ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE
Zeszyt 196 Prace Matematyczne XV 1998

AURELIO CANNIZZO

Geometrical convexity and the Artin functional equation

Abstract. We give uniqueness and existence results of Krull type for the Artin functional equation $F(x+1) = g(x)F(x)$, in the frame of geometrical convexity. Some of them are improvements of previous ones in the literature.

0. Introduction

The history of the characterizations of solutions of the Artin functional equation

$$F(x+1) = g(x)F(x), \quad x \in \mathbb{R}_+ \tag{0.1}$$

starts with the well known theorem of H. Bohr and J. Mollerup ([4], [2]), in which they characterized the Euler Γ-function as the *unique logarithmically convex solution of the functional equation*

$$F(x+1) = xF(x), \quad x \in \mathbb{R}_+, \tag{0.2}$$

with $F(1) = 1$.

Here and in what follows the *convexity* of a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is meant in the classical sense

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y), \quad \text{for all } \lambda \in (0,1) \text{ and } (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

and f is *logarithmically convex* (log-convex) means that $\log \circ f$ is convex (see e.g. [1] and [2]).

Many authors extended the above theorem in different ways, many of them involving equations (0.1) and (0.2). Even a short list of subsequent results would be too long anyway, therefore we restrict ourselves to mention only

AMS (1991) subject classification: Primary 39B12, Secondary 39B22, 33B15.

Supported by *Ministero Università e Ricerca Scientifica e Tecnologica*, Rome (Italy).

three which are directly connected with results presented in this paper. The first one is historically important, the second and the third are more recent and refer to equations (0.1) and (0.2).

We restate the first result in the following, equivalent, form:

THEOREM 1 (W. Krull [7], see also [8] p. 114)
Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be ultimately log-concave (log-convex) and

$$\lim_{x \to +\infty} \frac{g(x+1)}{g(x)} = 1. \tag{0.3}$$

Then equation (0.1) *admits one and only one ultimately log-convex (log-concave) solution F with $F(1) = 1$.*

As for the second result, we are interested in the main one in [5], which substantially improves the Bohr-Mollerup theorem:

THEOREM 2
The Γ-function is the unique ultimately geometrically convex solution of equation (0.2), *with $F(1) = 1$.*

Here: *a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is geometrically convex (g-convex)* means that:

$$f(x^\lambda \cdot y^{1-\lambda}) \le f(x)^\lambda \cdot f(y)^{1-\lambda}, \quad \text{for all } \lambda \in (0,1) \text{ and } (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{0.4}$$

The reverse inequality states that f is *geometrically concave.*

The third result generalizes Theorem 2 and improves Theorem 1. It is included in Theorems 2 and 3 of [6], but, for the sake of shortness, we restate it in the following unified equivalent form:

THEOREM 3
Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be ultimately log-concave and suppose

$$\lim_{x \to +\infty} \frac{g(x+1)}{g(x)} = 1.$$

Then there exists at most one ultimately g-convex solution F of equation (0.1), *with $F(1) = 1$. Moreover, if $g(y) \ge 1$ for some y in a neighbourhood of $+\infty$ in which g is log-concave, then the above solution F exists.*

The aim of the present paper is to give a Krull-analogous uniqueness result for solutions of (0.1) either log-convex or g-convex and a first existence result, only involving g-convex functions. The first gives an improvement of the uniqueness result of both Theorems 1 and 3, the second is obtained as an application of Theorem 3.

1. Some useful propositions

In this section we deal with some properties of g-convex functions and connections with log-convexity, which are convenient for the reader and useful for the following. Most of the proofs of propositions will not be given or simply sketched, because they are standard extensions of classical ones. The interested reader can find them in several books (see e.g. [1], [9]).

In the sequel $f : \mathbb{R}_+ \to \mathbb{R}_+$ will be always assumed.

PROPOSITION 1

A function f is g-convex (g-concave) on (c, ∞), $c > 0$, iff the function $\log \circ f \circ \exp$ *is convex (concave) on* $(\log c, \infty)$.

As in classical convexity theory one can prove the

PROPOSITION 2

A function f is g-convex on (c, ∞), $c > 0$, iff for every $c < x < y$, $t < z$, $x \leq t$, $y \leq z$ one has

$$\frac{\log f(x) - \log f(y)}{\log x - \log y} \leq \frac{\log f(t) - \log f(z)}{\log t - \log z}, \tag{1.1}$$

(or the opposite inequality if f is g-concave).

A classical results states the *ultimate monotonicity of convex and log-convex functions.* Such a results holds also for g-convex functions:

PROPOSITION 3

An ultimately g-convex or g-concave function f is ultimately monotone.

Starting from (1.1), the proof is quite analogous to the classical ones.

PROPOSITION 4

If f is ultimately log-convex (log-concave), then the ratio $\frac{f(x+1)}{f(x)}$ is ultimately increasing (decreasing).

If f is ultimately g-convex (g-concave), then the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Phi(x) := \left[\frac{f(x+1)}{f(x)}\right]^{\frac{1}{\log \frac{x+1}{x}}}$$

is ultimately increasing (decreasing), and either ultimately $\frac{f(x+1)}{f(x)} \geq 1$ or ultimately $\frac{f(x+1)}{f(x)} \leq 1$ holds.

Proof. The first claim is a well known result of the theory of log-convexity. As regards the second, the monotonicity of the function Φ follows from (1.1) with $y = x+1$ and $z = t+1$. Hence, ultimately either $\Phi(x) \geq 1$ or $\Phi(x) \leq 1$. From this the thesis follows immediately.

The following results state some connections between g-convexity and log-convexity.

PROPOSITION 5

a) *If f is log-convex (log-concave) and increasing (decreasing) in a neighbourhood of $+\infty$, then f is ultimately g-convex (g-concave).*

b) *If f is g-convex (g-concave) and decreasing (increasing) in a neighbourhood of $+\infty$, then f is ultimately log-convex (log-concave).*

Proof. One of the two statements in a) is proved in [5], the proof of the remaining one is quite analogous. To prove one of the two in b), assume that f is g-convex and decreasing.

Let $x, y \in \mathbb{R}_+$ and $\lambda \in (0,1)$. Since $x^\lambda y^{1-\lambda} \leq \lambda x + (1-\lambda)y$, from (0.4) we get

$$f(\lambda x + (1-\lambda)y) \leq f\left(x^\lambda y^{1-\lambda}\right) \leq f(x)^\lambda f(y)^{1-\lambda}.$$

Hence, by taking logarithms of the extreme sites, the thesis follows. The remaining statement can be proved in analogous way.

The next result gives information on the special behaviour at $+\infty$ of g-concave and increasing functions. It plays a role in section 3.

PROPOSITION 6

If f is g-concave and increasing in a neighbourhood of $+\infty$, then

$$\lim_{x\to+\infty} \frac{f(x+1)}{f(x)} = 1.$$

Proof. Because f is increasing, we have ultimately $\frac{f(x+1)}{f(x)} \geq 1$, and then also $\Phi(x) \geq 1$ (Proposition 4). From Proposition 4 it follows that the function $\Phi(x)$ is ultimately decreasing, hence $+\infty > \lim_{x\to+\infty} \Phi(x) \geq 1$. By $\frac{f(x+1)}{f(x)} = [\Phi(x)]^{\log \frac{x+1}{x}}$ we get $\lim_{x\to+\infty} \frac{f(x+1)}{f(x)} = 1$.

2. A uniqueness result

Now we can prove our uniqueness theorem.

THEOREM 4
Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ *be such that*

$$\lim_{n\to+\infty} \frac{g(n+1)}{g(n)} = 1. \tag{2.1}$$

Then equation (0.1) *admits at most one solution* F, *with* $F(1) = 1$, *which is either ultimately g-convex or ultimately g-concave or ultimately log-convex or ultimately log-concave.*

Proof. First we observe that (2.1) implies $\lim_{n\to+\infty} \sqrt[n]{g(n)} = 1$, hence $\lim_{n\to+\infty} \frac{\log g(n)}{n} = 0$. Let $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ be solutions of (0.1) with $F(1) = G(1)$. Then there is a positive and 1-periodic function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $F(x) = \alpha(x)G(x)$ and $\alpha(1) = 1$.

In the following, without loss of generality, we assume $x \in (1, 2]$ and $n \in \mathbb{N}$, large enough. We have ten cases

a) Assume *both* F *and* G *ultimately g-convex.* From (1.1) we get

$$\begin{aligned}\frac{\log F(n+1) - \log F(n)}{\log(n+1) - \log n} &\le \frac{\log F(n+x) - \log F(n)}{\log(n+x) - \log n}\\ &\le \frac{\log F(n+2) - \log F(n+1)}{\log(n+2) - \log(n+1)}\end{aligned}$$

and the same inequalities with G in place of F. Hence we obtain

$$\begin{aligned}\log\frac{n+x}{n}\left[\frac{\log\frac{F(n+1)}{F(n)}}{\log\frac{n+1}{n}} - \frac{\log\frac{G(n+2)}{G(n+1)}}{\log\frac{n+2}{n+1}}\right] &\le \log\frac{F(n+x)}{F(n)} - \log\frac{G(n+x)}{G(n)}\\ &= \log\frac{\alpha(n+x)}{\alpha(n)}\\ &\le \log\frac{n+x}{n}\left[\frac{\log\frac{F(n+2)}{F(n+1)}}{\log\frac{n+2}{n+1}} - \frac{\log\frac{G(n+1)}{G(n)}}{\log\frac{n+1}{n}}\right].\end{aligned}$$

This by the functonal equation (0.1) and after defining

$$\beta_n := \frac{\log\frac{n+1}{n}}{\log\frac{n+2}{n+1}} \cdot \log g(n+1) - \log g(n),$$

can be written as

$$|\log \alpha(x)| \cdot \frac{\log \frac{n+1}{n}}{\log \frac{n+x}{n}} \leq \beta_n.$$

Therefore it is sufficient to prove that $\lim_{n\to+\infty} \beta_n = 0$.

Because $g(t) = \frac{F(t+1)}{F(t)}$ for every $t \in \mathbb{R}_+$, from Proposition 4 one deduces that ultimately either $g(n) \leq 1$ or $g(n) \geq 1$. In the first case, from $\frac{\log \frac{n+1}{n}}{\log \frac{n+2}{n+1}} > 1$, we get

$$\beta_n \leq \log \frac{g(n+1)}{g(n)},$$

which, by (2.1), implies the thesis in this case.

In the second case, from $\frac{\log \frac{n+1}{n}}{\log \frac{n+2}{n+1}} < 1 + \frac{2}{n}$, we get

$$\beta_n \leq \frac{2}{n} \log g(n+1) + \log \frac{g(n+1)}{g(n)},$$

and the thesis follows in this case, too.

b) With suitable little modifications, this proof can be repeated in the case when *both F and G are ultimately g-concave.*

c) Assume that *both F and G are ultimately log-concave* and repeat the proof given in the case a), but by using the log-concavity in place of the g-concavity, namely

$$\begin{aligned} \log F(n+1) - \log F(n) &\geq \frac{\log F(n+x) - \log F(n)}{x} \\ &\geq \log F(n+2) - \log F(n+1) \end{aligned} \tag{2.2}$$

and analogously for G. We deduce

$$\begin{aligned} x \cdot \left[\log \frac{F(n+2)}{F(n+1)} - \log \frac{G(n+1)}{G(n)}\right] &\leq \log \alpha(x) \\ &\leq x \cdot \left[\log \frac{F(n+1)}{F(n)} - \log \frac{G(n+2)}{G(n+1)}\right]. \end{aligned}$$

This, by functional equation (0.1), can be written as

$$|\log \alpha(x)| \leq x \cdot \log \frac{g(n)}{g(n+1)},$$

hence, also in this case, $F = G$.

d) With obvious modifications, this proof holds also in the case where *both F and G are ultimately log-convex.*

e) Asume that F is *ultimately log-convex (g-convex) and G is ultimately log-concave (g-concave).* Then the function $\alpha(t) = \frac{F(t)}{G(t)}$, $t \in \mathbb{R}_+$, is a 1-periodic and ultimately log-convex (g-convex) function, this implies $\alpha(t) \equiv \alpha(1) = 1$ and then $F = G$.

f) Asume that F is *ultimately log-convex and G is ultimately g-concave.* From Proposition 4 we deduce that the function $\alpha(t) = \frac{F(t+1)}{F(t)}$, $t \in \mathbb{R}_+$, is ultimately increasing. There are two possibilities (both hold ultimately):

— $g(t) \geq 1$; hence $\frac{G(t+1)}{G(t)} \geq 1$, $t \in \mathbb{R}_+$, and, by Proposition 3, G is ultimately increasing. From Proposition 5-b) we deduce that G is ultimately log-concave. Then we get back into the previous case e).

— $g(t) \leq 1$; the special convexity assumptions on F and G yield (2.2) and

$$\begin{aligned}\frac{\log G(n+1) - \log G(n)}{\log(n+1) - \log n} &\geq \frac{\log G(n+x) - \log G(n)}{\log(n+x) - \log n}\\ &\geq \frac{\log G(n+2) - \log G(n+1)}{\log(n+2) - \log(n+1)}.\end{aligned}$$

From these inequalities we get

$$\begin{aligned}x \cdot \log \frac{F(n+1)}{F(n)} - \frac{\log \frac{n+x}{n}}{\log \frac{n+1}{n}} \log \frac{G(n+1)}{G(n)} &\leq \log \alpha(x)\\ \leq x \cdot \log \frac{F(n+2)}{F(n+1)} - \frac{\log \frac{n+x}{n}}{\log \frac{n+2}{n+1}} \log \frac{G(n+2)}{G(n+1)}&,\end{aligned}$$

which, by equation (0.1), becomes

$$\left[x - \frac{\log \frac{n+x}{n}}{\log \frac{n+1}{n}}\right] \cdot \log g(n) \leq \log \alpha(x) \leq \left[x - \frac{\log \frac{n+x}{n}}{\log \frac{n+2}{n+1}}\right] \cdot \log g(n+1).$$

As both the quantities between $[\cdot]$ vanish for $n \to +\infty$ and as $g(n)$ is ultimately increasing and upper-bounded, we get again $F = G$.

g) With suitable slight modifications the same proof can be supplied in the case: *F ultimately log-concave and G ultimately g-convex.*

h) Assume F *is ultimately log-convex and G ultimately g-convex*; with the same considerations as at the beginning of the case f), we deduce that g is ultimately increasing, and then we have two possibilities (both ultimately) again:

— $g(t) \geq 1$; therefore F is ultimately increasing and, via Proposition 5-a), g-convex: this is the case we just dealt with above in a);

— $g(t) \leq 1$; therefore G is decreasing and, via Proposition 5-b), log-convex: but this is the case we just dealt with above in d).

i) The proof of the remaining case: *F ultimately log-concave and G ultimately g-concave* is an obvious variation of that given for the preceding case h).

REMARK 1

This uniqueness result is an improvement of the ones in Theorems 1 and 3. There exist indeed ultimately *g-concave (g-convex)* functions F which are neither ultimately *log-concave* nor ultimately *log-convex* and which are solutions of some functional equation (0.1) with g fulfilling equation (0.3) and neither *log-concave* nor *log-convex.*

For example: for a fixed $\alpha > 0$ define

$$F(x) := \begin{cases} \exp\left[2+\alpha+\dfrac{(2+\alpha+\sin\log x)(1-x)}{\log x}\right] & \text{for } 0 < x \neq 1, \\ 1 & \text{for } x = 1; \end{cases}$$

and

$$g(x) := \frac{F(x+1)}{F(x)}.$$

Because of $\lim_{x\to+\infty} g(x) = 1$, (0.3) is fulfilled. To prove the remaining statements we consider the second derivatives of $\log F(e^x)$ (see Proposition 1) and of $\log F(x)$. After suitable calculations, we get

$$[\log F(e^x)]'' = -\frac{e^x}{x}\left\{\left[\frac{x-2}{x^2}(2+\alpha+\sin x)+\frac{\cos x}{x}+\sin x\right]e^{-x}\right. \\ \left.+2\frac{1-x}{x^2}(2+\alpha+\sin x)-2\frac{\cos x}{x}+(2+\alpha+2\cos x)\right\}$$

and

$$[\log F(x)]'' = -\frac{1}{x\log x}\left[\frac{2(x+1)+(1-x)\log x}{x\log^2 x}(2+\alpha+\sin\log x)\right. \\ -\frac{x+1}{x\log x}2\cos\log x-\frac{\sin\log x+\cos\log x}{x} \\ \left.+\sin\log x-\cos\log x\right].$$

Hence ultimately $[\log F(e^x)]'' < -\frac{\alpha e^x}{2x} < 0$, whereas $[\log F(x)]''$ does not have the same sign in any neighbourhood of $+\infty$. Then, from Theorem 4 it follows that F is the unique ultimately g-convex solution of equation (0.1) in the present case, and that this equation does not admit neither ultimately log-convex nor ultimately log-concave solutions. Hence, from the existence statement of Theorem 1, it follows that the function g is neither ultimately log-convex nor ultimately log-concave.

3. About existence results

The example in the above Remark 1 shows, among others, the existence of functional equations (0.1) with g-concave (g-convex) solutions F and with function g neither log-concave nor log-convex. This suggest to look for existence theorems for g-convex or g-concave solutions with assumptions e.g. of g-concavity (g-convexity) of g. However, if the function g is supposed to be only *g-concave*, the solution F can be either *g-concave* (of the *same type*) or *g-convex* (of the *reverse type*) as it will be subsequently showed via examples and a theorem. This does not happen in cases of Theorems 1 and 3 in which the *expected* solutions are *a priori* of the *reverse type* with respect to g (except of the trivial cases: $g(x) = \exp\{c\}$, $F(x) = k\exp\{cx\}$, c, k — constants). For, consider the case of Theorem 1: from log-concavity of g and (0.3), by Proposition 4 we get that $\frac{g(x+1)}{g(x)} \geq 1$; from equation (0.1) it easily follows that the ratio $\frac{F(x+1)}{F(x)}$ cannot be decreasing and then by Proposition 4 any solution F cannot be log-concave. Hence any *expected* solutions of (0.1), with some logarithmical convexity property, must be log-convex.

As regards the *expected* solutions of Theorem 3, we can do an analogous reasoning, holding in a suitable neighbourhood of $+\infty$: from assumptions on g we get

$$\frac{F(x+1)}{F(x)} = g(x) \geq 1 \quad \text{and} \quad \frac{g(x+1)}{g(x)} \geq 1,$$

and by equation (0.1)

$$\begin{aligned}\left[\frac{F(x+2)}{F(x+1)}\right]^{\frac{1}{\log\frac{x+2}{x+1}}} &\geq \left[\frac{F(x+2)}{F(x+1)}\right]^{\frac{1}{\log\frac{x+1}{x}}} \\ &= \left[\frac{g(x+1)}{g(x)}\right]^{\frac{1}{\log\frac{x+1}{x}}} \cdot \left[\frac{F(x+1)}{F(x)}\right]^{\frac{1}{\log\frac{x+1}{x}}} \\ &\geq \left[\frac{F(x+1)}{F(x)}\right]^{\frac{1}{\log\frac{x+1}{x}}},\end{aligned}$$

then from Proposition 4, as above, we conclude that any *expected* solution of (0.1), with some geometrical convexity property, must be g-convex.

Now we can deal with the aforesaid case of geometrical convexity both for g and for F. As an example of *same type* case, consider the *g-convex* function $g(x) := \exp\{\sqrt{x+1} - \sqrt{x}\}$ and the function $F(x) := \exp\{\sqrt{x} - 1\}$. Here F is the (*unique*) *g-convex* solution of equation (0.1) with $F(1) = 1$.

A *reverse type* case is dealt with in the following

THEOREM 5

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be g-concave, increasing and let $g(x) \geq 1$ in a neighbourhood of $+\infty$. Then equation (0.1) *admits exactly one ultimately g-convex solution F with $F(1) = 1$. This solution coincides with the unique normalized log-convex solution of the same equation.*

Proof. From Proposition 5-b) we deduce that g is ultimately log-concave and by Proposition 6 g fulfils the limit condition (0.3). Then, by Theorem 3, the existence claim follows, whereas the uniqueness one is attained via Theorem 4.

The following two couples of functions: $g(x) = x$, $F(x) = \Gamma(x)$, $x \in \mathbb{R}_+$ and $g(x) = \frac{1-q^x}{1-q}$, $F(x) = \Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$, $x \in \mathbb{R}_+$, $0 < q < 1$, with $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$, provide two examples for the *reverse type* case. The first one is motivated by Theorem 2 (but also by Theorem 5) and the second one from the fact that the function $g(x) = \frac{1-q^x}{1-q}$ fulfils the hypotheses of Theorem 5 and that in [3] the author gives, among others, the following result:

> *Let $0 < q < 1$. The function $F = \Gamma_q$ is the unique log-convex solution of the functional equation*
>
> $$F(x+1) = \frac{1-q^x}{1-q}F(x), \quad x \in \mathbb{R}_+ \tag{3.1}$$
>
> *with $F(1) = 1$.*

REMARK 2

In [6], Example 2, the function Γ_q is characterized as: *the unique normalized ultimately g-convex solution of equation* (3.1).

References

[1] J. Anastassiadis, *Définition des fonctions Eulériennes par des équations fonctionnelles*, Gauthier-Villars, Paris, 1964.

[2] E. Artin, *The Gamma-Function*, Holt, Rinehart and Winston, New York, 1964.

[3] R. Askey, *The q-gamma and the q-beta functions*, Applicable Anal. **8** (1978-79), 125-141.

[4] H. Bohr, J. Mollerup, *Lærebog i matematisk Analyse* III. Grænsenprocesser. Jul. Gjellerup, København, 1922.

[5] D. Gronau, J. Matkowski, *Geometrical convexity and generalization of the Bohr-Mollerup Theorem on the Gamma function*, Mathematica Pannonica, **4/2** (1993), 1-8.

[6] D. Gronau, J. Matkowski, *Geometrically convex solutions of certain difference equations and generalized Bohr-Mollerup Type Theorems*, Results Math. **26** (1994), 289-297.

[7] W. Krull, *Bemerkungen zur Differenzengleichung* $g(x+1) - g(x) = \varphi(x)$, Math. Nachr. **1** (1948), 365-376 and **2** (1949), 251-262.

[8] M. Kuczma, *Functional Equations in a Single Variable*, Polish Scientific Publishers, Warszawa, 1968.

[9] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Publishers, Warszawa – Kraków – Katowice, 1985.

Dipartimento di Matematica ed Applicazioni
Via Archirafi, 34
I-90123 Palermo
Italy
E-mail: cannizzo@dipmat.math.unipa.it