$$h(\hat{y}_s) = h(\hat{y}_{s-1}, 0) \qquad \text{for} \;\; \hat{y}_s \in \hat{P}_s^i\,.$$

Suppose that for some $k \in |i+2, s-1|$

$$h(\hat{x}_s) = h(\hat{x}_k, 0, \ldots, 0) \qquad \text{for } \hat{x}_s \in \tilde{P}^i_s .$$

From (3.3) we get

$$h(\hat{z}_k, 0, \ldots, 0) = h(\hat{x}_k, 0, \ldots, 0) h(\hat{y}_k, 0, \ldots, 0) \quad \text{for } \check{x}_k, \check{y}_k \in P^i_k .$$

Similarly like above we can show that

$$h(\hat{x}_k, 0, \ldots, 0) = h(\hat{x}_{k-1}, 0, \ldots, 0)$$

and so

$$h(\hat{x}_s) = h(\hat{x}_{k-1}, 0, \ldots, 0) \qquad \text{for } \hat{x}_s \in \tilde{P}^i_s .$$

Consequently

$$h(\hat{x}_s) = h(0, \ldots, 0) \qquad \text{for } \hat{x}_s \in \tilde{P}^i_s . \tag{3.6}$$

From (3.3) we get

$$h(0, \ldots, 0) = h(0, \ldots, 0)^2$$

so

$$h(0, \ldots, 0) = 1$$

and from (3.6)

$$h \equiv 1.$$

Obviously the function $h \equiv 1$ satisfies (3.3).

Thus we have

THEOREM 11. *A unique subsemigroup of the form* (3.1) *of the group* L^1_s, *on condition* (3.4), *is the set* $P^{s,i}_1(\hat{x}_s \to 1)$.

Consider a set $P^{s,i}_{1,s}(h_1, h_s)$. The set $P^{s,i}_{1,s}(h_1, h_s)$ is a subsemigroup of the group L^1_s if and only if the functions $h_1 : \bar{P}^i_{s-1} \to \mathbf{R}_0$, $h_s : \bar{P}^i_{s-1} \to \mathbf{R}$ satisfy the following system of functional equations

$$h_1(\hat{z}_{s-1}) = h_1(\hat{x}_{s-1}) h_1(\hat{y}_{s-1}) \tag{3.7}$$

$$\begin{aligned} h_s(\hat{z}_{s-1}) = h_1(\hat{x}_{s-1}) h_s(\hat{y}_{s-1}) + \sum_{k=2}^{s-1} x_k \sum_{\bar{u}_s \in U_{s,k}} A_{\bar{u}_s} h_1(\hat{y}_{s-1})^{u_1} \prod_{j=2}^{s-1} y_j^{u_j} \\ + h_s(\hat{x}_{s-1}) h_1(\hat{y}_{s-1})^s \quad \text{for } \check{x}_{s-1}, \check{y}_{s-1} \in P^i_{s-1} , \end{aligned} \tag{3.8}$$

where

$$z_n = h_1(\hat{x}_{s-1}) y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} h_1(\hat{y}_{s-1})^{u_1} \prod_{j=2}^{n-1} y_j^{u_j} + x_n h_1(\hat{y}_{s-1})^n$$

$$\text{for } n \in |i+2, s-1|.$$

Assume that

$$\text{the functions } x_j \to h_1(0,\dots,0,x_j,0,\dots,0) \text{ for } j \in |i+2, s-1| \text{ are continuous.} \tag{3.9}$$

From Theorem 10 we then get

$$h_1 \equiv 1.$$

Set in (3.8) $h_1 \equiv 1$. From Theorems 1, 7 and 8 we get

THEOREM 12. *Assume that* (3.9) *holds.*

(i) For $s \geq 4 + 2i$ the system of equations (3.7)-(3.8) *has not any solutions.*

(ii) The general solution of the system of equations (3.7)-(3.8) *for $s \in |i+3, 2i+2|$ is given by*

$$h_1 \equiv 1,$$

$$h_s(\hat{x}_{s-1}) = \sum_{k=i+2}^{s-1} \psi_k(x_k),$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

(iii) The general solution of the system of equations (3.7)-(3.8) *for $s = 3 + 2i$ is given by*

$$h_1 \equiv 1,$$

$$h_s(\hat{x}_{s-1}) = \frac{1}{2}\binom{s}{i+2} x_{i+2}^2 + \sum_{k=i+2}^{s-1} \psi_k(x_k),$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

Thus we have

THEOREM 13. *Assume that* (3.9) *holds.*

(i) The only subsemigroups of the form (3.2) *of L_s^1 for $s \in |i+3, 2i+2|$ are the sets*

$$P_{1,s}^{s,i}\left(\hat{x}_{s-1} \to 1, \hat{x}_{s-1} \to \sum_{k=i+2}^{s-1} \psi_k(x_k)\right),$$

where $\psi_k : \mathbf{R} \to \mathbf{R}$ for $k \in |i+2, s-1|$ are arbitrary additive functions.

(ii) *The only subsemigroups of the form* (3.2) *of* L_s^1 *for* $s = 3 + 2i$ *are the sets*

$$P_{1,s}^{s,i}\left(\hat{x}_{s-1}\to 1, \hat{x}_{s-1}\to \frac{1}{2}\binom{s}{i+2}x_{i+2}^2 + \sum_{k=i+2}^{s-1}\psi_k(x_k)\right),$$

where $\psi_k : \mathbf{R}\to\mathbf{R}$ *for* $k \in |i+2, s-1|$ *are arbitrary additive functions.*

(iii) *For* $s \geq 4 + 2i$ *here does not exist any subsemigroup* $P_{1,s}^{s,i}(h_1, h_s)$ *of the group* L_s^1.

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