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The stability of the Wigner equation in complex Hilbert spaces

Abstract. We prove the Hyers-Ulam type stability of the Wigner equation for complex Hilbert spaces.

Let E_1 and E_2 be *complex* inner product spaces. Let $\langle\cdot|\cdot\rangle$ denotes the inner product and $\|\cdot\|$ the norm associated with it (we shall not distinguish between the symbols used for E_1 and E_2).
A mapping $T : E_1 \to E_2$ is called *isometric* if it is a solution of the *orthogonality equation*:

$$\langle T(x)|T(y)\rangle = \langle x|y\rangle \qquad \text{for } x, y \in E_1. \tag{1}$$

Such an isometric operator T has necessarily to be linear and injective (cf. [5], p. 125). If, additionally, it is surjective, then it is called *unitary* mapping (*unitary isomorphism*). By *conjugate-linear isometry* we mean a solution of the functional equation

$$\langle T(x)|T(y)\rangle = \langle y|x\rangle \qquad \text{for } x, y \in E_1. \tag{2}$$

In particular, conjugate-linear isometry T has the property: $T(\lambda x) = \bar{\lambda}T(x)$ for $x \in E_1$, $\lambda \in \mathbb{C}$.
In what follows we will use the term *phase-equivalent* functions to $f, g : E_1 \to E_2$ such that for each $x \in E_1$ there is $g(x) = \sigma(x)f(x)$ where σ is a function mapping E_1 into the unit circle S on the complex plane. A functional equation

$$|\langle T(x)|T(y)\rangle| = |\langle x|y\rangle| \qquad \text{for } x, y \in E_1 \tag{3}$$

with unknown function $T : E_1 \to E_2$ is called the *Wigner equation* (cf. [7], p. 251). For solutions of this equation we have

Theorem 1 (Wigner)
If $T : E_1 \to E_2$ satisfies (3), *then T is phase-equivalent to a linear or a conjugate-linear isometry $S : E_1 \to E_2$.*

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The detailed proof of this theorem as well as the complete bibliography may be found in a recent paper of J. Rätz [6] (cf., in particular, Corollary 8 there).

We start with a lemma that will be of use later.

LEMMA 1

If $T_1, T_2 : E_1 \to E_2$ *satisfy* (3) *and* $\|T_1(x) - T_2(x)\| \le \delta$ *for all* $x \in E_1$ *(with some* $\delta \ge 0$*), then* T_1 *and* T_2 *are phase-equivalent.*

Proof. On account of Theorem 1 there exist mappings $S_1, S_2 : E_1 \to E_2$ satisfying (independently) equations (1) or (2) as well as functions $\sigma_1, \sigma_2 : E_1 \to S$ such that:

$$T_1(x) = \sigma_1(x)S_1(x), \quad T_2(x) = \sigma_2(x)S_2(x) \quad \text{for } x \in E_1.$$

Thus we have ($\Re$ denotes the real part of a complex number)

$$\begin{aligned}\delta^2 &\ge \|T_1(x) - T_2(x)\|^2\\ &= \|\sigma_1(x)S_1(x) - \sigma_2(x)S_2(x)\|^2\\ &= \|S_1(x)\|^2 + \|S_2(x)\|^2 - 2\Re(\langle\sigma_1(x)S_1(x)|\sigma_2(x)S_2(x)\rangle)\\ &\ge 2\|x\|^2 - 2|\,\langle\sigma_1(x)S_1(x)|\sigma_2(x)S_2(x)\rangle\,|\\ &= 2\|x\|^2 - 2|\,\langle S_1(x)|S_2(x)\rangle\,|\end{aligned}$$

whence

$$\|x\|^2 - \frac{\delta^2}{2} \le |\,\langle S_1(x)|S_2(x)\rangle\,| \quad \text{for } x \in E_1. \tag{4}$$

Suppose that there exists an $x_0 \ne 0$ such that $S_1(x_0)$ and $S_2(x_0)$ are linearly independent. From the Schwarz inequality we would have

$$|\,\langle S_1(x_0)|S_2(x_0)\rangle\,| < \|S_1(x_0)\| \cdot \|S_2(x_0)\| = \|x_0\|^2$$

whence

$$|\,\langle S_1(x_0)|S_2(x_0)\rangle\,| = \xi\|x_0\|^2$$

for some $\xi \in [0; 1)$. Take an arbitrary $\lambda \in \mathbb{R}$; we have

$$\begin{aligned}|\,\langle S_1(\lambda x_0)|S_2(\lambda x_0)\rangle\,| &= |\,\langle \lambda S_1(x_0)|\lambda S_2(x_0)\rangle\,|\\ &= \lambda^2|\,\langle S_1(x_0)|S_2(x_0)\rangle\,|\\ &= \lambda^2\xi\|x_0\|^2\end{aligned}$$

which together with (4) yields

$$\|\lambda x_0\|^2 - \frac{\delta^2}{2} \le \lambda^2\xi\|x_0\|^2 \quad \text{for } \lambda \in \mathbb{R}$$

and then

$$(1-\xi)\lambda^2\|x_0\|^2 \leq \frac{\delta^2}{2} \quad \text{for } \lambda \in \mathbb{R}.$$

Letting $\lambda \to \infty$ we obtain a contradiction; thus $S_1(x)$ and $S_2(x)$ are linearly dependent for all $x \in E_1$.

Now suppose that there exists an $x_0 \in E_1$ such that

$$S_1(x_0) = \sigma S_2(x_0) \quad \text{and} \quad |\sigma| \neq 1.$$

Without loss of generality we asssume that $|\sigma| < 1$ (if $|\sigma| > 1$, then we have $S_2(x_0) = \sigma^{-1}S_1(x_0)$ and $|\sigma^{-1}| < 1$). For arbitrary $\lambda \in \mathbb{R}$ we have, according to (4),

$$\begin{aligned}\|\lambda x_0\|^2 - \frac{\delta^2}{2} &\leq |\langle S_1(\lambda x_0)|S_2(\lambda x_0)\rangle| = |\langle \lambda S_1(x_0)|S_2(\lambda x_0)\rangle| \\ &= |\langle \lambda\sigma S_2(x_0)|S_2(\lambda x_0)\rangle| = |\sigma||\langle S_2(\lambda x_0)|S_2(\lambda x_0)\rangle| \\ &= |\sigma|\,\|\lambda x_0\|^2\end{aligned}$$

whence

$$(1-|\sigma|)\lambda^2\|x_0\|^2 \leq \frac{\delta^2}{2} \quad \text{for } \lambda \in \mathbb{R}.$$

Letting $\lambda \to \infty$ we get a contradiction again. This means there exists a function $\sigma : E_1 \to S$ such that $S_1(x) = \sigma(x)S_2(x)$ for $x \in E_1$. Consequently we have

$$T_1(x) = \sigma_1(x)S_1(x) = \sigma_1(x)\sigma(x)S_2(x) = \frac{\sigma_1(x)\sigma(x)}{\sigma_2(x)}T_2(x)$$

and hence T_1 and T_2 are phase-equivalent.

Apart from the class of solutions of the Wigner equation one can also consider a class of *approximate solutions* of this equation defined as the class of all solutions of the functional inequality

$$\big|\,|\langle T(x)|T(y)\rangle| - |\langle x|y\rangle|\,\big| \leq \varepsilon \quad \text{for } x, y \in E_1 \tag{5}$$

with some nonnegative ε. Now one can ask a question whether the Wigner equation (3) is stable with respect to the class of its approximate solutions (5). The definition of this class as well as the stability problem itself, follow the results of Hyers and Ulam — cf. [3], [4].

For T being a solution of (5) let us consider the *Hyers' sequence* T_n, i.e., a functional sequence given by

$$T_n(x) := 2^{-n}T(2^n x) \quad \text{for } n \in \mathbb{N},\ x \in E_1.$$

Putting $2^n x$ in place of x and x in place of y in inequality (5) one obtains easily

$$\|x\|^2 - \frac{\varepsilon}{2^n} \le |\langle T_n(x)|T(x)\rangle| \le \|x\|^2 + \frac{\varepsilon}{2^n}. \tag{6}$$

Similarly, putting $2^n x$ in place of x and y in inequality (5) one gets

$$\|x\|^2 - \frac{\varepsilon}{4^n} \le \|T_n(x)\|^2 \le \|x\|^2 + \frac{\varepsilon}{4^n}. \tag{7}$$

Unlike the Hyers' proof of the stability of the Cauchy equation in the present case the sequence $T_n(x)$ need not be convergent. Namely, one can consider an arbitrary Hilbert space E, an element $0 \ne x_0 \in E$ and a mapping $T : E \to E$ defined as follows

$$T(x) := \sigma(x) \cdot x \quad \text{for } x \in E$$

where

$$\sigma(x) = 1 \quad \text{for } x \notin \{x_0, 2x_0, 4x_0, 8x_0 \ldots\} \quad \text{and} \quad \sigma(2^n x_0) := (-1)^n \quad \text{for } n \in \mathbb{N}.$$

Such a T satisfies (3) but $(T_n(x_0))$ diverges.

LEMMA 2

For an arbitrary element $x \in E_1$ and for an arbitrary sequence $(l_n(x))$ of integers there exists a convergent sequence $(\sigma_n(x))$ of complex numbers from the unit circle $\mathcal{S}$ and a subsequence $(k_n(x))$ of the sequence $(l_n(x))$ such that

$$\langle \sigma_n(x) T_{k_n(x)}(x)|T(x)\rangle \ge \|x\|^2 - \frac{\varepsilon}{2^{k_n(x)}}. \tag{8}$$

Proof. We choose $\sigma_{l_n}(x) \in \mathcal{S}$ such that

$$\sigma_{l_n(x)} \langle T_{l_n(x)}(x)|T(x)\rangle \in \mathbb{R}_+$$

(for $z \in \mathbb{C} \setminus \{0\}$ and $\sigma := e^{-i \arg z}$, $\sigma z \in \mathbb{R}_+$).
As $\mathcal{S}$ is compact in $\mathbb{C}$ one can find a covergent subsequence $\sigma_{k_n(x)} =: \sigma_n(x)$. Thus we have

$$\langle \sigma_n(x) T_{k_n(x)}(x)|T(x)\rangle \in \mathbb{R}_+$$

which together with (6) gives us

$$\begin{aligned}\langle \sigma_n(x) T_{k_n(x)}(x)|T(x)\rangle &= |\langle \sigma_n(x) T_{k_n(x)}(x)|T(x)\rangle| \\ &= |\langle T_{k_n(x)}(x)|T(x)\rangle| \\ &\ge \|x\|^2 - \frac{\varepsilon}{2^{k_n(x)}}\end{aligned}$$

which completes the proof of the lemma.

The Hyers-Ulam type stability of the Wigner equation for a *real* Hilbert space has been proved in [1]; for Euclidean space $\mathbb{R}^n$ a stronger result, *superstability*, has been proved — cf. [2]. Now we are ready to prove the stability in the complex case.

THEOREM 2 (Stability)

Let E_1 be a complex inner product space and let E_2 be a complex Hilbert space. Then, for $T : E_1 \to E_2$ being a solution of (5) *there exists $T_* : E_1 \to E_2$ — a solution of* (3) *such that*

$$\|T(x) - T_*(x)\| \leq \sqrt{\varepsilon} \quad \text{for } x \in E_1.$$

Moreover, such a function T_* is unique up to phase-equivalence.

Proof. Inequality (7) implies that for an arbitrary $x \in E_1$ the sequence $(T_n(x))$ is bounded. Thus, there exists (cf. [5], p. 149, Theorem 2) a subsequence $\big(T_{l_n(x)}(x)\big)$ of $(T_n(x))$ weakly convergent in E_2. For this subsequence, due to Lemma 2, we can choose a sub-subsequence $\big(T_{k_n(x)}(x)\big)$ and suitable convergent sequence $(\sigma_n(x))$ such that (8) holds.
Let

$$T_*(x) := \operatorname*{w-lim}_{n\to\infty} \sigma_n(x) T_{k_n(x)}(x).$$

Using inequality (5) we can estimate, for any $x, y \in E_1$ and $m, n \in \mathbb{N}$,

$$\Big|\, \big|\big\langle \sigma_m(x)T_{k_m(x)}(x) \big| \sigma_n(y)T_{k_n(y)}(y) \big\rangle\big| - |\,\langle x|y\rangle\,| \,\Big| \leq 2^{-(k_m(x)+k_n(y))}\varepsilon.$$

We have

$$\big\langle\, . \,|\sigma_n(y)T_{k_n(y)}(y)\big\rangle \in E_2^* \quad \text{for } y \in E_1 \text{ and } n \in \mathbb{N} \text{ being fixed}$$

whence, on letting $m \to \infty$,

$$\big|\big\langle T_*(x) | \sigma_n(y)T_{k_n(y)}(y) \big\rangle\big| = |\langle x|y\rangle| \quad \text{for } x, y \in E_1,\ n \in \mathbb{N}$$

and also

$$\big|\big\langle \sigma_n(y)T_{k_n(y)}(y) | T_*(x) \big\rangle\big| = |\langle x|y\rangle| \quad \text{for } x, y \in E_1,\ n \in \mathbb{N}.$$

Similarly, for fixed $x \in E_1$, the functional $\langle\, . \,|T_*(x)\rangle$ belongs to the dual space E_2^* whence, on letting $n \to \infty$,

$$|\langle T_*(y)|T_*(x)\rangle| = |\langle x|y\rangle| \quad \text{for } x, y \in E_1,$$

i.e., T_* satisfies the Wigner equation.

For an arbitrary $x \in E_1$ from (8) we have

$$-2\Re\left(\left\langle \sigma_n(x)T_{k_n(x)}(x)|T(x)\right\rangle\right) = -2\left\langle \sigma_n(x)T_{k_n(x)}(x)|T(x)\right\rangle \leq -2\|x\|^2 + \frac{2\varepsilon}{2^{k_n(x)}}.$$

From (7) we obtain

$$\|\sigma_n(x)T_{k_n(x)}(x)\|^2 = \|T_{k_n(x)}(x)\|^2 \leq \|x\|^2 + \frac{\varepsilon}{4^{k_n(x)}}$$

and finally, from (5),

$$\|T(x)\|^2 \leq \|x\|^2 + \varepsilon.$$

Making use of the last three inequalities we get

$$\begin{aligned}\|\sigma_n(x)T_{k_n(x)}(x) - T(x)\|^2 &= \|\sigma_n(x)T_{k_n(x)}(x)\|^2 + \|T(x)\|^2 \\ &\quad -2\Re\left(\left\langle \sigma_n(x)T_{k_n(x)}(x)|T(x)\right\rangle\right) \\ &\leq \varepsilon + \frac{\varepsilon}{4^{k_n(x)}} + \frac{2\varepsilon}{2^{k_n(x)}}.\end{aligned}$$

Now, let us fix $x \in E_1$. If $T_*(x) \neq T(x)$ we define

$$\varphi := \left\langle \cdot \left| \frac{T_*(x) - T(x)}{\|T_*(x) - T(x)\|} \right.\right\rangle \in E_2^*$$

with $\|\varphi\| = 1$. Thus we have

$$\begin{aligned}\left|\varphi\left(\sigma_n(x)T_{k_n(x)}(x) - T(x)\right)\right| &\leq \|\sigma_n(x)T_{k_n(x)}(x) - T(x)\| \\ &\leq \sqrt{\varepsilon}\cdot\sqrt{1 + \frac{1}{4^{k_n(x)}} + \frac{2}{2^{k_n(x)}}}\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get

$$\|T_*(x) - T(x)\| \leq \sqrt{\varepsilon}.$$

Finally, note that the second part of the assertion follows directly from the Lemma 1 because any two functions T_* and T_*' satisfying the assertion of the theorem are solutions of the Wigner equation (3) such that $\|T_*(x) - T_*'(x)\| \leq 2\sqrt{\varepsilon}$ for all $x \in E_1$.

REMARK

The constant $\sqrt{\varepsilon}$ is the best possible. Indeed, for $E_1 = E_2 = l^2$ and for the mapping $T : l^2 \to l^2$ defined by $T(t_1, t_2, t_3, \ldots) = (\sqrt{\varepsilon}, t_1, t_2, t_3, \ldots)$ we have: T satisfies (5) and for any T_* satisfying the Wigner equation (3) we have $T_*(0) = 0$ and hence $\|T(0) - T_*(0)\|_{l^2} = \sqrt{\varepsilon}$. Thus $\sqrt{\varepsilon}$ is an optimal constant.

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