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Functional inequalities for infinite series

Abstract. The object of this paper is to consider a general functional inequality from which we obtain the earlier results. The positive sequences $\{x_n\}$ with $\sum_{i=n+1}^{\infty} x_i \le Mx_n$ $(n \in \mathbb{N})$ play a fundamental role in our investigations and therefore we also present a characterization theorem for these sequences.

1. In the paper [4] we generalized an inequality of J.-P. Allouche, M. Mendés France and G. Tenenbaum [2]. (This result was a correct generalization; see an incorrect one in [3].) The object of this paper is to give a very general functional inequality from which we obtain the earlier results as particular cases.

Theorem 1

Let $\{x_n\}$ be a positive sequence and let M be a positive real number. We denote by $\mathcal{F}$ the set of all functions $f :]0,\infty[\to \mathbb{R}$ for which the following conditions are true:

(i) $\lim_{x\to 0+} f(x) = 0$;

(ii) *f is strictly increasing ($\uparrow$), f' is strictly deacreasing ($\downarrow$), and f'' is strictly increasing ($\uparrow$) on $]0,\infty[$.*

If

$$r_{n+1} := \sum_{i=n+1}^{\infty} x_i \le Mx_n \quad \text{for all } n \in \mathbb{N}, \tag{1}$$

then for any $f \in \mathcal{F}$ we have

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$$f\left(\sum_{n=1}^{\infty} x_n\right) \geq \sum_{n=1}^{\infty} [f((M+1)x_n) - f(Mx_n)] \\ + \sum_{n=1}^{\infty} (Mx_n - r_{n+1}) \left[f'(Mx_n) - f'((M+1)x_n)\right], \tag{2}$$

where the series on the right hand side of (2) *are convergent and have non-negative terms. Equality holds in* (2) *iff*

$$x_n = \left(\frac{M}{M+1}\right)^{n-1} x_1 \qquad (n \in \mathbb{N}). \tag{3}$$

Proof. Let u be a fixed real number. The function

$$g_u(x) := f(x+u) - f(x)$$

is strictly convex on $]0,\infty[$ because $g_u''(x) = f''(x+u) - f''(x) > 0$ for all $x > 0$. This implies

$$g_u(x) \geq g_u(y) - (x-y)g_u'(y) \tag{4}$$

for all $x, y > 0$. Equality holds in (4) if and only if $x = y$. Setting

$$x = r_{n+1}, \quad y = Mx_n \quad \text{and} \quad u = x_n$$

in (4), we obtain

$$f(r_n) - f(r_{n+1}) \geq f((M+1)x_n) - f(Mx_n) \\ +(Mx_n - r_{n+1}) \left[f'(Mx_n) - f'((M+1)x_n)\right] \tag{5}$$

for all $n \in \mathbb{N}$. Using the notation

$$A(f) := \sum_{n=1}^{\infty} [f((M+1)x_n) - f(Mx_n)]$$

and

$$B(f) := \sum_{n=1}^{\infty} (Mx_n - r_{n+1}) \left[f'(Mx_n) - f'((M+1)x_n)\right]$$

we infer by (1) and (ii) that the series $A(f)$ and $B(f)$ have nonnegative terms. Since due to (i) the series

$$\sum_{n=1}^{\infty} [f(r_n) - f(r_{n+1})] = f\left(\sum_{n=1}^{\infty} x_n\right)$$

is convergent and by (5) it majorizes both $A(f)$ and $B(f)$, therefore $A(f)$ and $B(f)$ are also convergent. Summing up the inequalities (5) for $n \in \mathbb{N}$, we have (2). Since equality holds in (4) if and only if $x = y$, we conclude that there is equality in (2) if and only if $r_{n+1} = Mx_n$ for all $n \in \mathbb{N}$, which is equivalent to (3).

COROLLARY 1

If the positive sequence $\{x_n\}$ *fulfils the condition* (1), *then we have for all* $c \in]0,1[$ *that*

$$\left(\sum_{n=1}^{\infty} x_n\right)^c \geq A(c) + B(c) \tag{6}$$

where the series

$$A(c) := [(M+1)^c - M^c] \sum_{n=1}^{\infty} x_n^c \tag{7}$$

and

$$B(c) := c\left[M^{c-1} - (M+1)^{c-1}\right] \sum_{n=1}^{\infty} (Mx_n - r_{n+1}) x_n^{c-1} \tag{8}$$

are convergent and have nonnegative terms. Equality holds in (6) *iff* (3) *is valid.*

REMARKS

(a) Corollary 1 is a trivial consequence of Theorem 1 with $f(x) = x^c$ ($x > 0$, $0 < c < 1$) (see [4]).

(b) From Corollary 1 it follows

$$\left(\sum_{n=1}^{\infty} x_n\right)^c \geq A(c)$$

which was proved in [2].

2. As an application ot Theorem 1 or Corollary 1 in the information theory we give a short proof of the following

THEOREM 2

Let $\mathcal{P} := \{p_k \mid p_k > 0, \sum_{n=1}^{\infty} p_k = 1\}$ *be an infinite probability distribution with the property*

$$M := \sup_{n \in \mathbb{N}} \left\{ \frac{1}{p_n} \sum_{i=n+1}^{\infty} p_i \right\} < \infty. \tag{9}$$

Then the Shannon entropy of $\mathcal{P}$

$$H(\mathcal{P}) := -\sum_{k=1}^{\infty} p_k \log p_k$$

exists and the inequality

$$H(\mathcal{P}) + \log\left(1 + \frac{1}{M}\right)\left(M - \sum_{n=1}^{\infty} P_{n+1}\right) \le F(M) \tag{10}$$

is true, where $F(x) := (x+1)\log(x+1) - x\log x$ $(x > 0)$ *and* $P_{n+1} := \sum_{k=n+1}^{\infty} p_k$ $(n \in \mathbb{N})$. *Equality holds in* (10) *iff* $p_k = \frac{M^{k-1}}{(M+1)^k}$ $(k \in \mathbb{N})$.

Proof. For $0 < c < 1$ we define $A(c)$ and $B(c)$ by (7) and (8), where $x_n = p_n$ $(n \in \mathbb{N})$. By Corollary 1, (6) is true which implies

$$0 < A(c) + B(c) \le 1 \quad \text{for all } 0 < c < 1.$$

From this inequality we obtain

$$\frac{1}{1-c}\log\sum_{n=1}^{\infty} p_n^c - \frac{1}{1-c}\log\left[1 - B(c)\right] \le \frac{1}{1-c}\log\frac{1}{(M+1)^c - M^c}. \tag{11}$$

The quantity $H_c(\mathcal{P}) := \frac{1}{1-c}\log\sum_{n=1}^{\infty} p_n^c$ is the well known Rényi entropy of order c $(c \ne 1)$ (see [7] and [1]) which has the limit-property that $\lim_{c\to 1} H_c(\mathcal{P}) = H(\mathcal{P})$. If we let c tend to 1 in (11), then by a direct computation we obtain the inequality (10), where by (9) $M - \sum_{n=1}^{\infty} P_{n+1} \ge 0$. Equality holds in (10) iff $p_k = \frac{M^{k-1}}{(M+1)^k}$ $(n \in \mathbb{N})$.

REMARK

The inequality (10) is an essential generalization of the inequality (3) in [2] (see [4]).

3. The positive sequences $\{x_n\}$ for which the property (1) holds play fundamental role in our investigations. In what follows we present a characterization theorem for these sequences which, in the monotone decreasing case, was first proved in [5]. This characterization theorem in such a general case has been formulated and verified in [6]. The proof which is given below uses the method from [5].

DEFINITION 1 (see [5])

The positive sequence $\{x_n\}$ is called *smooth* if there exists a constant $M > 0$ such that

$$\sum_{k=n+1}^{\infty} x_k \leq M x_n \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

DEFINITION 2 (see [6])

The positive sequence $\{x_n\}$ is called *quasi-geometrically decreasing* if there exists a $T \in \mathbb{N}$ and a $K > 0$ such that

$$x_{n+T} < \frac{1}{2} x_n \quad \text{and} \quad x_{n+1} < K x_n \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

THEOREM 3

A positive sequence $\{x_n\}$ is smooth if and only if it is quasi-geometrically decreasing.

Proof. (i) We suppose that $\{x_n\}$ is positive and smooth. Then (12) implies

$$x_{n+1} < M x_n \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the second inequality of (13) holds true with $K := M$. Furthermore, by (12) we have for any $t \in \mathbb{N}$ that

$$M x_n > \sum_{k=n+1}^{n+t} x_k \geq t \frac{1}{M} x_{n+t} \tag{14}$$

and if $T := t > 2M^2$, then (14) yields that

$$\frac{1}{2} x_n > x_{n+T}$$

for any $n \in \mathbb{N}$. Herewith we have proved the first inequality of (13), i.e., $\{x_n\}$ is quasi-geometrically decreasing.

(ii) Suppose that $\{x_n\}$ is a positive and quasi-geometrically decreasing sequence. From (13) it follows

$$x_{n+kT} < \frac{1}{2^k} x_n \tag{15}$$

and

$$x_{n+l} < K^l x_n \tag{16}$$

for any natural numbers k, l and n. Hence, by (15) and (16)

$$x_{n+1} + x_{n+2} + \cdots + x_{n+T} < (K + K^2 + \cdots + K^T) x_n \,,$$
$$x_{n+T+1} + x_{n+T+2} + \cdots + x_{n+2T} < \frac{1}{2}(K + K^2 + \cdots + K^T) x_n \,,$$
$$x_{n+2T+1} + x_{n+2T+2} + \cdots + x_{n+3T} < \frac{1}{2^2}(K + K^2 + \cdots + K^T) x_n \,,$$
$$\cdots \qquad \cdots \qquad \cdots$$

These inequalities imply

$$\sum_{k=n+1}^{\infty} x_k < \left(\sum_{i=1}^{T} K^i\right)\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)x_n = 2\left(\sum_{i=1}^{T} K^i\right)x_n\,,$$

i.e., (12) holds for $M := 2\sum_{i=1}^{T} K^i$ $(K > 0)$.

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