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On the stability of the cosine functional equation

Abstract. In the present paper we study the stability problem of the cosine functional equation for complex- and vector-valued functions.

1. Introduction

Classical d'Alembert's functional equation (the cosine equation) for a complex function f defined on a group $(G, +)$ has the form:

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G. \tag{1}$$

The two most general results concerning equation (1) have been proved by Kannappan [7] and Dacić [4]. The first one, due to Kannappan asserts that for any solution f of the cosine functional equation satisfying the condition:

$$f(x+y+z) = f(x+z+y), \quad x, y, z \in G, \tag{2}$$

there exists a homomorphism m of the group G into the multiplicative group of the complex field $\mathbb{C}$, i.e., m satisfies the exponential functional equation:

$$m(x+y) = m(x)m(y), \quad x, y \in G \tag{3}$$

such that

$$f(x) = \frac{1}{2}(m(x) + m(-x)), \quad x \in G. \tag{4}$$

Dacić replaced hypothesis (2) by the following assumption:

$$\bigvee_{z_0 \in G} \left(f(z_0) = 0 \ \wedge \bigwedge_{x,y \in G} f(x+y+z_0) = f(x+z_0+y) \right). \tag{5}$$

It was shown in [4] that under the new hypothesis the assertion of Kannappan result remains valid.

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The aim of this note is to examine the stability problem of the cosine functional equation. It is known that equation (1) for complex functions defined on Abelian group is stable in the sense of Baker ("superstable").

THEOREM (J. A. Baker, [1])
Let $\varepsilon \geq 0$ *be a given number and let* $(G, +)$ *be an Abelian group. Then any unbounded solution* $f : G \to \mathbb{C}$ *of the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon, \quad x, y \in G$$

satisfies d'Alembert's equation (1).

We present a new, short proof of Baker's result. Next, in the second part of the present paper, we consider the problem of the stability of equation (1) for vector-valued mappings.

2. The scalar case

J. A. Baker in [1] showed that the cosine functional equation for complex functions defined on an Abelian group is superstable. Instead of the commutativity of the group G we assume a condition of the type assumed by Kannappan. Then we have the following generalization of Baker's result:

THEOREM 1
Let $\varepsilon \geq 0$, $\delta \geq 0$ *and let* $(G, +)$ *be a group. Suppose that* $f : G \to \mathbb{C}$ *satisfies*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon, \quad x, y \in G. \tag{6}$$

and

$$|f(x+y+z) - f(x+z+y)| \leq \delta, \quad x, y, z \in G. \tag{7}$$

Then either f *is bounded or* f *satisfies d'Alembert's equation* (1).

Our proof of this theorem is based on the idea presented by A. Bil and J. Tabor in [3].

Proof. If f is unbounded then there exists a sequence $(x_n : n \in \mathbb{N})$ in G such that

$$f(x_n) \neq 0, \quad n \in \mathbb{N} \tag{8}$$

and

$$\lim_{n \to \infty} |f(x_n)| = +\infty. \tag{9}$$

Inequality (6) implies that

$$|f(x+x_n)+f(x-x_n)-2f(x)f(x_n)|\le\varepsilon,\quad x\in G,\ n\in\mathbb{N}.$$

Therefore, by (8), we have

$$\left|\frac{f(x+x_n)+f(x-x_n)}{2f(x_n)}-f(x)\right|\le\frac{\varepsilon}{|2f(x_n)|},\quad x\in G,\ n\in\mathbb{N}.$$

Hence and from (9) we infer that

$$\lim_{n\to\infty}\frac{f(x+x_n)+f(x-x_n)}{2f(x_n)}=f(x),\quad x\in G. \tag{10}$$

Now, applying again (6), we get

$$\begin{aligned}&|f(x+(y+x_n))+f(x-(y+x_n))-2f(x)f(y+x_n)\\&+f(x+(y-x_n))+f(x-(y-x_n))-2f(x)f(y-x_n)|\le 2\varepsilon,\\&\qquad\qquad x,y\in G,\ n\in\mathbb{N},\end{aligned}$$

whence,

$$\begin{aligned}&|f(x+y+x_n)+f(x-x_n-y)-2f(x)f(y+x_n)\\&+f(x+y-x_n)+f(x+x_n-y)-2f(x)f(y-x_n)|\le 2\varepsilon,\\&\qquad\qquad x,y\in G,\ n\in\mathbb{N}.\end{aligned}$$

Further, using (7), we obtain

$$\begin{aligned}&|f(x+y+x_n)+f(x+y-x_n)+f(x-y+x_n)+f(x-y-x_n)\\&\qquad -2f(x)(f(y+x_n)+f(y-x_n))|\le 2\varepsilon+2\delta,\\&\qquad\qquad x,y\in G,\ n\in\mathbb{N}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\frac{f(x+y+x_n)+f(x+y-x_n)}{2f(x_n)}+\frac{f(x-y+x_n)+f(x-y-x_n)}{2f(x_n)}\right.\\&\qquad\left.-2f(x)\frac{f(y+x_n)+f(y-x_n)}{2f(x_n)}\right|\le\frac{2\varepsilon+2\delta}{2|f(x_n)|},\\&\qquad\qquad x,y\in G,\ n\in\mathbb{N}.\end{aligned}$$

This inequality combined with (9) and (10) implies that f satisfies equation (1) and ends the proof.

3. The vector case

In the vector-valued case the problem of the stability of equation (3) was considered by J. Lawrence in [8] and by R. Ger and P. Šemrl in [6]. Moreover, R. Ger in [5] considered the stability problem of a system of trygonometric functional equations.

At the beginning of the present section we observe that the superstability of the cosine functional equation fails to hold in the case of vector-valued mappings. Our counter-example reads as follows (see also J. A. Baker [1]): take a function f defined on a group G with values in the algebra $M_2(\mathbb{C})$ of all complex (2×2)-matrices given by the formula

$$f(x) = \begin{bmatrix} f_0(x) & 0 \\ 0 & c \end{bmatrix}, \quad x \in G, \tag{11}$$

where $f_0 : G \to \mathbb{C}$ is an unbounded function fulfilling equation (1) and $c \neq 1$ is a positive constant. Then

$$\|f(x+y) + f(x-y) - 2f(x)f(y)\| = \text{const} > 0 \quad x, y \in G.$$

Therefore, this difference is bounded but f is neither bounded nor satisfying equation (1).

For vector-valued mappings we have the following stability result:

THEOREM 2

Let $(G, +)$ be an Abelian group and let $\mathcal{A}$ be a complex normed algebra. Assume that the function $f : G \to \mathcal{A}$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x)f(y)\| \leq \varepsilon, \quad x, y \in G \tag{12}$$

and

$$\|f(x) - f(-x)\| \leq \eta, \quad x \in G, \tag{13}$$

for some $\varepsilon, \eta \geq 0$. If

$$\begin{cases} \text{there exists a } z_0 \in G \text{ such that the map} \\ \qquad G \ni x \mapsto \|f(x)f(z_0)\| \in \mathbb{R} \\ \text{is bounded} \end{cases} \tag{14}$$

then there exist a function $m : G \to \mathcal{A}$ and constants $c_1, c_2 \in \mathbb{R}$ such that

$$\|m(x+y) - m(x)m(y)\| \leq c_1, \quad x, y \in G$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(-x)) \right\| \leq c_2, \quad x \in G.$$

Proof. Let us observe that

$$\|f(x)f(-z_0)\| \le \delta + \varepsilon, \quad x \in G, \tag{15}$$

where $\delta := \sup_{x\in G} \|f(x)f(z_0)\|$. In fact, by (12) we have

$$\begin{aligned}\|f(x)f(-z_0)\| &\le \|f(x)f(z_0)\| + \|f(x)f(-z_0) - f(x)f(z_0)\| \\ &\le \delta + \Big\|\frac{1}{2}(f(x+z_0) + f(x-z_0) - 2f(x)f(z_0) \\ &\quad -\frac{1}{2}(f(x-z_0) + f(x+z_0) - 2f(x)f(-z_0)\Big\| \\ &\le \delta + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \delta + \varepsilon, \quad x \in G.\end{aligned}$$

Now, we define a function $h : G \to \mathcal{A}$ as follows:

$$h(x) := \frac{1}{2}(f(x) + f(-x)), \quad x \in G.$$

Then

$$h(-x) = h(x), \quad x \in G \tag{16}$$

and by (13), we have

$$\|h(x) - f(x)\| \le \frac{1}{2}\eta, \quad x \in G. \tag{17}$$

Moreover,

$$\|h(x+y) + h(x-y) - 2h(x)h(y)\| \le \varepsilon + \frac{1}{2}\eta^2 =: \varepsilon_1, \quad x, y \in G. \tag{18}$$

Indeed, from (12) and (13), we deduce that

$$\begin{aligned}&\|h(x+y) + h(x-y) - 2h(x)h(y)\| \\ &\quad = \frac{1}{2}\|f(x+y) + f(-y-x) + f(x-y) + f(y-x) \\ &\qquad -f(x)f(y) - f(x)f(-y) - f(-x)f(y) - f(-x)f(-y)\| \\ &\quad \le \frac{1}{2}\|f(x+y) + f(x-y) - 2f(x)f(y)\| \\ &\qquad +\frac{1}{2}\|f(-x-y) + f(-x+y) - 2f(-x)f(-y)\| \\ &\qquad +\frac{1}{2}\|f(x)f(y) - f(x)f(-y) + f(-x)f(-y) - f(-x)f(y)\| \\ &\quad \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon + \frac{1}{2}\|(f(x) - f(-x))(f(y) - f(-y))\| \\ &\quad \le \varepsilon + \frac{1}{2}\|f(x) - f(-x)\|\|f(y) - f(-y)\| \\ &\quad \le \varepsilon + \frac{1}{2}\eta^2, \quad x, y \in G.\end{aligned}$$

We define the map $m : G \to \mathcal{A}$ by the following formula:

$$m(x) := h(x) + ih(x + z_0), \quad x \in G.$$

Then, conditions (18), (14) and (15) imply that

$$\begin{aligned}
&\|2m(x+y) - 2m(x)m(y)\| \\
&\quad = \|2h(x+y) + 2ih(x+y+z_0) \\
&\qquad -h(x+y) - h(x-y) + (h(x+y) + h(x-y) - 2h(x)h(y)) \\
&\qquad -ih(x+y+z_0) - ih(x-y+z_0) \\
&\qquad +i(h(x+y+z_0) + h(x-y+z_0) - 2h(x+z_0)h(y)) \\
&\qquad -ih(x+y+z_0) - ih(x-y-z_0) \\
&\qquad +i(h(x+y+z_0) + h(x-y-z_0) - 2h(x)h(y+z_0)) \\
&\qquad +h(x+y+2z_0) + h(x-y) \\
&\qquad -(h(x+y+2z_0) + h(x-y) - 2h(x+z_0)h(y+z_0))\| \\
&\quad \leq 4\varepsilon_1 + \|h(x+y+2z_0) + h(x+y) - 2h(x+y+z_0)h(z_0) \\
&\qquad -i(h(x-y+z_0) + h(x-y-z_0) + 2h(x-y)h(z_0)) \\
&\qquad +2h(x+y+z_0)h(z_0) + 2ih(x-y)h(z_0)\| \\
&\quad \leq 4\varepsilon_1 + 2\varepsilon_1 + 2\|h(x+y+z_0)h(z_0)\| + 2\|h(x-y)h(z_0)\| \\
&\quad = 6\varepsilon_1 + \frac{1}{2}\|(f(x+y+z_0) + f(-x-y-z_0))(f(z_0) + f(-z_0))\| \\
&\qquad +\frac{1}{2}\|(f(x-y) + f(-x+y))(f(z_0) + f(-z_0))\| \\
&\quad \leq 6\varepsilon_1 + 4 \cdot \frac{1}{2}\delta + 4 \cdot \frac{1}{2}(\delta + \varepsilon) \\
&\quad = 6\varepsilon_1 + 4\delta + 2\varepsilon, \quad x, y \in G,
\end{aligned}$$

whence,

$$\|m(x+y) - m(x)m(y)\| \leq 3\varepsilon_1 + 2\delta + \varepsilon =: c_1, \quad x, y \in G.$$

Moreover, using (16), (17), (18), (14) and (15) we obtain

$$\begin{aligned}
&\|(m(x) + m(-x)) - 2f(x)\| \\
&\quad \leq \|m(x) + m(-x) - 2h(x)\| + \|2h(x) - 2f(x)\| \\
&\quad = \|h(x) + ih(x+z_0) + h(-x) + ih(-x+z_0) - 2h(x)\| \\
&\qquad +2\|h(x) - f(x)\| \\
&\quad = \|ih(x+z_0) + ih(-x+z_0)\| + 2\|h(x) - f(x)\|
\end{aligned}$$

$$\begin{aligned}
&\le \|h(x+z_0)+h(-x+z_0)-2h(x)h(z_0)\|+\|2h(x)h(z_0)\|+2\cdot\frac{1}{2}\eta \\
&= \|h(x+z_0)+h(x-z_0)-2h(x)h(z_0)\|+\|2h(x)h(z_0)\|+\eta \\
&\le \varepsilon_1+\frac{1}{2}\|(f(x)+f(-x))(f(z_0)+f(-z_0))\|+\eta \\
&\le \varepsilon_1+2\cdot\frac{1}{2}\delta+2\cdot\frac{1}{2}(\delta+\varepsilon)+\eta, \quad x\in G,
\end{aligned}$$

which yields

$$\left\|f(x)-\frac{1}{2}(m(x)+m(-x))\right\| \le \frac{1}{2}\varepsilon_1+\delta+\frac{1}{2}\varepsilon+\frac{1}{2}\eta =: c_2, \quad x\in G$$

and completes the proof of the Theorem.

For functions fulfilling conditions (13) and (14) Theorem 2 reduces the problem of the stability of d'Alembert's functional equation to the problem of the stability of Cauchy's multiplicative functional equation (3) which was considered by J. Lawrence in [8] and by R. Ger and P. Šemrl in [6] (see also R. Ger [5]).

Remark 1

Since

$$\|f(x)-f(-x)\| \le \left\|f(x)-\frac{1}{2}(m(x)+m(-x))\right\| + \left\|f(-x)-\frac{1}{2}(m(-x)+m(x))\right\|, \quad x\in G,$$

for every $m: G\to\mathcal{A}$, we infer that if the map

$$G\ni x\longmapsto \|f(x)-f(-x)\|\in\mathbb{R}$$

is unbounded, then the stability theorem of our type fails to hold, which means that assumption (13) is a necessary condition in our study.

Remark 2

If $\mathcal{A}$ is a complex normed algebra with the identity e and if $f(x_0)^{-1}$ exists for some $x_0\in G$, then hypothesis (13) is fulfiled with $\eta=\|f(x_0)^{-1}\|\frac{\varepsilon}{2}$. Indeed, by (12), we have

$$\|f(x_0+x)+f(x_0-x)-2f(x_0)f(x)\|\le\varepsilon, \quad x\in G$$

and

$$\|f(x_0-x)+f(x_0+x)-2f(x_0)f(-x)\|\le\varepsilon, \quad x\in G.$$

Therefore,

$$\|f(x_0)(f(x)-f(-x))\|\le\varepsilon, \quad x\in G$$

and

$$\begin{aligned}\|f(x)-f(-x)\| &= \|f(x_0)^{-1}f(x_0)(f(x)-f(-x))\| \\ &\le \|f(x_0)^{-1}\|\frac{1}{2}\|2f(x_0)(f(x)-f(-x))\| \\ &\le \|f(x_0)^{-1}\|\frac{\varepsilon}{2}, \quad x\in G.\end{aligned}$$

REMARK 3

For a complex normed algebra $\mathcal{A}$ with the identity e condition (13) can be replaced by the following assumption:

$$\|(f(0)-e)f(x)\| \le \xi, \quad x\in G. \tag{19}$$

In fact, in the proof of Theorem 2 first we observe that if $z_0 = 0$ then (12) with $y = 0$ yields

$$\|2f(x)-2f(x)f(0)\| \le \varepsilon, \quad x\in G$$

and from (14) we deduce that f is bounded. This concludes the proof in the case $z_0 = 0$ beacause we take then the function $m : G \to \mathcal{A}$ defined as follows:

$$m(x) = 0, \quad x\in G.$$

Let $z_0 \ne 0$. We define a function $g : G \to \mathcal{A}$ as follows

$$g(x) := \begin{cases} f(x), & x \ne 0 \\ e, & x = 0. \end{cases}$$

Then the function $g : G \to \mathcal{A}$ satisfies:

$$\|g(x)-f(x)\| \le \|e-f(0)\| =: \gamma, \quad x\in G. \tag{20}$$

We are going to show that g satisfies all the assumptions of Theorem 2. Obviously

$$\|g(x)g(z_0)\| \le \max\{\sup_{x\in G}\|f(x)f(z_0)\|,\ \|f(z_0)\|\} =: \zeta, \quad x\in G. \tag{21}$$

From (12) and our assumption (19), we get

$$\|g(x+y)+g(x-y)-2g(x)g(y)\| \le \varepsilon + 2\xi + 2\gamma =: \tilde{\varepsilon}, \quad x,y\in G. \tag{22}$$

To prove this, suppose that $x\ne 0$, $y \ne 0$, $x+y\ne 0$ and $x-y\ne 0$. Then (22) is trivially fulfilled. Next assume that $x = 0$. Then

$$\begin{aligned}&\|g(0+y)+g(0-y)-2g(0)g(y)\| \\ &\quad\le \|f(0+y)+f(0-y)-2f(0)f(y)\| + \|2f(0)f(y)-2ef(y)\| \\ &\quad\le \varepsilon + 2\xi, \quad y\in G\setminus\{0\}.\end{aligned}$$

If $y = 0$, then $\|g(x+0) + g(x-0) - 2g(x)g(0)\| = 0$, $x \in G$. Finally, assume that $x + y = 0$ or $x - y = 0$ and $x \neq 0 \neq y$. Then

$$\begin{aligned}&\|g(x+y) + g(x-y) - 2g(x)g(y)\| \\ &\qquad \leq \|f(x+y) + f(x-y) - 2f(x)f(y)\| + 2\|g(0) - f(0)\| \\ &\qquad \leq \varepsilon + 2\gamma.\end{aligned}$$

Inserting $x = 0$ to (22), we get

$$\|g(y) + g(-y) - 2g(y)\| \leq \tilde{\varepsilon}, \quad y \in G,$$

whence

$$\|g(y) - g(-y)\| \leq \tilde{\varepsilon}, \quad y \in G. \tag{23}$$

From Theorem 2 we deduce that there exist a function $m : G \to \mathcal{A}$ and constants $c_1, c_2 \in \mathbb{R}$ such that

$$\|m(x+y) - m(x)m(y)\| \leq c_1, \quad x, y \in G$$

and

$$\left\|g(x) - \frac{1}{2}(m(x) + m(-x))\right\| \leq c_2, \quad x \in G.$$

Now, from (20), we get

$$\left\|f(x) - \frac{1}{2}(m(x) + m(-x))\right\| \leq \gamma + c_2, \quad x \in G.$$

Moreover, let us observe that every solution of inequality (12) satisfies the following condition:

$$\|f(x)(f(0) - e)\| \leq \frac{1}{2}\varepsilon, \quad x \in G$$

(taking $y = 0$ in (12), we have $\|2f(x) - 2f(x)f(0)\| \leq \varepsilon$, $x \in G$).

So, our last remark yields the following

Remark 4

Assumption (13) can be omitted for a complex commutative normed algebra with the identity.

Remark 5

Condition (14) is fulfilled by the function f defined by (11) when the function f_0 satisfies the cosine functional equation and Dacić's condition (5).

Finally let us observe, that our stability theorem leads to the following version of the Dacić result for vector valued mappings:

THEOREM 3

Let $(G,+)$ be an Abelian group and let $\mathcal{A}$ be a complex normed algebra. Then every even solution $f : G \to \mathcal{A}$ of the d'Alembert functional equation (1) *has the form:*

$$f(x) = \frac{1}{2}(m(x) + m(-x)), \quad x \in G,$$

where $m : G \to \mathcal{A}$ satisfies Cauchy's equation (3).

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