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# On a class of linear differential operators of first order with singular point

**Abstract.** The aim of this paper is to give a background to study spectral properties of some operators occurring in differential geometry. We deal with the problem of existence and uniqueness of local solutions of partial differential equations of type $D_X u - Bu = f$. An integral formula for the solutions and conditions of its feasibility are given in terms of dynamics of the vector field $X$. A generalization to equations with polynomials in the operator $D_X$ is presented.

## 1. Introduction

Let $C^k(n,s)$, for $k = 0, 1, 2, \ldots, \infty, \omega$, denote the space of germs at the origin $0 \in \mathbb{R}^n$ of $C^k$-maps (analytic if $k = \omega$) from $\mathbb{R}^n$ into $\mathbb{R}^s$.

Consider a linear differential operator $L : C^{k+1}(n,s) \to C^k(n,s)$ defined by

$$Lu = D_X u - Bu,$$

where $X \in C^k(n,n)$ is a vector-function, $B \in C^k(n, s \times s)$ is a matrix-function, and $D_X u$ stands for the directional derivative of $u$ in direction $X$.

The operators of this type represent a local form (for specific $B$) of a Lie derivative $L_X$ or covariant derivative $\nabla_X$, which are widely used in differential geometry on manifolds. In both these cases $B(x)$ depends uniquely on a certain jet of $X$ at the point $x$, and $u$ stands usualy for a differentiable section of a vector bundle over the manifold.

For example, if $u = Y$ is another vector field, then $Lu = [X, Y]$ is the Poisson bracket of vector fields, and then $B = DX$.

In this paper we are interested in local solvability of the operator $L$. In other words we ask for the existence and the uniqueness of local solutions of

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the partial differential equation

$$D_X u - Bu = f, \tag{1.1}$$

or in coordinates

$$\sum_{j=1}^{n} X^j(x)\, D_j u^i - B^i_j(x)\, u^j = f^i(x) \tag{1.2}$$

where $i = 1, \dots, s$.

REMARK 1. In case $u = Y$ and $B = DX$, equation (1.1) takes the form

$$D_X\, Y - D_Y\, X = f.$$

In linear case $X(x) = Ax$ this is called a *homological* equation, and the problem of existence of its formal and analytic solutions was solved by Poincaré and Siegel.

REMARK 2. If the origin 0 is not a critical point of $X$, i.e., $X(0) \neq 0$, then $X$ is a non-zero characteristic vector and the existence of local solutions is well known. Thus, henceforth, we shall consider only the singular case $X(0) = 0$.

## 2. An integral form of solution

We are going to present a dynamical method to give an integral formula for a solution of equation (1.1), useful in case when the method works.

We shall assume in this section that the coefficients in (1.2) are of class $C^1$ and the vector field $X$ is *halfcomplete* in the sense that the flow $\phi_t(x)$ generated by $X$ is defined in a half-cylindrical neighborhood $C_\delta \subset \mathbb{R}^n \times \mathbb{R}$

$$\|x\| \leq \delta, \qquad t \geq 0.$$

The flow of $X$ is the solution of the initial value problem

$$x' = X(x), \qquad x(0) = x, \tag{2.1}$$

which means that

$$\phi_t' = X \circ \phi_t, \qquad \phi_0(x) = x, \tag{2.2}$$

where $' = \frac{d}{dt}$.

LEMMA 1. *For all $s$, $t$ such that $s \geq 0$, $t \geq 0$ and $s - t \geq 0$ we have*

$$\phi_{s-t}^{-1} = \phi_t \circ \phi_s^{-1}. \tag{2.3}$$

*Proof.* In fact, since $t \to \phi_t$ is a semigroup of local diffeomorphisms of $\mathbb{R}^n$, so $\phi_s \circ \phi_t = \phi_{s+t}$ for positive $s$, $t$, and hence

$$\phi_{s-t} \circ (\phi_t \circ \phi_s^{-1}) = (\phi_{s-t} \circ \phi_t) \circ \phi_s^{-1} = \phi_o = \mathrm{id}$$

which completes the proof.

Consider another auxiliary differential equation

$$w' = -B(\phi_t^{-1}(x))\, w \tag{2.4}$$

where $w \in \mathbb{R}^m$ and $B$ is the matrix function from equation (1.1). $x$ plays here a role of a parameter in respect to which the right hand side is $C^1$. Thus there exists a normalized fundamental solution $R(t,x)$ of (2.4), satisfying

$$R'(t,x) = -B(\phi_t^{-1}(x))\, R(t,x), \qquad R(0,x) = I. \tag{2.5}$$

We shall prove now

THEOREM 1. *If the integral on the right hand side of the formula*

$$u(x) = -\int_0^\infty R(s, \phi_s(x))\, f(\phi_s(x))\, ds \tag{2.6}$$

*is uniformly convergent in a neighborhood of the origin, then* $u(x)$ *is a local solution of equation* (1.1).

*Proof.* The function $u(x)$ is continuous in a neighborhood of $0 \in \mathbb{R}^n$. In order to show that it satisfies (1.1) we compute first $u(\phi_t(x))$ since its derivative at $t = 0$ is equal to $D_X u$, by (2.2).

$$\begin{aligned} u(\phi_t(x)) &= -\int_0^\infty R(s, \phi_s(\phi_t(x))\, f(\phi_s(\phi_t(x))ds \\ &= -\int_0^\infty R(s, \phi_{s+t}(x))\, f(\phi_{s+t}(x))\, ds \\ &= -\int_t^\infty R(\tau - t, \phi_\tau(x))\, f(\phi_\tau(x)))\, d\tau. \end{aligned}$$

By Lemma 1 and (2.4) we get

$$\begin{aligned} \frac{d}{dt}u(\phi_t(x)) &= R(0, \phi_t(x))\, f(\phi_t(x)) + \int_t^\infty R'(\tau - t, \phi_\tau(x))\, f(\phi_\tau(x))\, d\tau \\ &= f(\phi_t(x)) - \int_t^\infty B(\phi_{\tau-t}^{-1}(\phi_\tau(x))R(\tau - t, \phi_\tau(x))\, f(\phi_\tau(x))\, d\tau \\ &= f(\phi_t(x)) - B(\phi_t(x)) \int_t^\infty R(\tau - t, \phi_\tau(x) f(\phi_\tau(x)))\, d\tau \\ &= f(\phi_t(x)) + B(\phi_t(x))\, u(\phi_t(x). \end{aligned}$$

Setting $t = 0$ and knowing that $\phi_o(x) = x$, we get finally $D_X u = f + Bu$ which yields (1.1).

Consider a particular case $B = bI$, $b = \text{const}$. Then equation (2.4) is $w' = -bw$ and $R(t,x) = e^{-bt}I$, and (2.6) writes formally

$$u(x) = -\int_0^\infty e^{-bt} f(\phi_t(x))\, dt$$

Thus, if $b$ is positive and $f$ bounded, the integral is uniformly convergent and the formula gives a solution of (1.1). We shall see in the next section that this solution is unique.

The integral formula (2.6) works to get effectively a local solution near a wandering point of the vector field $X$. Recall that $a \in \mathbb{R}^n$ is a *wandering point* of $X$ if there is a neighborhood $U$ of $a$ and some $T > 0$ such that $\phi_t(U) \cap U$ is empty for $t > T$.

THEOREM 2. *For each wandering point of $X$ there is a neighborhood in which the integral in* (2.6) *can be modified to define a local solution of eq.* (1.1).

In fact, since a local solution is wanted, we may replace $f$ by $\tilde{f} := \alpha f$, where $\alpha$ is a test function which is 1 in a neighborhood $W$ of the point $a$ with compact support in $U$ mentioned above. Then

$$\tilde{u}(x) = -\int_0^T R(s, \phi_s(x))\tilde{f}(\phi_s(x))ds$$

is a solution of equation

$$D_X\tilde{u} + B(x)\tilde{u} = \tilde{f}$$

in $W$. But in this neighborhood $f$ and $\tilde{f}$ coincide, so that $\tilde{u}$ is a local solution of (1.1).

## 3. Uniqueness theorems

Consider the homogenous version of equation (1.1)

$$D_X\, v - Bv = 0, \qquad v(0) = 0, \tag{3.1}$$

The solutions described in Section 2 are unique if eq. (3.1) has only trivial solution $v(x) = 0$ in a neighborhood of the origin.

Let $\alpha(t,x)$ and $\beta(t,x)$ denote respectively the least and the greatest real parts of the eigenvalues occurring in the spectrum of the matrix $B(\phi_t(x))$ (with fixed $t$ and $x$).

Let $S(t,x)$ be the normalized fundamental matrix of the auxiliary equation

$$v' = B(\phi_t(x))\,v. \tag{3.2}$$

The following estimates, in version without $x$ are well known (cf. Wintner [2])

$$\exp\int_0^t \alpha(s,x)ds \le \|S(t,x)\| \le \exp\int_0^t \beta(s,x)ds \tag{3.3}$$

for $t \ge 0$ and $x$ fixed, and for every solution $v(t,x)$ of (3.2) with $v(0,x) = v(x)$ it holds

$$\|v(x)\|\exp\int_0^t \alpha(s,x)ds \le \|v(t,x)\| \le \|v(x)\|\exp\int_0^t \beta(s,x)ds \tag{3.3$^*$}$$

Setting $\phi_t(x)$ as argument in eq. (3.1) yields

$$v(\phi_t(x))' = B(\phi_t(x))\,v(\phi_t(x)).$$

This is because by (2.2)

$$(D_X\,v)\circ\phi_t = \frac{d}{dt}(v\circ\phi_t). \tag{3.4}$$

If a solution $v(x)$ satisfies (3.1) then $v(t,x) := v(\phi_t(x))$ satifies (3.2), and consequently $v(t,x) = S(t,x)v(x)$, since $v(0,x) = v(x)$. Combining this we get the equality

$$v(\phi_t(x)) = S(t,x)\,v(x), \quad t \ge 0. \tag{3.5}$$

Suppose that both integrals

$$\int_0^\infty \alpha(t,x)dt, \quad \int_0^\infty \beta(t,x)dt, \tag{3.6}$$

are convergent, then

(i) both $\|S(t,x)\|$ and $\|S^{-1}(t,x)\|$ are bounded as $t \to \infty$,

(ii) the norm $\|v(t,x)\|$ of every non-zero solution vector of (3.2) tends to a finite and non-vanishing limit as $t \to \infty$ and $x$ remains fixed.

If moreover

$$\int_0^\infty \|B(\phi_t(x)\|\,dt < \infty, \tag{3.7}$$

then there exists

$$S(x) := \lim_{t\to\infty} S(t,x), \tag{3.8}$$

which is a non-singular matrix.

The facts above follow from (3.3) and (3,3)* by arguments as in [2, sec. 9, 10]. Recall that a flow $\phi_t$ is said to be *quasi-asymptotically stable* if there exists $\delta > 0$ such that if $\|x\| < \delta$ then $\phi_t(x) \to 0$ as $t \to \infty$.

THEOREM 3. *Suppose that the flow $\phi_t$ generated by $X$ is quasi-asymptotically stable and either integral*

$$\int_0^\infty \alpha(s,x)ds \quad or \quad \int_0^\infty \|B(\phi_t(x))\|dt$$

*converges for $x$ from a neighborhood of the origin. Then every solution of* (3.1) *is locally trivial.*

*Proof.* Note that if the second integral is convergent then so is the first one. By the assumption there exists $\delta > 0$ such that if $\|x\| < \delta$ then $v(\phi_t(x)) \to 0$ as $t \to \infty$. From (3.3)* we have

$$\|v(\phi_t(x))\| \geq \|v(x)\| \exp \int_0^t \alpha(s,x)ds. \tag{3.9}$$

Letting $t \to \infty$ we get $v(x) = 0$ for $x$ sufficiently small.

THEOREM 4. *Suppose that $\phi_t(x)$ is bounded as $t \to \infty$ and $x$ is small (e.g., if $X$ is finitely supported). If for such $x$*

$$\sup_{t>0} \int_0^t \alpha(s,x)ds = +\infty$$

*then $v(x) = 0$ for every solution of* (3.1).

*Proof.* It follows directly from (3.9).

COROLLARY. *Suppose that $B(x) = B$ is a constant matrix. If either hypothesis below holds*

(a) *$\phi_t$ is quasi-asymptotically stable and* $\mathrm{Re}\,\lambda \geq 0$ *for all $\lambda$ from the spectrum of $B$, or*

(b) *$\phi_t$ is bounded and* $\mathrm{Re}\,\lambda > 0$ *for* $\lambda \in \mathrm{Spect}\,B$,
*then every solution of equation* (3.1) *vanishes in a neighborhood of the origin.*

*In particular, the conclusion is true if $B = 0$ and $\phi_t$ is quasi-asymptotically stable.*

## 4. Solvability of the operator $D_X$

We return to the non-homogenous equation (1.1) in the case when $B = 0$,

$$D_X u = f. \tag{4.1}$$

With $X(0) = 0$ the origin 0 is a *hyperbolic* point of $X$ if the Jacobian matrix $DX(0)$ has no eigenvalue with real part equal to zero. In this case there exists a decomposition $\mathbb{R}^n = E_+ \oplus E_-$ in which $DX(0)$ has a quasi-diagonal form with blocks $C$, $D$, and thare are positive constants $c$, $\delta$, $K$ such that

$$\|e^{tC}\| \leq e^{-ct}, \quad \|e^{-tD}\| \leq e^{-ct}, \quad \|e^{tDX(0)}x\| \geq \delta\|x\|, \tag{4.2}$$

$$\|\phi_t(x)\| \leq K\|x\|e^{-ct} \quad \text{if } x \in E_+ \cap B_\delta\,, \tag{4.3}$$

and

$$\|\phi_{-t}(x)\| \leq K\|x\|e^{-ct} \quad \text{if } x \in E_- \cap B_\delta\,, \tag{4.4}$$

where $B_\delta$ denotes the ball $\{\|x\| < \delta\}$.

The subspace $E_+$ is called *contracting* and the subspace $E_-$ *expanding.* If both are non-trivial the critical point is of *saddle type.* In restriction to $E_+$ the eigenvalues of $DX(0)$ have negative real parts (notation: $\operatorname{Re}\lambda < 0$) and the restriction to $E_-$ satisfies $\operatorname{Re}\lambda > 0$.

We know that hyperbolic critical points of any vector field are necessarily isolated critical points (cf. Smale [1])

A critical point will be said a *contracting critical point* if the whole space $\mathbb{R}^n$ is contracting. We introduce a new notion by

DEFINITION. A hyperbolic critical point is said to be *strongly hyperbolic* if it is of saddle type and the contracting subspace $E_+$ is invariant under linear maps $DX(x)$ for $x$ from a neighborhood of the point.

Note that $E_+$ is invariant under $DX(0)$ by definition.

Clearly, every hyperbolic point of a linear vector field on $\mathbb{R}^n$ is strongly hyperbolic. Denote by $p$ any projection in $\mathbb{R}^n$ the kernel of which is exactly $E_+$; e.g., the projection on $E_-$.

LEMMA 2. *A critical hyperbolic point is strongly hyperbolic if and only if it satisfies locally one of the following equivalent properties*

(1) $DX\, E_+ \subset E_+\,,$

(2) $p \circ DX = p \circ DX \circ p,$

(3) $pX = p(X \circ p),$

(4) $p\phi_t = p(\phi_t \circ p),$

(5) $p \circ D\phi_t = p \circ D\phi_t \circ p,$

*where $DX$ and $D\phi_t$ are taken at $x$.*

*Proof.* The proof goes (1) $\iff$ (2) $\Rightarrow \ldots$ (5) $\Rightarrow$ (2). Trivially (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2); the latter by differentiation at $t = 0$.

In order to show (1) $\Rightarrow$ (2) we decompose $v = w + z$ where $w \in E_+$ and $z = p(v)$. Then

$$p(DXv) = p(DXw) + p(DXp(v)) = p(DXp(v))$$

because $DXw \in E_+$ by (1).

For (2) $\Rightarrow$ (1) let $v \in E_+$; then $pv = 0$ and we have $0 = pDX(pv) = pDXv$ which means that $DXv$ is in $E_+$.

Now we show (3) $\Rightarrow$ (4). By (3) we have

$$\frac{d}{dt}p\phi_t = pX \circ \phi_t = pX \circ p\phi_t$$

and

$$\frac{d}{dt}p\phi_t \circ p = pX \circ p\phi_t \circ p.$$

We see that both sides of (4) satisfy the same differential equation

$$z' = pX(z)$$

with same initial value, since $\phi_0 = \mathrm{id}$ and $p^2 = p$. Therefore they coincide.

PROPOSITION 1. *Let the origin be a contracting critical point of a $C^\infty$ vector field $X$ on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^s$ be $C^\infty$ and satisfying $f(0) = 0$.*

*Then equation* (4.1) *has a $C^\infty$ solution $u(x)$, uniquely defined in a neighborhood of the origin.*

*Proof.* Without loss of generality we can assume that $X$ has a global Lipschitz constant, hence its flow $\phi_t$ is defined for all $t \in \mathbb{R}$. To make use of the integral formula (2.6) we should set $R(t,x) = I$ since here $B = 0$. Therefore the only problem is to show that the map

$$u(x) = \int_0^\infty f(\phi_t(x))\,dt \tag{4.5}$$

is well defined and $C^\infty$ near the origin. The local uniqueness then follows from Corollary (a). For this it should be noted that if a point is contracting then it is obviously quasi-asympotically stable.

Now, we have to show that all the integrals

$$\int_0^\infty D^k f(\phi_t(x))\,dt$$

are uniformly convergent in a neighborhood of the origin. We have

$$D^k(f \circ \phi_t) = \sum_{s=1}^{k} (D^s f) \circ \phi_t \sum_{\substack{j_1 + \ldots + j_s = k \\ j_i > 0}} D^{j_1}\phi_t \ldots D^{j_s}\phi_t\,. \tag{4.6}$$

The estimate (4.3) now reads $\|\phi_t(x)\| \le K\|x\|e^{-ct}$ for $x$ near the origin. Therefore, for $t$ sufficiently large and $x$ small

$$\|f \circ \phi_t(x))\| \leq C\|\phi_t(x))\| \leq CK\|x\|e^{-ct}.$$

In [3] it was shown that if $\phi_t$ has an exponential bound of order $e^{-ct}$, then so do all the derivatives $D^k\phi_t$, $k \geq 1$. Therefore by (4.6) we easily get

$$\|D^k(f \circ \phi_t)(x)\| \leq \text{const} \cdot e^{-ct}, \tag{4.7}$$

where we assumed $(D^s f) \circ \phi_t$ as bounded by a constant in a ball $\|x\| \leq \delta$. This completes the proof.

Actually, we proved the following

FACT. *In a neighborhood of a contracting critical point of a vector field $X$ on $\mathbb{R}^n$, the linear differential operator $D_X$, acting on the space of smooth maps $f : \mathbb{R}^n \to \mathbb{R}^s$ vanishing at the point, is surjective and injective.*

The above properties are strictly connected with the fact that there are no non-trivial germs of first integrals of a vector field at a contracting critical point. We have namely

LEMMA 3. *If a vector field $X$ has bounded right branches of its flow $\phi_t$, e.g., if $X$ is finitely supported, and there exists a non-zero germ of first integral, say $f$, of $X$ at a point of $\mathbb{R}^n$ then the equation* (4.1) *has no $C^1$ local solution.*

*Proof.* Substituting $\phi_t(x)$ for $x$ in $(D_X u)(x) = f(x)$, we obtain using (3.4)

$$\frac{d}{dt}u(\phi_t(x)) = f(\phi_t(x)) = f(x).$$

Hence

$$u(\phi_t(x)) - u(x) = tf(x), \quad t \geq 0.$$

Letting $t \to +\infty$ we get $f(x) = 0$ because the left hand side is bounded, which proves the lemma.

Since non-trivial first integrals may exist around a hyperbolic critical point of saddle type, we can not expect an assertion like Proposition 1 in this case. However, under stronger hypotheses we can obtain the following

PROPOSITION 2. *Let $X$ be a $C^\infty$ vector field on $\mathbb{R}^n$ for which the origin* 0 *is a strongly hyperbolic point. Suppose that $f \in C^\infty(n, s)$ vanishes to infinite order on the contracting subspace $E_+$. Then the differential equation* (4.1) *has a $C^\infty$ local solution near the origin.*

*Proof.* Since $f$ is infinitely flat on the contracting subspace, for all non-negative integers $k$, $m$ there is a $\delta > 0$ such that if $\|x\| < \delta$ then

$$\|D^k f(x)\| \leq M_{k,m}\|x - E_+\|^m, \tag{4.8}$$

where $M_{k,m}$ are positive constants and $x - E_+$ stands for the projection $p(x)$ on the expanding subspace $E_-$. It should be noted that $\delta$ can be chosen independently of $k$ and $m$. It follows from the fact that each derivative $D^k f$ is also zero to infinite order on $E_+$, and hence the map $x \to \|x - E_+\|^{-m} D^k f(x)$ is $C^\infty$ and equally flat . Therefore it is bounded in the ball $B_\delta$ and $M_{k,m}$ is a bound for it.

Instead of (4.1) we consider the equivalent equation

$$D_{-X} u = -f$$

because $-X$ is contracting on $E_-$ with the estimate (4.4) which now can be written, using property (4) of Lemma 2,

$$\|p\phi_{-t}(x)\| = \|p\phi_{-t}(p(x))\| \leq K\|p(x)\|e^{-ct}. \tag{4.9}$$

The integral formula (2.6) writes now

$$u(x) = \int_0^\infty f(\phi_{-t}(x))\, dt.$$

The uniform convergence of all the integrals

$$\int_0^\infty D^k f(\phi_{-t}(x))\, dt$$

follows directly from the estimates (4.8) and (4.9), as it was shown in the proof of Proposition 1.

## 5. Generalization to polynomials in $D_X$

Let $P(\xi) = \xi^r + a_{r-1}\xi^{r-1} + \ldots + a_1\xi + a_0$ be a polynomial of degree $r$ with real coefficients. The differential operator $P(D_X)$ of order $r$ gives rise to equation

$$(D_X)^r u + \ldots + a_1 D_X u + a_0 u = f. \tag{5.1}$$

THEOREM 5. *Let $k(t)$ be the solution of the ordinary differential equation $P\left(\frac{d}{dt}\right) k = 0$ with initial conditions*

$$k(0) = \ldots = k^{(r-2)}(0) = 0,\ \ k^{(r-1)}(0) = -1$$

*for $r \geq 2$ and $k(0) = -1$ for $r = 1$. If the integrals*

$$\int_0^\infty k^{(j)}(t-s)\, f(\phi_s(x))\, ds,$$

*are uniformly convergent in a ball $B_\delta$, for $j = 0, 1, \ldots, r$, then the map*

$$u(x) = \int_0^\infty k(-s)\, f(\phi_s(x))\, ds$$

*is a local solution of* (5.1).

*Proof.* Denote $u_t = u \circ \phi_t$ and $f_t = f \circ \phi_t$. Then $u_0 = u$, $f_0 = f$, and using (3.4) we get by iteration

$$u_t^{(j)} = (D_X u)^j \circ \phi_t\,.$$

Applying this to $P(D_X)u = f$ we obtain

$$P(D_X)u_t = P\left(\frac{d}{dt}\right) u_t = f_t\,. \tag{5.2}$$

In order to verify that $u(x)$ defined in the theorem satisfies equation (5.1) for $x$ in $B_\delta$ we show that $u_t$ satisfies (5.2) and set $t = 0$. We have

$$u_t = \int_0^\infty k(-s) f \circ \phi_s \circ \phi_t\, ds = \int_0^\infty k(-s) f_{t+s} ds = \int_t^\infty k(t-s) f_s ds.$$

Hence, using initial conditions one has

$$u_t' = -k(0) f_t + \int_t^\infty k'(t-s) f_s ds = \int_t^\infty k'(t-s) f_s ds$$

etc.,

$$u_t^{(r-1)} = \int_t^\infty k^{(r-1)}(t-s) f_s ds,$$

$$u_t^{(r)} = f_t + \int_t^\infty k^{(r)}(t-s) f_s ds.$$

Taking the sum of these terms with due coefficients yields

$$P\left(\frac{d}{dt}\right) u_t = f_t + \int_t^\infty P\left(\frac{d}{dt}\right) k(t-s) f_s ds.$$

Setting $t = 0$ and applying the definition of function $k(t)$ we get

$$P(D_X)u = f + \int_0^\infty \left(P\left(\frac{d}{dt}\right) k\right)(-s) f_s ds = f$$

which proves the theorem.

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