ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE
Zeszyt 196 Prace Matematyczne XV 1998

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The characterization of the indicator plurality function

Abstract. The characterization of the rational and real indicator plurality function in the sense of M. F. S. Roberts is obtained.
The answer to M. F. S. Roberts' question is presented.

Introduction

To state the characterization of the indicator plurality function we introduce the following notations:

$\underline{0}=(0,0,\dots,0) \in \mathbb{R}^p$,

$\mathbb{R}(p)$ — the set of all p-vectors of non-negative real numbers except $\underline{0}$,

$\mathbb{Q}(p)$ — the subset of $\mathbb{R}(p)$ of all p-vectors of non-negative rational numbers,

$0(p)$ — the subset of $\mathbb{R}(p)$ of all p-vectors in which each component is 0 or 1,

$\mathbb{R}_+$ — the set of positive real numbers,

$\mathbb{Q}_+$ — the set of positive rational numbers.

The definition and a great number of properties of the indicator plurality function were given by M. F. S. Roberts in [2].

Definition 1

Let $U \subset \mathbb{R}(p)$. A function $f : U \to 0(p)$ is called the *plurality function* on U iff

$$\forall\, a=(a_1,a_2,\dots,a_p) \in U \ \forall\, j \in \{1,2,\dots,p\} \quad [f_j(a_1,\dots,a_p)=1 \Leftrightarrow \forall\, i \in \{1,2,\dots,p\}\ a_i \le a_j].$$

In this paper we give the characterization of the rational ($U=\mathbb{Q}(p)$) and real ($U=\mathbb{R}(p)$) indicator plurality function.

We use the following denotations:

AMS (1991) subject classification: 39B40.

$e_{\leq k} = (1, 1, \ldots, \overset{(k)}{1}, 0, \ldots, 0);$

$e_{i_1, i_2, \ldots, i_k}$ — the vector from $0(p)$ with 1 in the $i_1, i_2, \ldots, i_k$ positions, where $k \in \{1, 2, \ldots, p\}$, and 0 in the others;

$a = (a_1, a_2, \ldots, a_k), b = (b_1, b_2, \ldots, b_k) \in U.$

For $a, b \in \mathbb{R}(p)$, $t \in \mathbb{R}_+$ we set,

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_p + b_p),$$
$$ab = (a_1 b_1, a_2 b_2, \ldots, a_p b_p),$$
$$ta = (ta_1, ta_2, \ldots, ta_p).$$

The characterization of the rational indicator plurality function

Let $f : U \to \mathbb{R}(p)$, where $U = \mathbb{Q}(p)$ or $U = \mathbb{R}(p)$. We introduce the following definitions:

DEFINITION 2
A function f is called *neutral* iff

$$\forall a \in U \ \forall \pi - \text{ permutation of } \{1, 2, \ldots, p\} \ \ f_i(a_{\pi(1)}, \ldots, a_{\pi(p)}) = f_{\pi(i)}(a).$$

DEFINITION 3
A function f is called *weakly neutral* iff

$$\forall a \in U \ \forall i, m \in \{1, 2, ..., p\} \ \ [a_i = a_m \Rightarrow f_i(a) = f_m(a)].$$

DEFINITION 4
A function f is called *consistent* iff

$$\forall j \in \{1, 2, \ldots, p\} \ \forall a, b \in U \ \ [f(a)f(b) \neq \underline{0} \Rightarrow f_j(a + b) = f_j(a) f_j(b)].$$

DEFINITION 5
A function f is called *weakly consistent* iff

$$\exists j \in \{1, 2, \ldots, p\} \ \forall a, b \in U \ \ [f(a)f(b) \neq \underline{0} \Rightarrow f_j(a + b) = f_j(a) f_j(b)].$$

DEFINITION 6
A function f is called *faithful* iff

$$\forall j \in \{1, 2, \ldots, p\} \ \ f(e_j) = e_j.$$

DEFINITION 7

A function f is called *weakly faithful* iff

$$\exists j \in \{1,2,\dots,p\} \quad f(e_j) = e_j.$$

DEFINITION 8

A function f is called 2-*homogeneous* iff

$$\forall j \in \{1,2,\dots,p\} \; \forall a \in U \quad f_j(2a) = f_j(a).$$

DEFINITION 9

A function f is called *weakly* 2-*homogeneous* iff

$$\exists j \in \{1,2,\dots,p\} \; \forall a \in U \quad f_j(2a) = f_j(a).$$

DEFINITION 10

A function f is called *partially faithful* iff

$$\forall k \in \{1,2,\dots,p\} \quad f(e_{\leq k}) = e_{\leq k}\,.$$

DEFINITION 11

A function f is called *weakly partially faithful* iff

$$f(e_1) = e_1 \quad \text{and} \quad \exists k \in \{2,\dots,p\} \quad f(e_{\leq k}) = e_{\leq k}\,.$$

LEMMA 1

If a function $f : \mathbb{Q}(p) \to \mathbb{Q}(p)$ is consistent then

$$f(qa) = [f(a)]^q \quad \textit{for } a \in \mathbb{Q}(p),\; q \in \mathbb{Q}_+ .$$

Proof. It is easy to prove by induction that $f(ma) = [f(a)]^m$ for all $m \in \mathbb{N}$ and all $a \in \mathbb{Q}(p)$. Hence, for all $b \in \mathbb{Q}(p)$ and all $n \in \mathbb{N}$, we get $f(b) = f(n\frac{1}{n}b) = [f(\frac{1}{n}b)]^n$ and thus $f(\frac{1}{n}b) = [f(b)]^{\frac{1}{n}}$.
Take a $q \in \mathbb{Q}_+$ and an $a \in \mathbb{Q}(p)$. For $q = \frac{m}{n}$ we have

$$\begin{aligned} f(qa) &= f\left(\frac{m}{n}a\right) = f\left(m\frac{1}{n}a\right) = \left[f\left(\frac{1}{n}a\right)\right]^m = \left([f(a)]^{\frac{1}{n}}\right)^m \\ &= [f(a)]^{\frac{m}{n}} = [f(a)]^q. \end{aligned}$$

The following result was proved by Roberts in [2].

THEOREM A

If $f : \mathbb{Q}(p) \to 0(p)$ then the following conditions are equivalent:

(1) *f is the indicator plurality function on $\mathbb{Q}(p)$;*

(2) *f is neutral, consistent and faithful.*

The following theorem is also true.

THEOREM 1

If $f : \mathbb{Q}(p) \to 0(p)$ then the following conditions are equivalent:

(1) *f is the indicator plurality function on $\mathbb{Q}(p)$;*

(3) *f is neutral, weakly consistent and weakly faithful;*

(4) *f is weakly neutral, consistent and weakly faihtful.*

Proof. The proof of the fact that (1) implies (3) is analogous to the one that (1) implies (2) in Theorem A.

We shall show that (3) implies (4). It is sufficient to prove that f is consistent. Take arbitrary $a, b \in \mathbb{Q}(p)$ such that $f(a)f(b) \neq \underline{0}$. There exists an $l \in \{1, 2, \ldots, p\}$ such that $f_l(a)f_l(b) \neq 0$. By weak consistency of f there exists an $i \in \{1, 2, \ldots, p\}$ such that $f_i(a+b) = f_i(a)f_i(b)$. Take a $j \in \{1, \ldots, p\}$. We shall show that $f_j(a+b) = f_j(a)f_j(b)$. Let π be a permutation of $\{1, 2, \ldots, p\}$ such that $\pi(i) = j$, $\pi(j) = i$ and $\pi(k) = k$ for $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$. By the definition of the permutation π and the neutrality of f we have the following equalities:

$$\begin{aligned}
f_j(a+b) &= f_{\pi(i)}(a_1 + b_1, a_2 + b_2, \ldots, a_p + b_p) \\
&= f_i(a_{\pi(1)} + b_{\pi(1)}, a_{\pi(2)} + b_{\pi(2)}, \ldots, a_{\pi(p)} + b_{\pi(p)}), \\
f_j(a) &= f_{\pi(i)}(a_1, a_2, \ldots, a_p) = f_i(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(p)}), \\
f_j(b) &= f_{\pi(i)}(b_1, b_2, \ldots, b_p) = f_i(b_{\pi(1)}, b_{\pi(2)}, \ldots, b_{\pi(p)}), \\
f_l(a) &= f_{\pi(l)}(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(p)}), \\
f_l(b) &= f_{\pi(l)}(b_{\pi(1)}, b_{\pi(2)}, \ldots, b_{\pi(p)}).
\end{aligned}$$

Since $f_{\pi(l)}(a_{\pi(1)}, \ldots, a_{\pi(p)})f_{\pi(l)}(b_{\pi(1)}, \ldots, b_{\pi(p)}) \neq 0$ and f is weakly consistent, we obtain

$$\begin{aligned}
&f_i(a_{\pi(1)} + b_{\pi(1)}, a_{\pi(2)}, +b_{\pi(2)}, \ldots, a_{\pi(p)} + b_{\pi(p)}) \\
&\quad = f_i(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(p)})f_i(b_{\pi(1)}, b_{\pi(2)}, \ldots, b_{\pi(p)}).
\end{aligned}$$

From the above equalities we have $f_j(a+b) = f_j(a)f_j(b)$.

We shall show now that (4) implies (1). First we prove that the range of f is contained in $0(p)$. Suppose that $f_i(a) = q \in \mathbb{Q}_+ \setminus \{1\}$ for $a \in \mathbb{Q}(p)$. There exist relatively prime positive integers n and m such that at least one of them is greater than 1 and $q = \frac{n}{m}$. Let $r(s)$ denote the number of prime numbers appearing in the decomposition of the number n (number m) into prime factors. Each prime number in this decomposition is counted as many times as it appears. Since $n > 1$ or $m > 1$ we have $r > 0$ or $s > 0$. Put $t = \max\{s, r\}$. By Lemma 1 we have $f_i(\frac{1}{t+1}a) = [f_i(a)]^{\frac{1}{t+1}} = q^{\frac{1}{t+1}}$, hence $f_i(\frac{1}{t+1}a)$ is an irrational number, which contradicts the fact that $f : \mathbb{Q}(p) \to \mathbb{Q}(p)$.

We shall show that for every positive rational number q there is $f(qe_{i_1,\dots,i_k}) = e_{i_1,\dots,i_k}$. Since f is weakly faihtful, there exists a $j \in \{1, 2, \dots, p\}$ such that $f(e_j) = e_j$. Hence and by Lemma 1 we obtain, that $f(qe_j) = [f(e_j)]^q = e_j$. By the weak neutrality and consistency of f and since the range of f is contained in $0(p)$ it follows that for all $i \in \{1, 2, \dots, p\} \setminus \{j\}$ there is $f(qe_i) = e_i$ and $f(qe_{\leq p}) = e_{\leq p}$. Take a $k \in \{2, \dots, p-1\}$ and let $i_1, i_2, \dots, i_k \in \{1, 2, \dots, p\}$ be pairwise different. By the weak neutrality $f(qe_{i_1,\dots,i_k})$ is a vector such that there exists $(c, d) \in 0(2)$ with d in the $i_1, \dots, i_k$ positions and c in the others. Suppose that $c = 1$. Take an $m \in \{1, 2, \dots, p\} \setminus \{i_1, i_2, \dots, i_k\}$. By the consistency of f there is $f(qe_{i_1,\dots,i_k,m}) = f(qe_{i_1,\dots,i_k})f(qe_m) = e_m$, which contradicts the fact that f is weakly neutral. We deduce that $c = 0$ and hence $d = 1$. This shows that $f(qe_{i_1,\dots,i_k}) = e_{i_1,\dots,i_k}$.

Take an $a \in \mathbb{Q}(p)$. We can order the set $\{a_1, \dots, a_p\}$ in the following way:

$$a_{j_1} = \cdots = a_{j_{k_1}} < a_{j_{k_1+1}} = \cdots = a_{j_{k_1+k_2}} < \cdots < a_{j_{k_1+k_2+\cdots+k_m+1}} = \cdots = a_{j_p},$$

where k_1 can be equal to p.
Put $n = k_1 + \cdots + k_m$. Notice that if $k_1 = p$ then $f(a) = e_{\leq p}$.

We shall show that if $k_1 \neq p$ then $f(a) = e_{j_{n+1},\dots,j_p}$. Consider the case when $a_{j_1} \neq 0$. By the consistency of f we obtain

$$\begin{aligned} f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) &= f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p})f(a_{j_1}e_{\leq p}) \\ &= e_{j_{k_1+1},\dots,j_p}. \end{aligned}$$

If $a_{j_1} = 0$, then

$$f(a_{j_{k_1+1}}e_{j_{k_1+1},\dots,j_p}) = f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) = e_{j_{k_1+1},\dots,j_p}.$$

Thus $f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) = e_{j_{k_1+1},\dots,j_p}$.
If $a_{j_{k_1+1}} = a_{j_{n+1}}$, then

$$f(a) = f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) = e_{j_{k_1+1},\dots,j_p}.$$

If $a_{j_{k_1+1}} \neq a_{j_p}$, then by the consistency of f we have

$$\begin{aligned} &f((a_{j_{k_1+k_2+1}} - a_{j_{k_1+k_2}})e_{j_{k_1+k_2+1},\dots,j_p} + (a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) \\ &\quad = f((a_{j_{k_1+k_2+1}} - a_{j_{k_1+k_2}})e_{j_{k_1+k_2+1},\dots,j_p})f((a_{j_{k_1+1}} - a_{j_1})e_{j_{k_1+1},\dots,j_p} + a_{j_1}e_{\leq p}) \\ &\quad = e_{j_{k_1+k_2+1},\dots,j_p}. \end{aligned}$$

Following the analogous way we obtain after at most p steps that $f(a) = e_{n+1,\dots,p}$. Thus f is the indicator plurality function.

REMARK 1

Let us observe that if a function $f : \mathbb{Q}(2) \to 0(2)$ is weakly neutral, weakly consistent and faithful then f is the indicator plurality function.

Namely, consider $f = (f_1, f_2)$. By the weak neutrality of f and by the fact that the range of f is contained in $0(2)$, for every $(a_1, a_1) \in \mathbb{Q}(2)$, we have $f(a_1, a_1) = (1, 1)$. The function f is consistent, thus there exists an $i \in \{1, 2\}$ such that for every $a, b \in \mathbb{Q}(2)$ if $f(a)f(b) \neq \underline{0}$ then $f_i(a_1, a_2)f_i(b_1, b_2) = f_i(a_1 + b_1, a_2 + b_2)$. Put $l \in \{1, 2\}$ and $l \neq i$.

We shall show that for every positive rational number q, $f_i(qe_i) = 1$ and $f_i(qe_l) = 0$. Let n be a positive integer.
Suppose that $f_i(\frac{1}{n}e_i) = 0$. Since the range of function f is contained in $0(2)$ we have $f_l(\frac{1}{n}e_i) = 1$. By the weak consistency of f we get $f_i(\frac{2}{n}e_i) = f_i(\frac{1}{n}e_i)f_i(\frac{1}{n}e_i) = 0$. Thus $f_l(\frac{2}{n}e_i) = 1$ and hence $f_l(\frac{2}{n}e_i)f_l(\frac{1}{n}e_i) = 1$. From the weak consistency of f, $f_i(\frac{3}{n}e_i) = f_i(\frac{2}{n}e_i)f_i(\frac{1}{n}e_i) = 0$. In the analogous way, after $n - 1$ steps, we obtain $f_i(\frac{n}{n}e_i) = f_i(\frac{n-1}{n}e_i)f_i(\frac{1}{n}e_i) = 0$. Thus $f_i(e_i) = 0$, which contradicts the fact that f is faithful. Hence $f_i(\frac{1}{n}e_i) = 1$.
Suppose that $f_i(\frac{1}{n}e_l) = 1$. By the weak consistency of f we have $f_i(\frac{2}{n}e_l) = f_i(\frac{1}{n}e_l)f_i(\frac{1}{n}e_l) = 1$. Thus $f_i(\frac{2}{n}e_l) = 1$ and hence $f_i(\frac{2}{n}e_l)f_i(\frac{1}{n}e_l) = 1$. From the weak consistency of f, $f_i(\frac{3}{n}e_l) = f_i(\frac{2}{n}e_l)f_i(\frac{1}{n}e_l) = 1$. In the analogous way, after $n - 1$ steps, we obtain $f_i(\frac{n}{n}e_l) = f_i(\frac{n-1}{n}e_l)f_i(\frac{1}{n}e_l) = 1$. Thus $f_i(e_l) = 1$, which contradicts the fact that f is faithful. Hence $f_i(\frac{1}{n}e_l) = 0$. On the other hand, the range of f is contained in set $0(2)$. Consequently, for every positive integer n we obtain $f_l(\frac{1}{n}e_l) = 1$.
It is easy to prove by induction with respect to m that for every $n, m \in \mathbb{N}$ and for every $k \in \{1, 2\}$, $f_i(\frac{m}{n}e_k) = [f_i(\frac{1}{n}e_k)]^m$. Now, fix a $k \in \{1, 2\}$ and a $q \in \mathbb{Q}_+$. Since $q \in \mathbb{Q}_+$, there exist $m, n \in \mathbb{N}$ such that $q = \frac{m}{n}$. Notice that $f_i(qe_k) = f_i(\frac{m}{n}e_k) = [f_i(\frac{1}{n}e_k)]^m$. Since $f_i(\frac{1}{n}e_k) = 1$ for $k = i$ and $f_i(\frac{1}{n}e_k) = 0$ for $k \neq i$, we have $f_i(qe_i) = 1^m = 1$ and $f_i(qe_l) = 0^m = 0$.
Take an $a \in \mathbb{Q}(2)$ with a_i in the i-th position and with a_l in the l-th position. Let us consider three cases:

I. $a_i = a_l$. Then $f(a) = (1, 1)$.

II. $a_i < a_l$. Then $a_l - a_i > 0$. Since $f_i((a_l - a_i)e_l) = 0$ and the range of f is contained in $0(2)$, we have $f_l((a_l - a_i)e_l) = 1$. Thus $f_l(a_i, a_i) = 1$ and $f_l((a_l - a_i)e_l) = 1$. Hence and by the weak consistency of f we obtain $f_i(a) = f_i(a_i, a_i)f_i((a_l - a_i)e_l) = 1 \cdot 0 = 0$. The range of the function f is contained in $0(2)$, whence $f_l(a) = 1$ and consequently $f(a) = e_l$.

III. $a_i > a_l$. Then $a_i - a_l > 0$. Since $f_i(a_l, a_l) = 1$ and $f_i((a_i - a_l)e_i) = 1$, and by the weak consistency of f, we get $f_i(a) = f_i(a_l, a_l)f_i((a_i - a_l)e_i) = 1$. Suppose that $f_l(a) = 1$. Since $f_l(a) = 1$ and $f_l((a_i - a_l)e_l) = 1$, and by the weak consistency of f, we obtain that $f_i(a_i, a_i) = f_i(a) f_i((a_i - a_l)e_l) = 1 \cdot 0 = 0$, which contradicts the fact that $f(a_i, a_i) = (1, 1)$, whence $f_l(a) = 0$ and consequently $f(a) = e_i$.

REMARK 2

Let us observe that the weak neutrality, weak consistency and faithfulness of a function $f : \mathbb{Q}(p) \to 0(p)$, where $p > 2$, do not imply that f is the indicator plurality function.
For, put

$$S = \{(a_1, \dots, a_p) \in \mathbb{Q}(p) : a_1 \le a_j \text{ for every } j \in \{1, \dots, p\} \text{ and } a_1 < a_n \text{ and } a_1 < a_m \text{ for some } m, n \in \{1, \dots, p\} \text{ such that } n \ne m\}.$$

We define a function $f : \mathbb{Q}(p) \to 0(p)$, where $p > 2$, as follows: f is the indicator plurality function on $\mathbb{Q}(p) \setminus S$ and if $a \in S$ then $f(a) = e_{i_1, i_2, \dots, i_k}$, where $i_1, \dots, i_k$ mean all the position numbers such that $a_1 < a_{i_j}$ for $j \in \{1, \dots, k\}$.

It follows from Theorem 1 that the function f is weakly neutral on $\mathbb{Q}(p) \setminus S$. Take an arbitrary $a \in S$ such that $a_j = a_m$ for some $j, m \in \{1, \dots, p\}$. Consider two cases:

I. $a_j = a_m = a_1$. Then we get $f_j(a) = f_m(a) = 0$.

II. $a_j = a_m \ne a_1$. Then we have $f_j(a) = f_m(a) = 1$.

Thus f is weakly neutral on S and consequently f is weakly neutral.
The function f is not neutral. Namely,

$$f(1, 3, 2, \dots, 2) = (0, 1, \dots, 1) \quad \text{and} \quad f(3, 1, 2, \dots, 2) = (1, 0, \dots, 0).$$

We shall show that f is weakly consistent. Take $a, b \in \mathbb{Q}(p)$ such that $f(a)f(b) \ne \underline{0}$. Let us consider three cases:

I. $a + b \in S$. Then there exist $i, m \in \{2, 3, \dots, p\}$, $i \ne m$, such that $a_1 + b_1 < a_i + b_i$ and $a_1 + b_1 < a_m + b_m$. Take a $t \in \{i, m\}$. Suppose that $a_1 \ge a_t$ and $b_1 \ge b_t$. It follows that $a_1 + b_1 \ge a_t + b_t$, which contradicts the fact that $a_1 + b_1 < a_t + b_t$. Thus for every $t \in \{i, m\}$: $a_1 < a_t$ or $b_1 < b_t$. Hence we get $f_1(a) = 0$ or $f_1(b) = 0$. Since $a + b \in S$, we obtain $f_1(a + b) = 0$, thus $f_1(a) f_1(b) = f_1(a + b)$.

II. $a + b \in \mathbb{Q}(p) \setminus S$. Consider a few cases:

(a) $a, b \in \mathbb{Q}(p) \setminus S$. Then $f_1(a + b) = f_1(a) f_1(b)$ because f is the indicator plurality function on $\mathbb{Q}(p) \setminus S$.

(b) $a \in S$ and $b \in \mathbb{Q}(p) \setminus S$. There exists a $k \in \{2, \ldots, p\}$ such that $f_k(a)f_k(b) = 1$ ($k \neq 1$ because $f_1(a) = 0$). Hence we obtain $b_j \leq b_k$ for every $j \in \{1, \ldots, p\}$ and $a_1 < a_k$. It follows from the above inequalities that $a_1 + b_1 < a_k + b_k$. Hence we get $f_1(a + b) = 0$. Since $a \in S$ and from the definition of the function f we have $f_1(a) = 0$. Consequently $f_1(a)f_1(b) = f_1(a + b)$.

(c) $a \in \mathbb{Q}(p) \setminus S$ and $b \in S$. The property of the weak consistency is symmetrical with respect to a, b. Thus, the case (c) reduces to the case (b).

(d) $a, b \in S$. Then $a_1 \leq a_j$ and $b_1 \leq b_j$ for every $j \in \{1, 2, \ldots, 3\}$ and there exist $m, n \in \{1, 2, \ldots, p\}$ such that $m \neq n$, $a_1 < a_m$ and $a_1 < a_n$. Hence $a_1 + b_1 \leq a_j + b_j$ for every $j \in \{1, 2, \ldots, p\}$, $a_1 + b_1 < a_m + b_m$ and $a_1 + b_1 < a_n + b_n$, thus $a + b \in S$. Then the case (d) does not hold.

Therefore the function f is weakly consistent.

The function f is not consistent. Indeed: $f(0, 1, 2, \ldots, 2) = (0, 1, \ldots, 1)$ and $f(0, 1, 0, \ldots, 0) = e_2$, thus $f(0, 1, 2, \ldots, 2)f(0, 1, 0, \ldots, 0) = e_2$. On the other hand, $f((0, 1, 2, \ldots, 2) + (0, 1, 0, \ldots, 0)) = f(0, 2, \ldots, 2) = (0, 1, \ldots, 1)$, thus $f(0, 1, 2, \ldots, 2)f(0, 1, 0, \ldots, 0) \neq \underline{0}$ and $f_3((0, 1, 2, \ldots, 2) + (0, 1, 0, \ldots, 0)) \neq f_3(0, 1, 2, \ldots, 2)f_3(0, 1, 0, \ldots, 0)$.

We shall show that f is faithful. By the definition of the set S, for every $k \in \{1, 2, \ldots, p\}$, we have $e_k \notin S$. It follows from Theorem 1 that the indicator plurality function is faithful on $\mathbb{Q}(p)$, whence so it is on $\mathbb{Q}(p) \setminus S$. Consequently, f is faithful.

The function f is not the indicator plurality function because it is not neutral (it is not consistent as well).

REMARK 3

A function $f : \mathbb{Q}(2) \to \mathbb{Q}(2)$, which is weakly neutral, weakly consistent and faithful, need not be the indicator plurality function. Namely, it is sufficient to consider a function $f : \mathbb{Q}(2) \to \mathbb{Q}(2)$ such that f is the indicator plurality function on $\mathbb{Q}(2) \setminus \{2e_2\}$ and $f(2e_2) = 2e_2$.

The following result was proved by Roberts in [2].

THEOREM B

For $f : \mathbb{Q}(p) \to \mathbb{R}(p)$ the following conditions are equivalent:

(1) *f is the indicator plurality function on $\mathbb{Q}(p)$;*

(5) *f is neutral, consistent, faithful and 2-homogeneous.*

The following result is also true:

THEOREM 2

For $f : \mathbb{Q}(p) \to \mathbb{R}(p)$ the following conditions are equivalent:

(1) *f is the indicator plurality function on $\mathbb{Q}(p)$;*

(6) *f is neutral, weakly consistent and weakly partially faithful;*

(7) *f is neutral, consistent and partially faihtful;*

(8) *f is neutral, weakly consistent, weakly faithful and weakly 2-homogenous;*

(9) *f is weakly neutral, consistent, weakly faithful and 2-homogenous.*

Proof. The fact that (1) implies (6) follows from Theorem B.

(6) $\Longrightarrow$ (7). From the proof that (3) implies (4) in Theorem 1, it follows that the function f is consistent (we have shown that a function defined on $\mathbb{Q}(p)$, neutral and weakly consistent has the consistency property). It is sufficient to show that f is partially faithful. Take an $m \in \{2, 3, \dots, p\}$. If $m = p$ then by the neutrality of f we obtain $f(e_{\leq m}) = (c_m, c_m, \dots, c_m)$. Since the range of f is contained in $\mathbb{R}(p)$ we have $c_m \in \mathbb{R}_+$. If $m < p$ then, by the neutrality of f, there exist $c_m, d_m \in \mathbb{R}_+ \cup \{0\}$ such that $f(e_{\leq m}) = (c_m, \dots, c_m, d_m, \dots, d_m)$. Suppose that $d_m \neq 0$. Since $f(e_1) = e_1$ and f is neutral we obtain $f(e_{m+1}) = e_{m+1}$. By the consistency of f we have $f(e_{\leq m+1}) = f(e_{\leq m})f(e_{m+1}) = d_m e_{m+1}$, which contradicts the fact that f is neutral. Therefore $d_m = 0$ and $c_m \in \mathbb{R}_+$. We will prove that $c_m = 1$ for every $m \in \{1, 2, \dots, p\}$. Since $f(e_1) = e_1$ and f is neutral we get

$$f(1, 0, \dots, 0, \overset{(k)}{1}, 0, \dots, 0) = (c_2, 0, \dots, 0, \overset{(k)}{c_2}, 0, \dots, 0)$$

for every $k \in \{1, 2, \dots, p\}$.

By the consistency of f we obtain the following equalities:

$$\begin{aligned}
c_2 e_1 &= f(e_{\leq 2})f(e_1) = f(2, 1, 0, \dots, 0),\\
c_3 e_1 &= f(e_{\leq 3})f(e_1) = f(2, 1, 1, 0, \dots, 0) = f(e_{\leq 2})f(1, 0, 1, 0, \dots, 0)\\
&= c_2 c_2 e_1,\\
c_4 e_1 &= f(e_{\leq 4})f(e_1) = f(2, 1, 1, 1, 0, \dots, 0) = f(e_{\leq 3})f(1, 0, 0, 1, 0, \dots, 0)\\
&= c_3 c_2 e_1,\\
&\vdots\\
c_p e_1 &= f(e_{\leq p})f(e_1) = f(2, 1, \dots, 1) = f(e_{\leq p-1})f(1, 0, \dots, 0, 1)\\
&= c_{p-1} c_2 e_1.
\end{aligned}$$

Then, for every $n \in \{2,3,\dots,p\}$, $c_n = c_2^{n-1}$. The function f is weakly partially faithful, thus there exists a $k \in \{1,2,\dots,p\}$ such that $f(e_{\leq k}) = e_{\leq k}$. Hence $c_k = 1$. Since $c_2^{k-1} = c_k = 1$, thus $c_2 = 1$. Hence $c_m = 1$ for every $m \in \{1,2,\dots,p\}$.

(7) $\Longrightarrow$ (8). It is sufficient to show that is f weakly 2-homogenous. By the consistency of f we have $f(2a) = [f(a)]^2$. Therefore, it is sufficient to show that $f_1(a) \in \{0,1\}$ for every $a \in \mathbb{Q}(p)$. Consider three cases:

I. $a_1 < a_k$ for some $k \in \{2,\dots,p\}$. Suppose that $f_1(a) \neq 0$. From the consistency of f we obtain $f(a+(a_k-a_1)e_1) = f(a)f((a_k-a_1)e_1) = re_1$, where $r \in \mathbb{R}_+$, which contradicts the fact that f is neutral. Consequently, $f_1(a) = 0$.

II. $a_1 = a_k$ for every $k \in \{2,3,\dots,p\}$. Then $f_1(a_1 e_{\leq p}) = [f_1(e_{\leq p})]^{a_1} = 1$.

III. $a_1 \geq a_k$ for every $k \in \{2,3,\dots,p\}$ and there exist $k_1,\dots,k_l \in \{2,\dots,p\}$ such that $a_{k_j} < a_1$ for $j \in \{1,\dots,l\}$. Suppose that $f_1(a) = 0$. There exists a $j \in \{1,\dots,l\}$ such that $f_{k_j}(a) \neq 0$. By the consistency of f we get that $f(a+(a_1-a_{k_j})e_{k_j}) = f(a)f((a_1-a_{k_j})e_{k_j}) = re_{k_j}$, where $r \in \mathbb{R}_+$, which contradicts the fact that f is neutral. Consequently, $f_1(a) = r$, where r is some positive real number. On the other hand, for every $j \in \{1,2,\dots,p\}$ and $q \in \mathbb{Q}_+$: $f_1(qe_{1,j}) = [f_1(e_{1,j})]^q = 1$.
By the consistency of f we obtain:

$$\begin{aligned}
&f_1(a+(a_1-a_{k_1})e_{1,k_1})\\
&\qquad = f_1(a)f_1((a_1-a_{k_1})e_{1,k_1}) = r,\\
&f_1(a+(a_1-a_{k_1})e_{1,k_1}+(a_1-a_{k_2})e_{1,k_2})\\
&\qquad = f_1(a+(a_1-a_{k_1})e_{1,k_1})f_1((a_1-a_{k_2})e_{1,k_2}) = r,\\
&\qquad\vdots\\
&f_1\left(a+\sum_{j=1}^{l}(a_1-a_{k_j})e_{1,k_j}\right)\\
&\qquad = f_1\left(a+\sum_{j=1}^{l-1}(a_1-a_{k_j})e_{1,k_j}\right) f_1((a_1-a_{k_l})e_{1,k_l}) = r.
\end{aligned}$$

Let us notice that

$$f_1\left(a+\sum_{j=1}^{l}(a_1-a_{k_j})e_{1,k_j}\right)=f_1\left(a_1e_{\leq p}+\sum_{j=1}^{l}(a_1-a_{k_j})e_1\right)$$
$$=f_1(a_1e_{\leq p})f_1\left(\sum_{j=1}^{l}(a_1-a_{k_j})e_1\right)=1,$$

whence $r=1$. Consequently, $f_1(a)\in\{0,1\}$.

(8) $\Longrightarrow$ (9). From the proof that (3) implies (4) in Theorem 1 follows that f is consistent (we showed that a function defined on $\mathbb{Q}(p)$, neutral and weakly consistent has the consistency property). It is sufficient to show that f is 2-homogeneous. Take an arbitrary $a\in\mathbb{Q}(p)$ and a $j\in\{1,2,\ldots,p\}$. By the weak 2-homogeneity of f we obtain that there exists a $k\in\{1,2,\ldots,p\}$ such that $f_k(2a)=f_k(a)$. Let π be a permutation of $\{1,\ldots,p\}$ such that $\pi(k)=j$, $\pi(j)=k$ and $\pi(n)=n$ for every $n\in\{1,2,\ldots,p\}\setminus\{k,j\}$. By the neutrality and weak 2-homogeneity of f we have $f_j(2a)=f_{\pi(k)}(2a)=f_k(2a_{\pi(1)},2a_{\pi(2)},\ldots,2a_{\pi(p)})=f_k(a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(p)})=f_{\pi(k)}(a)=f_j(a)$.

It is sufficient to show that (9) implies (1). It follows from the 2-homogeneity and consistency of f that $f(a)=f(2a)=f(a)f(a)=[f(a)]^2$. Consequently, $f(a)\in 0(p)$ and, according to Theorem 1, f is the indicator plurality function on $\mathbb{Q}(p)$.

Remark 4

Notice that the neutrality of the function f in condition (7) in Theorem 2 cannot be replaced by the weak neutrality.
The following function $f=(f_1,f_2):\mathbb{Q}(2)\to\mathbb{R}(2)$ is weakly neutral, consistent, partially faithful and is not the indicator plurality function on $\mathbb{Q}(p)$:

$$f_1(a_1,a_2)=\begin{cases}1 & \text{if } a_1\geq a_2,\\ 0 & \text{if } a_1<a_2,\end{cases}$$
$$f_2(a_1,a_2)=\begin{cases}0 & \text{if } a_1>a_2,\\ 2^{a_2-a_1} & \text{if } a_1\leq a_2.\end{cases}$$

The characterization of the real indicator plurality function

Consider a function $f:\mathbb{R}(p)\to\mathbb{R}(p)$. We introduce the following definitions:

Definition 12

A function f is called *homogeneously faithful* iff

$$\forall j\in\{1,\ldots,p\}\ \forall r\in\mathbb{R}_+\quad f(re_j)=f(e_j).$$

Definition 13

A function f is called *partially homogeneously faithful* iff

$$\forall k \in \{1,\dots,p\}\ \forall r \in \mathbb{R}_+ \quad f(re_{\leq k}) = e_{\leq k}\,.$$

Definition 14

A function f is called *strongly faithful* iff

$$\forall j \in \{1,\dots,p\}\ \forall a \in \mathbb{R}(p) \quad [a_j = 0 \Rightarrow f_j(a) = 0].$$

Definition 15

A function f is called *monotonic* iff

$$\forall j \in \{1,\dots,p\}\ \forall a,b \in \mathbb{R}(p) \quad [(f_j(a) \neq 0 \ \wedge\ b_j > a_j \ \wedge \forall k \neq j\ b_k \leq a_k) \Rightarrow f_j(b) \neq 0].$$

Definition 16

A function f is called *s-monotonic* iff

$$\forall j \in \{1,\dots,p\}\ \forall a,b \in \mathbb{R}(p) \quad [(f_j(a) \neq 0 \ \wedge\ b_j = a_j \ \wedge \forall k \neq j\ b_k < a_k) \Rightarrow f_j(b) \neq 0].$$

Definition 17

A function f is called *peculiarly partially homogeneously faithful* iff

$$\exists k \in \{1,\dots,p-1\}\ \forall r \in \mathbb{R}_+\ f(re_{\leq k}) = e_{\leq k}\,.$$

Definition 18

A function f is called *weakly partially homogeneously faithful* iff

$$\forall r \in \mathbb{R}_+\ [f(re_1) = e_1 \ \wedge\ \exists k \in \{2,\dots,p\}\ f(re_{\leq k}) = e_{\leq k}].$$

The following properties: homogeneous irrational faithfulness, partial homogeneous irrational faithfulness, peculiar partial homogeneous irrational faithfulness, weakly partial homogeneous irrational faithfulness of a function f we define in this way that in Definitions 12, 13, 17 and 18; we postulate the respectiwe conditions for positive irrational number r in place of positive real one.

The following properties: weak homogeneous irrational faithfulness, weak homogeneous faithfulness, peculiar strong faithfulness, weak monotonicity, weak s-monotonicity, weak homogeneity of a function f we define in this way that in the respective definitions described above, we take the small quantifier related to j in place of the big one.

REMARK 5

We show via examples that the monotonicity and s-monotonicity of a function f are two different properties.

EXAMPLE 1

We define a function $f : \mathbb{R}(p) \to 0(p)$ as follows:

$$f(a) = \begin{cases} (0, 1, \dots, 1) & \text{if } a \in W, \\ (1, 1, \dots, 1) & \text{if } a \in \mathbb{R}(p) \setminus W, \end{cases}$$

where

$$W = \{a \in \mathbb{R}(p) : \ a_1 = 0, \ a_2 \in \mathbb{Q}_+ \cup \{0\} \text{ and } a_k \in \mathbb{R}_+ \cup \{0\} \text{ for every } k \in \{3, 4, \dots, p\}\}.$$

We shall show that f is monotonic but not s-monotonic. Take arbitrary $c, d \in \mathbb{R}(p)$ such that there exists a $j \in \{1, 2, \dots, p\}$ such that $f_j(c) \neq 0$, $d_j > c_j$ and $d_k \leq c_k$ for every $k \neq j$. Since $d_j > c_j$, we have $d_j > 0$. By the definition of f we obtain $f_j(d) \neq 0$; consequently, f is monotonic.
Put $c = (0, \sqrt{2}, 1, \dots, 1)$, $d = (0, 1, 0, \dots, 0) \in \mathbb{R}(p)$. Then $f_1(c) = 1$, $d_1 = c_1$, $d_k < c_k$ for every $k \in \{2, 3, \dots, p\}$ and from the definition of f we get $f_1(d) = 0$. Then the function f is not s-monotonic.

EXAMPLE 2

We define a function $f : \mathbb{R}(p) \to 0(p)$ as follows:

$$f(a) = \begin{cases} (0, 1, \dots, 1) & \text{if } a \in Z, \\ (1, 1, \dots, 1) & \text{if } a \in \mathbb{R}(p) \setminus Z, \end{cases}$$

where $Z = \{a \in \mathbb{R}(p) : \ a_1 = 2 \text{ and } a_k \in \mathbb{R}_+ \cup \{0\} \text{ for every } k \in \{2, 3, \dots, p\}\}$.

We shall show that function f is s-monotonic but not monotonic. Take arbitrary $c, d \in \mathbb{R}(p)$ such that there exists a $j \in \{1, 2, \dots, p\}$ such that $f_j(c) \neq 0$, $d_j = c_j$ and $d_k < c_k$ for every $k \neq j$. Consider two cases:

I. $j = 1$. Then $c_1 \neq 2$, whence $d_1 \neq 2$. It follows directly from the definition of f that $f_1(d) = 1 \neq 0$.

II. $j \neq 1$. Then by the definition of f $f_j(d) = 1 \neq 0$.
This shows that f is s-monotonic.

Put $c = (1, 0, \dots, 0)$, $d = (2, 0, \dots, 0) \in \mathbb{R}(p)$. Then $f_1(c) = 1$, $d_1 > c_1$, $d_k = c_k$ for every $k \in \{2, 3, \dots, p\}$ and $f_1(d) = 0$. Consequently, f is not monotonic.

Let us notice that a partially homogeneously faithful function f is weakly partially homogeneously faithful, which implies that f is peculiarly partially homogeneously faithful. However, the converse implication is not true.

REMARK 6

The definitions of the homogeneous irrational faithfulness, homogeneous faithfulness, partial homogeneous irrational faithfulness and strong faithfulness were formulated by M. F. S. Roberts in [2]. He gave also in [2] the following definition of monotonicity. A function $f : \mathbb{R}(p) \to \mathbb{R}(p)$ is called *monotonic* iff the following conditions hold:

(a) if $f_i(c) \neq 0$ for $i = i_1, i_2, \ldots, i_k$, $d_i > c_i$ for $i = i_1, i_2, \ldots, i_k$, and $d_i \leq c_i$ otherwise, then $f_i(d) \neq 0$ for $i = i_1, \ldots, i_k$,;

(b) if $f_i(c) = 0$ for $i = i_1, i_2, \ldots, i_k$, $d_i < c_i$ for $i = i_1, i_2, \ldots, i_k$, and $d_i \geq c_i$ otherwise, then $f_i(d) = 0$ for $i = i_1, i_2, \ldots, i_k$.

THEOREM 3

Let $f : \mathbb{R}(p) \to 0(p)$ be an arbitrary function. Then the following conditions are equivalent:

(10) *f is the indicator plurality function on $\mathbb{R}(p)$;*

(11) *f is neutral* (p1), *consistent* (p2), *faithful* (p3) *and homogeneous* (p4)*;*

(12) *f is neutral* (p1), *consistent* (p2), *faithful* (p3) *and monotonic* (p5)*;*

(13) *f is neutral* (p1), *consistent* (p2), *and homogeneously irrational faithful* (p6)*;*

(14) *f is neutral* (p1), *consistent* (p2), *and homogeneously faithful* (p7)*;*

(15) *f is neutral* (p1), *consistent* (p2), *faithful and s-monotonic* (p8)*;*

(16) *f is neutral* (p1), *consistent* (p2), *and strongly faithful* (p9)*;*

(17) *f is neutral* (p1), *consistent* (p2), *and peculiarly partially homogeneously irrational faithful* (p10)*;*

(18) *f is neutral* (p1), *consistent* (p2), *and peculiarly partially homogeneously faithful* (p11)*;*

(19) *f is neutral* (p1), *consistent* (p2), *and weakly partially homogeneously faithful* (p12)*;*

(20) *f is neutral* (p1), *consistent* (p2), *and weakly partially homogeneously irrational faithful* (p13)*;*

(21) *f is neutral* (p1), *consistent* (p2), *and partially homogeneously irrational faithful* (p14)*;*

(22) *f is neutral* (p1), *consistent* (p2), *and partially homogeneously faithful* (p15).

In [2] M. F. S. Roberts formulates Theorem 6, which assumptions are the same as in Theorem 3 and the assertion says that the conditions: (10), (11), (12), (13), (14), (16), (21), (22). However, with Roberts' definition of the monotonicity the implication (10) $\Longrightarrow$ (12) is not true.
In [1] Z. Moszner gives an example of a function, which is monotonic in the sense of Roberts and is not the indicator plurality function. He also proposed another definition of monotonicity, which is accepted in this paper (Def. 15).

Proof. The proof that (10) implies (11) was given by Roberts in [2] p. 171.

(11) $\Longrightarrow$ (12). Take arbitrary $a, b \in \mathbb{R}(p)$ and an arbitrary $j \in \{1, 2, \dots, p\}$ such that $f_j(a) \neq 0$, $b_j > a_j$ and $b_k \leq a_k$ for $k \neq j$.

First we will prove that $f(b') = e_j$, where $b' = (a_1, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_p)$. By (p4) and (p3) we have $f((b_j - a_j)e_j) = f(e_j) = e_j$, thus from (p2) we obtain $f(b') = f(a)f((b_j - a_j)e_j) = e_j$. Consequently, if a and b are such vectors that $b_j > a_j$ and $b_k = a_k$ for every $k \neq j$ then $f(b) = f(b') = e_j$.

We shall show that $b_j > a_k$ for every $k \in \{1, 2, \dots, p\}$. The case $k = j$ is obvious. Suppose that there exists a $k \neq j$ such that $a_k > b_j$ or $a_k = b_j$. If $a_k = b_j$, then by (p1) we get equality $f_j(b') = f_k(b')$, which contradicts the fact that $f(b') = e_j$. If $a_k > b_j$ then $a_k - b_j > 0$. Since $f_j(b') = 1$ and $f_j((a_k - b_j)e_j) = 1$, from (p2) we obtain $f(b' + (a_k - b_j)e_j) = f(b')f((a_k - b_j)e_j) = e_j$, which contradicts the property (p1).
Take an arbitrary $c = (c_1, c_2, \dots, c_p) \in \mathbb{R}(p)$ for which there exists a $j \in \{1, 2, \dots, p\}$ such that $f(c) = e_j$ and $c_j > c_n$ for every $n \in \{1, 2, \dots, p\} \setminus \{j\}$. We shall show that if we replace c_i, for $i \neq j$, by a non-negative real number d_i such that $d_i < c_i$ then $f(c_1, \dots, c_{i-1}, d_j, c_{i+1}, \dots, c_p) = e_j$. Fix an $i \in \{1, \dots, p\} \setminus \{j\}$ and let d_i be a non-negative real number such that $d_i < c_i$. Put $c' = (c_1, \dots, c_{i-1}, d_i, c_{i+1}, \dots, c_p)$. Suppose that there exists a $k \neq j$ such that $f_k(c') = 1$. From (p2) we have $f(c' + (c_j - c_k)e_k) = f(c')f((c_j - c_k)e_k) = e_k$, which contradicts therefore the property (p1). Consequently, for every $k \neq j$ we get $f_k(c') = 0$ and since $f(c') \in 0(p)$ we obtain $f(c') = e_j$.
We have $f(b') = e_j$ and $b_j > a_k$ for $k \in \{1, 2, \dots, p\} \setminus \{j\}$. If there exists an i such that $b_i < a_i$ then $f(a_1, a_2 \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_p) = e_j$ (i can be less than j). Following the analogous way, after at most $p - 1$ steps, we obtain $f(b) = e_j$, whence $f_j(b) = 1$.

(12) $\Longrightarrow$ (13). Let s be a positive irrational number, and let q be a positive rational number such that $s > 2q$ and $i \in \{1, 2, \dots, p\}$. It follows from Lemma 1 and (p3) that $f(qe_i) = [f(e_i)]^q = [e_i]^q = e_i$. From (p5) we obtain $f_i((s-q)e_i) = f_i([q + (s-q)]e_i) = 1$ and, by (p2), $f(se_i) = f((s-q)e_i)f(qe_i) = e_i$.

The proof that (13) implies (14) is analogous to the proof that (4) implies (5) in Roberts' theorem.

(14) $\Longrightarrow$ (15). Take arbitrary $a, b \in \mathbb{R}(p)$ and an arbitrary $i \in \{1, 2, \dots, p\}$ such that $f_i(a) \neq 0$, $b_i = a_i$ and $b_k < a_k$ for every $k \neq i$.

At first we shall show that $a_k \leq a_i$ for every $k \in \{1, 2, \dots, p\}$. Suppose that there exists an $n \in \{1, 2, \dots, p\} \setminus \{i\}$ such that $a_n > a_i$. By (p2) we obtain $f(a + (a_n - a_i)e_i) = f(a)f((a_n - a_i)e_i) = e_i$, which contradicts the property (p1). Consequently, $a_k \leq a_i$ for every $k \in \{1, 2, \dots, p\}$.
Take an arbitrary $c = (c_1, c_2, \dots, c_p) \in \mathbb{R}(p)$ such that there exists an $i \in \{1, 2, \dots, p\}$ such that $f_i(c) = 1$ and $c_i \geq c_n$ for every $n \in \{1, 2, \dots, p\}$. We shall show that if we replace c_j for $j \neq i$ by a non-negative real number x_j such that $x_j < c_j$ then $f_i(c_1, \dots, c_{j-1}, x_j, c_{j+1}, \dots, c_p) = 1$. Fix an $j \in \{1, 2, \dots, p\} \setminus \{i\}$ and let x_j be a non-negative real number such that $x_j < c_j$. Put $c' = (c_1, c_2, \dots, c_{j-1}, x_j, c_{j+1}, \dots, c_p)$. Suppose that $f_i(c') = 0$. The range of the function f is contained in $0(p)$, thus there exists a $n \in \{1, 2, \dots, p\}$ such that $f_n(c') = 1$. Then $d_n = c_n$ if $n \neq j$ and $d_n = x_j$ if $n = j$. Notice that $c_i - d_n > 0$. From (p2) we have $f(c' + (c_i - d_n)e_n) = f(c')f((c_i - d_n)e_n) = e_n$, which contradicts the property (p1). Consequently, we get $f_i(c') = 1$.
Next notice that $f_i(a) = 1$ and $a_k \leq a_i$ for every $k \in \{1, 2, \dots, p\}$, $a_i = b_i$ and $b_n < a_n$ for every $n \neq i$. Take an $m \in \{1, 2, \dots, p\}$. Since $f_i(a) = 1$ and $a_k \leq b_i$ for every $k \in \{1, 2, \dots, p\}$ and $b_m < a_m$, then

$$f_i(a_1, \dots, a_{m-1}, b_m, a_{m+1}, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_p) = 1$$

(i can be less than m). Since $b_m < a_m$ and $a_k \leq b_i$ for every $k \in \{1, 2, \dots, p\}$, thus $b_m < b_i$ and $a_k \leq b_i$ for every $k \in \{1, 2, \dots, p\} \setminus \{m\}$. Take an $l \in \{1, 2, \dots, p\} \setminus \{i, m\}$. Let us observe that

$$f_i(a_1, \dots, a_{m-1}, b_m, a_{m+1}, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_p) = 1,$$

$b_m < b_i$ and $a_k \leq b_i$ for all $k \in \{1, 2, \dots, p\} \setminus \{m\}$ and $b_l < a_l$ thus $f_i(b_{i,l,m}) = 1$, where $b_{i,l,m}$ is a vector with b_i, b_l, b_m in the i-th, l-th, m-th positions, respectively, and the other components are the same as suitable components of the vector a. Following the analogous way, after $p - 1$ steps, we obtain $f_i(b) = 1 \neq 0$.

(15) $\Longrightarrow$ (16). Take arbitrary $a \in \mathbb{R}(p)$ and $i \in \{1, 2, \dots, p\}$ such that $a_i = 0$. Suppose that $f_i(a) \neq 0$. Since $a \in \mathbb{R}(p)$, there exists an indicator j such that $a_j > 0$. Let q be a positive rational number such that $a_j > q$. It follows from Lemma 1 and (p3) that $f(qe_i) = [f(e_i)]^q = [e_i]^q = e_i$. Function f has the property (p1) and the range of f is contained in $0(p)$, thus $f(r, r, \dots, r) = (1, 1, \dots, 1)$ for all $r \in \mathbb{R}_+$. By (p8) we obtain $f_i(re_i) = 1$ for

all $r \in \mathbb{R}_+$. Since $f_i((a_j - q)e_i) = 1$ and $f_i(qe_i) = 1$, thus by (p2), we have that $f(a_j e_i) = f((a_j - q)e_i)f(qe_i) = e_i$. Since $f_i(a) = 1$ and $f_i(a_j e_i) = 1$ from (p2) we get $f(a + a_j e_j) = f(a)f(a_j e_i) = e_i$, which contradicts the property (p1). Consequently, $f_i(a) = 0$.

It is obvious that (16) implies(17).

The proof that (17) implies (18) is analogous to the proof that (4) implies (5) in Roberts' theorem.

(18) $\Longrightarrow$ (19). By (p11) there exists a $k \in \{1, 2, \ldots, p-1\}$ such that $f(re_{\leq k}) = e_{\leq k}$ for all $r \in \mathbb{R}_+$. Let r be positive real number.

First we shall show that $f(re_1) = e_1$. It follows from (p1) that there exists a $(c, d) \in 0(2)$ such that $f(re_{k+1}) = (c, \ldots, c, \overset{(k+1)}{d}, c, \ldots, c)$. Suppose that $c = 1$. By (p2) we obtain $f(re_{\leq k+1}) = f(re_{\leq k})f(re_{k+1}) = e_{\leq k}$, which contradicts the property (p1). Thus $c = 0$, whence $d = 1$ and then $f(re_{k+1}) = e_{k+1}$. Hence and by (p1) we get that $f(re_1) = e_1$. It is easily seen that $f(re_{\leq p}) = e_{\leq p}$.

It is obvious that (19) implies (20).

(20) $\Longrightarrow$ (21). Let s be a positive irrational number. From the asumption we have $f(se_1) = e_1$. It is obvious that $f(se_{\leq p}) = e_{\leq p}$. Take a $k \in \{2, 3, \ldots, p-1\}$. Making use of (p1) and the range of f we obtain that there exists a $(c, d) \in 0(2)$ such that $f(se_{\leq k}) = (c, \ldots, \overset{(k)}{c}, d, \ldots, d)$. Suppose that $d \neq 0$. From (p1) we have $f(se_{k+1}) = e_{k+1}$. By (p2) we obtain $f(se_{\leq k+1}) = f(se_{\leq k})f(se_{k+1}) = e_{k+1}$, which contradicts the property (p1). Thus $d = 0$, whence $c = 1$ and then $f(se_{\leq k}) = e_{\leq k}$.

The proof that (21) implies (22) was given by Roberts in [2] p. 171.

(22) $\Longrightarrow$ (10). Take an $a \in \mathbb{R}(p)$. Let us observe that by (p15) and (p1) we obtain $f(re_{i_1,\ldots,i_k}) = e_{i_1,\ldots,i_k}$ for all $r \in \mathbb{R}_+$ and for all pairwise different $i_1, \ldots, i_k \in \{1, 2, \ldots, p\}$. We may order a set $\{a_1, \ldots, a_p\}$ in the following way:

$$a_{j_1} = \cdots = a_{j_{k_1}} < a_{j_{k_1+1}} = \cdots = a_{j_{k_1+k_2}} < \cdots < a_{j_{k_1+k_2+\cdots+k_m+1}} = \cdots = a_{j_p},$$

where k_1 can be equal to p.

The further part of the proof is analogous to the suitable part of the proof that (4) implies (1) in Theorem 1. Thus we obtain $f(a) = e_{k_1+k_2+\cdots+k_m+1,\ldots,p}$. Consequently, f is the indicator plurality function on $\mathbb{R}(p)$.

Remark 7

Theorem 3 holds true if we replace the following properties: faithfulness, homogenous irrational faithfulness, homogenous faithfulness, strong faithfulness, monotonicity, s-monotonicity, homogeneity and consistency, by the corresponding *weak* properties.

We shall prove that it follows from the neutrality and weak homogeneous faithfulness of the function f that f is homogeneously faithful. Take an arbitrary $r \in \mathbb{R}_+$ and $j \in \{1, 2, \dots, p\}$.
We shall show that $f(re_j) = e_j$. It follows by the weak homogeneous faithfulness that there exists an $i \in \{1, 2, \dots, p\}$ such that $f(re_i) = e_i$. Let π be a permutation of $\{1, 2, \dots, p\}$ such that $\pi(i) = j$ and $\pi(j) = i$ and $\pi(k) = k$ for $k \in \{1, 2, \dots, p\} \setminus \{i, j\}$. Take an $m \in \{1, 2, \dots, p\}$. From the definition of the permutation π, neutrality and weak homogeneous faithfulness of f we obtain:

$$f_m(re_j) = f_m(re_{\pi(i)}) = f_{\pi(m)}(re_i) = \begin{cases} 1 & \text{if } \pi(m) = i, \\ 0 & \text{if } \pi(m) \neq i. \end{cases}$$

Then $f(re_j) = e_j$, thus f is homogeneously faithful.

The proof that the neutrality and weak faithfulness imply that f is faithful and that the neutrality and weak homogeneous irrational faithfulness imply that f is homogeneous irrational faithful runs in the same way. It is sufficient to take $r = 1$ in the first case and r positive irrational in the second case in the above proof.

In the similar way we may prove that:

1) from the neutrality and peculiar strong faithfulness of the function f it follows that f is strong faithful,
2) from the neutrality and weak monotonicity of the function f it follows that f is monotonic,
3) from the neutrality and weak s-monotonicity of the function f it follows that f is s-monotonic,
4) from the neutrality and weak homogenity of the function f it follows that f is homogeneous.

The proof that from the neutrality and weak consistency of the function f it follows that f is consistent is analogous to the suitable part of the proof that (3) implies (4) in Theorem 1.

REMARK 8

Note that the neutrality, faithfulness, homogenous irrational faithfulness and homogeneity of the function f in Theorem 3 can be replaced by the weak neutrality, weak faithfulness, weak homogeneity irrational faithfulness and weak homogeneous, respectively.

If we weaken the assumption that a range function f is contained in $0(p)$ in Theorem 3 then we shall obtain the following theorem.

THEOREM 4

Let $f : \mathbb{R}(p) \to \mathbb{R}(p)$ *be an arbitrary function. Then the following conditions are equivalent:*

(10) f *is the indicator plurality function on* $\mathbb{R}(p)$*;*

(11) f *is neutral* (p1), *consistent* (p2), *faithful* (p3) *and homogeneous* (p4)*;*

(19) f *is neutral* (p1), *consistent* (p2), *and weakly partially homogeneously faithful* (p12)*;*

(20) f *is neutral* (p1), *consistent* (p2), *and weakly partially homogeneously irrational faithful* (p13)*;*

(21) f *is neutral* (p1), *consistent* (p2), *and partially homogeneously irrational faithful* (p14)*;*

(22) f *is neutral* (p1), *consistent* (p2), *and partially homogeneously faithful* (p15).

Proof. The proof that (10) implies (11) is the same as the proof that (10) implies (11) in Theorem 3.

(11) $\Longrightarrow$ (19). Let r be a positive real number.
We take into consideration $f(re_1)$ and $f(re_{\leq p})$. From (p4) and (p3) we obtain $f(re_1) = f(e_1) = e_1$. By (p1) we get that there exists $c \in \mathbb{R}$ such that $f(re_{\leq p}) = (c, c, \dots, c)$. Since the range of the function f is contained in $\mathbb{R}(p)$, c is a positive real number. It follows from (p4) and (p2) that

$$(c, c, \dots, c) = f(2re_{\leq p}) = f(re_{\leq p})f(re_{\leq p}) = (c^2, c^2, \dots, c^2).$$

We conclude that $c^2 = c$ and $c \in \mathbb{R}_+$, whence $c = 1$.

It is obvious that (19) implies (20).

The proof that (20) implies (21) is analogous to that of implication (7) $\Longrightarrow$ (8) in Theorem 2.
The next part of the proof is the same as in the proof of Theorem 3.

In [2] Roberts formulates Theorem 7, which assumptions are the same as Theorem 3 and in the assertions there is stated the equivalence of the conditions: (10), (11), (21) and (22).

REMARK 9

Note that the faithfulness, homogeneity and consistency of the function in Theorem 4 can be replaced suitably by the weak faithfulness, weak homogeneity and weak consistency.

REMARK 10

Several problems remain open:

1) Can the consistency and neutrality be replaced by the weak consistency and weak neutrality, respectively, in any of the conditions of Theorem 3?
2) Which conditions are equivalent to the condition that f is the indicator plurality function if we assume that $f : \mathbb{R}(p) \to \mathbb{Q}(p)$?
3) Can the neutrality be replaced by the weak neutrality in any of the conditions of Theorem 4?

The problem of Roberts

In [2] Roberts asks if every neutral, consistent and homogeneous faithful function $f : \mathbb{R}(p) \to \mathbb{R}(p)$ is the indicator plurality function.
The answer to this question is negative. An example for $p = 2$ was formulated by Z. Moszner in [1].
Below we give an example for an arbitrary p.

We define a function $f = (f_1, f_2, \dots, f_p) : \mathbb{R}(p) \to \mathbb{R}(p)$ as follows:

$$f_i(a) = \begin{cases} \exp[a_1 + a_2 + \cdots + a_p - a_i] & \text{if } a \in Z_i, \\ 0 & \text{if } a \in \mathbb{R}(p) \setminus Z_i, \end{cases} \qquad (*)$$

where $Z_i = \{a \in \mathbb{R}(p) : \ a_i \geq a_j \text{ for } j \in \{1, 2, \dots, p\}\}$ and $i \in \{1, 2, \dots, p\}$.

Take an $a \in \mathbb{R}(p)$, $i \in \{1, 2, \dots, p\}$ and let π be a permutation of $\{1, 2, \dots, p\}$. Let us consider two cases:

1) $a_{\pi(i)} \geq a_k$ for every $k \in \{1, 2, \dots, p\}$. Then, from $(*)$, we get $f_i(a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(p)}) = \exp[a_1 + a_2 + \cdots + a_p - a_{\pi(i)}]$ and $f_{\pi(i)}(a_1, a_2, \dots, a_p) = \exp[a_1 + a_2 + \cdots + a_p - a_{\pi(i)}]$.
2) $a_{\pi(i)} < a_j$ for some $j \in \{1, 2, \dots, p\}$. Then, by $(*)$, we have $f_i(a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(p)}) = 0$ and $f_{\pi(i)}(a_1, a_2, \dots, a_p) = 0$.

From 1) and 2) we obtain $f_i(a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(p)}) = f_{\pi(i)}(a_1, a_2, \dots, a_p)$. Consequently, f is neutral.

Take arbitrary $a, b \in \mathbb{R}(p)$ such that $f(a)f(b) \neq \underline{0}$. There exists an $l \in \{1, 2, \dots, p\}$ such that $f_l(a)f_l(b) \neq 0$. Take a $k \in \{1, 2, \dots, p\}$. Consider three cases :

1) $f_k(a) \neq 0$ and $f_k(b) \neq 0$. Then, from $(*)$, we get $f_k(a) = \exp[a_1 + a_2 + \cdots + a_p - a_k]$ and $f_k(b) = \exp[b_1 + b_2 + \cdots + b_p - b_k]$, $a_k \geq a_j$ and $b_k \geq b_j$ for every $j \in \{1, 2, \dots, p\}$. Hence $a_k + b_k \geq a_j + b_j$ for every $j \in \{1, 2, \dots, p\}$. According to $(*)$ we obtain $f_k(a + b) = \exp[a_1 + b_1 + a_2 + b_2 + \cdots + a_p + b_p - (a_k + b_k)]$.
2) $f_k(a) \neq 0$ and $f_k(b) = 0$. Then, by the definition of f, we get $a_k = a_l$ and $b_k < b_l$. Thus $a_k + b_k < a_l + b_l$. From $(*)$, we have $f_k(a + b) = 0$.

3) $f_k(a) = 0$ and $f_k(b) = 0$. Then by $(*)$, we get $a_k < a_l$ and $b_k < b_l$. Hence $a_k + b_k < a_l + b_l$. It follows from $(*)$ that $f_k(a + b) = 0$.

From 1), 2) and 3) we obtain $f_k(a)f_k(b) = f_k(a + b)$. Then the function f is consistent.

It follows from $(*)$ that for every $r \in \mathbb{R}_+$ and $j \in \{1, 2, \dots, p\}$ we have $f(re_j) = (0, 0, \dots, 0, \overset{(j)}{e^{r-r}}, 0, \dots, 0) = e_j$. Then the function f is homogeneously faithful.

Note that f is not the indicator plurality function because the range of f is not contained in $0(p)$. For example: $f(1, 1, 0, \dots, 0) = (e, e, 0, \dots, 0)$.

Remark 11

It is easily seen that the f mentioned above is also strongly faithful, monotonic and s-monotonic. This example shows that conditions (3), (4), (5), (6), (7), (8) and (9) of Theorem 3 do not imply that $f : \mathbb{R}(p) \to \mathbb{R}(p)$ is the indicator plurality function.
Let us observe that if the domain of f is restricted to the set $\mathbb{Q}(p)$ then $f : \mathbb{Q}(p) \to \mathbb{R}(p)$ is neutral, consistent, faithful but not the indicator plurality function on $\mathbb{Q}(p)$.

Acknowledgement

I wish to thank Professor Zenon Moszner for his helpful comments and suggestions.

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