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On the functional equation of exit laws for lattice semigroups

Abstract. Let $\mathbb{P} = (P_t)_{t>0}$ be a semigroup of kernels on an LCD space $(X, \mathcal{B})$ such that $V := \int_0^\infty P_t\,dt$ exists. An exit law for $\mathbb{P}$ is a family of positive measurable functions $f = (f_t)_{t>0}$ which verifies the following functional equation: $P_s f_t = f_{s+t}$, for all $s, t > 0$. Let $\mathcal{R} := \{u \in \mathcal{F} : u = \int_0^\infty f_t\,dt$ for some exit law f for $\mathbb{P}\}$ and $\operatorname{Im} V := \{Vu : u \in \mathcal{B}\}$. In general we have $\operatorname{Im} V \subset \mathcal{R}$.

If $\mathbb{P}$ is a lattice and submarkovian semigroup then we prove in this paper the equality $\operatorname{Im} V = \mathcal{R}$. For this purpose, we associate a semi-dynamical system to $\mathbb{P}$. Moreover, we give an example of not lattice semigroup for which $\operatorname{Im} V = \mathcal{R}$.

1. Introduction

Let X be a locally compact space with countable base, and let $\mathcal{B} = \mathcal{B}(X)$ be the set of all positive and real Borel functions on X (which contains the Borel subsets of X). The product space $[0, +\infty[\, \times X$ is endowed with the product topology. Moreover, we use the notations $f_t(x) := f(t, x)$ and $f = (f_t)_{t \geq 0}$, for every function f which is defined on $[0, +\infty[\, \times X$. Some of the following notions given in this paragraph are familiar in potential theory, see for example [2], [3]. However, no knowledge of potential theory is presupposed in this paper.

A *kernel* K on the measurable space $(X, \mathcal{B})$ is an additive and positively homogeneous mapping $K : \mathcal{B} \to \mathcal{B}$ such that $K(\sup u_n) = \sup K(u_n)$ for every increasing sequence $(u_n) \subset \mathcal{B}$. In particular $A \mapsto K(x, A) := K1_A(x)$ is a measure on $(X, \mathcal{B})$ for every $x \in X$. We say that a kernel K is *submarkovian* if $K1 \leq 1$. If $x \mapsto K(x, A)$ is bounded for every compact set $A \subset X$ then K is called *proper*.

Let $\mathbb{P} = (P_t)_{t \geq 0}$ be a measurable *semigroup* of submarkovian kernels on $(X, \mathcal{B})$. The associated *resolvent* is the family of kernels $\mathcal{V} := (V_\alpha)_{\alpha > 0}$ defined

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by $V_\alpha := \int_0^\infty \exp(-\alpha t) P_t \, dt$. (Note that kernels V_α are proper — they are even bounded — because $\mathbb{P}$ is submarkovian).

A measurable subset A of X is called *$\mathcal{V}$-negligible* if $V_\alpha 1_A = 0$, for every $\alpha > 0$. We say that a property holds *$\mathcal{V}$-a.e.* if the set for which this property does not hold is $\mathcal{V}$-negligible. We set

$$\mathcal{F} := \{u : X \to [0, \infty] \text{ such that } u \text{ is measurable and } u < \infty, \ \mathcal{V}\text{-a.e.}\}.$$

Set $\mathcal{E} := \{u \in \mathcal{F} : \ P_t u \leq u \text{ and } \sup_{t \to 0} P_t u = u\}$.

The *potential cone* associated to $\mathbb{P}$ is the cone of functions defined by

$$\mathcal{P} := \{u \in \mathcal{E} : \ \inf_{t \to \infty} P_t u = 0, \ \mathcal{V}\text{-a.e.}\}.$$

The *potential kernel* associated to $\mathbb{P}$ is given by $V := \int_0^\infty P_t \, dt$. If V is proper then it is clear that $\operatorname{Im} V := \{u \in \mathcal{F} : \ u = V\phi, \ \phi \in \mathcal{F}\}$ is a subset of $\mathcal{P}$ and $A \in \mathcal{B}$ is $\mathcal{V}$-negligible iff $V 1_A = 0$ because $V = \sup_{\alpha > 0} V_\alpha$.

EXAMPLE 1

Translation semigroup on $\mathbb{R}$: Let $X = \mathbb{R}$ and $\mathbb{P}$ defined by $P_t u(x) = u(x+t)$, for all $x \in \mathbb{R}$, $t \geq 0$ and $u \in \mathcal{B}$. For this example the $\mathcal{V}$-negligible sets are exactly the negligible sets for the Lebesgue measure on $\mathbb{R}$. On the other hand $\mathcal{P}$ is the cone of all positive functions on $\mathbb{R}$ which are decreasing, right-continuous and vanishing at $+\infty$. Moreover, the potential kernel V is proper ($Vu(x) = \int_x^\infty u(y) \, dy$). Thus all elements of $\operatorname{Im} V$ are absolutely continuous functions and consequently $\operatorname{Im} V \neq \mathcal{P}$.

EXAMPLE 2

Brownian semigroup on $\mathbb{R}^n$: Let $X = \mathbb{R}^n$ ($n \geq 1$) and $\mathbb{P}$ defined by

$$P_t u(x) = \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}} \int \exp\left(-\frac{\|x-y\|^2}{2t}\right) u(y) \, dy \tag{1}$$

for all $x \in \mathbb{R}^n$, $t \geq 0$ and $u \in \mathcal{B}$. Then $\mathcal{P}$ is the cone of all positive functions u on $\mathbb{R}^n$ which are superharmonic and for which the constant zero is the only positive harmonic minorant. The potential kernel V is proper iff $n \geq 3$.

Let $n \geq 3$ and let $N(x,y) = \|x-y\|^{2-n}$ be the well known Newton kernel on $\mathbb{R}^n$, then it is easy to see that

$$\forall \, x, y \in \mathbb{R}^n : \ N(x,y) = \int_0^\infty \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}} \exp\left(-\frac{\|x-y\|^2}{2t}\right) dt, \tag{2}$$

$$\forall \, x \in \mathbb{R}^n ; \ \forall \, u \in \mathcal{B} : \ Vu(x) = \int N(x,y) u(y) \, dy. \tag{3}$$

It can be also shown by calculation (on (1) and (2)) that $N(.,y) \in \mathcal{P} \setminus \operatorname{Im} V$ for all $y \in \mathbb{R}^n$. In view of (3), it is clear that the $\mathcal{V}$-negligible sets are the negligible sets for the Lebesgue measure on $\mathbb{R}^n$.

For more details about this two examples see [3] (Chap. 0, I, III and V).

A positive measurable function $f = (f_t)_{t>0}$ defined on $]0,\infty[\times X$ is called an *exit law* for $\mathbb{P}$ if

$$\forall\ s,t > 0 :\ P_s f_t = f_{s+t}\,. \tag{4}$$

This notion plays an important role in potential theory (see [3], pp 38-52). Let

$$\mathcal{R} := \left\{ u \in \mathcal{F} :\ u = \int_0^\infty f_t\,dt, \text{ for some exit law } f = (f_t)_{t>0} \text{ for } \mathbb{P} \right\}.$$

Let $u \in \mathcal{R}$, then the Fubini theorem and (4) imply that

$$\forall\ t > 0 :\ P_t u = \int_t^\infty f_s\,ds. \tag{5}$$

REMARKS

Let $x \in X$, by (5) the function $t \mapsto f_t(x)$ appears as the derivative of $t \mapsto -P_t u(x)$. So the important problem concerning the functional equation (4) is to determine the cone $\mathcal{R}$ for a given semigroup of kernels $\mathbb{P}$.

Suppose that $V := \int_0^\infty P_t\,dt$ is proper. Then, in view of (5), we have $\operatorname{Im} V \subset \mathcal{R} \subset \mathcal{P}$. In order to study $\mathcal{R}$, it is also a natural question to compare $\mathcal{R}$ with $\operatorname{Im} V$ or with $\mathcal{P}$.

If we suppose the existence of a reference measure i.e. a positive σ-finite measure m on $(X,\mathcal{B})$ such that $m(A) = 0$ iff $V1_A = 0$ for all $A \in \mathcal{B}$. Then we have proved in [7] that $\mathcal{R} = \mathcal{P}$ iff $\mathbb{P}$ is absolutely continuous with respect to m. This is the case of the semigroup of the Brownian motion on $\mathbb{R}^n$, but not for translation semigroup on $\mathbb{R}$.

Moreover, the existence of a reference measure is not necessary to have $\mathcal{R} = \mathcal{P}$. For example, it is shown in [4] that if $\mathbb{P}$ is symmetric with respect to a (not necessarily reference) measure then we have $\mathcal{R} = \mathcal{P}$.

The aim of this paper is to study the other extreme case, namely semigroups for which $\mathcal{R} = \operatorname{Im} V$.

2. Exit laws for lattice semigroups

A *semidynamical system* (SDS) on $(X,\mathcal{B})$ is a mapping $\Phi : \mathbb{R}^+ \times X \supset \Omega \to X$ which satisfies

1. Ω and Φ are measurable,

2. for every $x \in X$, the set $I(x) := \{t \geq 0 : (t,x) \in \Omega\}$ is an interval which contains 0,
3. $\Phi(0,x) = x$ for every $x \in X$,
4. if $(t,x) \in \Omega$ and $(s,\Phi(t,x)) \in \Omega$ then $(s+t,x) \in \Omega$ and $\Phi(s+t,x) = \Phi(s,\Phi(t,x))$.

Such a system is denoted by the pair (Ω,Φ) and if $\Omega \neq \mathbb{R}^+ \times X$ it is called *local* SDS. These definitions are adapted from [1].

Let (Ω,Φ) be an SDS on $(X,\mathcal{B})$. A *cocycle* of (Ω,Φ) is a measurable mapping $C : \Omega \to]0,\infty[$ which verifies

$$\forall\ (t,x) \in \Omega\,;\ \forall\ (s,\Phi(t,x)) \in \Omega\ :\ C(s+t,x) = C(t,x)\ \cdot\ C(s,\Phi(t,x)). \tag{6}$$

(The functional equation (6) has been studied in my papers [5], [6]).

Let (Ω,Φ) be an SDS on $(X,\mathcal{B})$ and C be a cocycle of (Ω,Φ). Then the family of operators $\mathbb{P} = (P_t)$ defined by

$$\forall\ u \in \mathcal{B}\,;\ x \in X;\ t \geq 0\ :\ P_t u(x) := 1_\Omega(t,x)\ \cdot\ C(t,x)\ \cdot\ u(\Phi(t,x)) \tag{7}$$

is a measurable semigroup of (not necessarily submarkovian) kernels on $(X,\mathcal{B})$. Moreover, $\mathbb{P}$ is also a *lattice* semigroup, i.e. $P_t|u| = |P_t u|$ for every $t \geq 0$ and for every measurable bounded function u. In fact, the converse is also true (see [5]):

PROPOSITION

Let $\mathbb{P}$ be a measurable lattice semigroup of kernels on $(X,\mathcal{B})$, then there exists a unique SDS (Ω,Φ) and a unique cocycle C of (Ω,Φ) such that $\mathbb{P}$ is of the form (7).

Proof. Let $\mathbb{P}$ be a measurable lattice semigroup of kernels on $(X,\mathcal{B})$. Then, using the fact that $\mathbb{P}$ is a lattice, we have

$$\forall\ (t,x) \in \mathbb{R}^+ \times X\,;\ \forall\ A \in \mathcal{B}\ :\ P_t 1(x) = \left|P_t 1_A(x) - P_t 1_{X\setminus A}(x)\right|. \tag{8}$$

Set $\Omega := \{(t,x) \in \mathbb{R}^+ \times X\ :\ P_t 1(x) \neq 0\}$ and $C(t,x) := P_t 1(x)$ for every $(t,x) \in \Omega$, then Ω and C are measurable. Moreover, if $P_t 1(x) = 0$ then the semigroup property implies that $P_s 1(x) = 0$ for all $s \geq t$, hence $I(x) := \{t \geq 0\ :\ (t,x) \in \Omega\}$ is an interval.

Now, let $(t,x) \in \Omega$, in view of (8) and the separability of the space $(X,\mathcal{B})$, we have

$$\forall\ (t,x) \in \Omega\,,\ \exists!\ \Phi(t,x) \in X\ :\ \varepsilon_x P_t = C(t,x)\ \cdot\ \varepsilon_{\Phi(t,x)}\,. \tag{9}$$

In particular, (9) defines a mapping $\Phi : \Omega \to X$ which is measurable because

$$\forall\ A \in \mathcal{B} \ : \ \Phi^{-1}(A) = \{(t,x) \in \Omega : \ \frac{P_t 1_A(x)}{P_t 1(x)} = 1\}.$$

Since P_0 is the identity, it follows that $0 \in I(x)$ and $\Phi(0,x) = x$. Furthermore, from the semigroup property and (8), we have the formula (6) and the relation 4. in the definition of a semidynamical system. This shows that (Ω, Φ) is an SDS on $(X, \mathcal{B})$ and C is an associated cocycle.

EXAMPLE 3

Let $\mathbb{P}$ be a measurable semigroup of kernels on $(X, \mathcal{B})$ which is *stable by product*, i.e. $P_t(uv) = P_t u \cdot P_t v$ for all $t \geq 0$ and $u \in \mathcal{B}$. Then $\mathbb{P}$ is a lattice semigroup of submarkovian kernels.

The proof is similar to the proof of the Proposition: The product stability implies that $(P_t 1)^2 = P_t 1$ for every $t \geq 0$, thus $\mathbb{P}$ is submarkovian. Set $\Omega := \{(t,x) \in \mathbb{R}^+ \times X \ : \ P_t 1(x) \neq 0\}$ and let $(t,x) \in \Omega$, then in view of the product stability and the separability of the space $(X, \mathcal{B})$ the support of the measure $P_t(x,.)$ cannot contain two different points of X. Therefore $P_t(x,.)$ is a Dirac measure, but this implies that $\mathbb{P}$ is a lattice semigroup.

THEOREM

Let $\mathbb{P}$ be a measurable lattice semigroup of submarkovian kernels on $(X, \mathcal{B})$ and let $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ the associated resolvent. Then for every measurable exit law $f = (f_t)_{t>0}$ for $\mathbb{P}$ there is some $\varphi \in \mathcal{F}$ such that

$$\forall\ \alpha > 0 ; \ x \in X \ : \ h_\alpha(x) := \int_0^\infty \exp(-\alpha t) f_t(x)\, dt = V_\alpha \varphi(x). \tag{10}$$

Proof. Let $\alpha > 0$ and set $\mathbb{P}^\alpha = (P_t^\alpha)$ where $P_t^\alpha := \exp(-\alpha t) P_t$ for every $t \geq 0$. Then $\mathbb{P}^\alpha$ is a measurable semigroup of submarkovian kernels; moreover, the potential kernel of $\mathbb{P}^\alpha$ is proper, it is equal to V_α. From (5) and the fact that $f^\alpha := (\exp(-\alpha t) f_t)$ is an exit law for $\mathbb{P}^\alpha$, it is clear that h_α defined as in (10) is a potential for $\mathbb{P}^\alpha$. So if we set

$$\varphi_\alpha(x) := D^\alpha h_\alpha(x) := \limsup_{t \to 0} \frac{h_\alpha(x) - P_t^\alpha h_\alpha(x)}{t} \tag{11}$$

for all $x \in X$, then φ_α is positive and measurable because we can take the lim sup in (11) for $t \in \mathbb{Q}$. Now let $s, t > 0$ and $x \in X$, from (5) and the definition in (10) we have

$$P_t^\alpha h_\alpha(x) - P_{s+t}^\alpha h_\alpha(x) = \int_t^{s+t} \exp(-\alpha r) f_r(x)\, dr$$

which implies that (in view of (11))

$$\forall\ x \in X \ : \ f_t^\alpha(x) := \exp(-\alpha t) f_t(x) = D^\alpha P_t^\alpha h_\alpha(x) \quad \text{a.e. in } (0, \infty). \tag{12}$$

Now we use the assumption that $\mathbb{P}$ is lattice: This implies that $\mathbb{P}$ is of the form (7) in view of the Proposition. But for such a semigroup, it is easy to see that

$$\forall\ t > 0\ :\ D^\alpha P_t^\alpha h_\alpha = P_t^\alpha(D^\alpha h_\alpha). \tag{13}$$

From (12), (13) and the expression of $\mathbb{P}^\alpha$ we have

$$\forall\ x \in X\ :\ P_t D^\alpha h_\alpha(x) = f_t(x) \quad \text{a.e. in } (0,\infty). \tag{14}$$

Choose (for example) $\varphi := D^1 h_1$, then from (14), we have

$$\forall\ x \in X\ :\ P_t\varphi(x) = f_t(x) \quad \text{a.e. in } (0,\infty).$$

which implies (10).

COROLLARY

Let $\mathbb{P}$ be a measurable lattice semigroup of submarkovian kernels on $(X,\mathcal{B})$ such that its potential kernel V is proper, then $\mathcal{R} = \operatorname{Im} V$.

Proof. It suffices to tend with α to 0 in the relation (10) of the Theorem.

3. Particular case

A global dynamical system on $(X,\mathcal{B})$ is a measurable mapping $\Psi : \mathbb{R}\times X \to X$ which verifies $\Psi(0,x) = x$ and $\Psi(s+t,x) = \Psi(s,\Psi(t,x))$ for every $x \in X$ and $s,t \in \mathbb{R}$.

Let $\mathbb{P}$ be a lattice semigroup of submarkovian kernels on $(X,\mathcal{B})$, (Ω,Φ) the associated SDS and C the associated cocycle. Suppose that (Ω,Φ) is a restriction of a global dynamical system Ψ on $(X,\mathcal{B})$, i.e. Φ is the restriction of Ψ on $\Omega = [0,\infty[\times X$. Also the function C can be extended to $\mathbb{R}\times X$.

Let $f = (f_t)$ be an exit law for $\mathbb{P}$, set $\varphi(x) := C(-1,x)\cdot f_1(\Psi(-1,x))$ for every $x \in X$. Then we have for every $\alpha > 0$ and $x \in X$

$$\begin{aligned}
&\int_0^\infty \exp(-\alpha t) P_t\varphi(x)\,dt \\
&\quad = \int_0^\infty \exp(-\alpha t)\cdot C(t,x)\cdot \varphi(\Psi(t,x))\,dt \\
&\quad = \int_0^\infty \exp(-\alpha t)\cdot C(t,x)\cdot C(-1,\Psi(t,x))\cdot f_1(\Psi(-1,\Psi(t,x)))\,dt \\
&\quad = \int_0^\infty \exp(-\alpha t)\cdot C(-1+t,x)\cdot f_1(\Psi(-1+t,x))\,dt
\end{aligned}$$

$$= \int_0^\infty \exp(-\alpha t) \cdot C(0,x) \cdot f_t(x)\,dt$$
$$= \int_0^\infty \exp(-\alpha t) \cdot f_t(x)\,dt.$$

Thus this is an easy proof of the Theorem but only in a very particular case.

REMARKS

1. Let $\mathbb{P}$ be a lattice semigroup of kernels on $(X, \mathcal{B})$. As in (14), it is not proved that $f_t = P_t\varphi$, for all $t > 0$. It is even wrong in general; an easy example is given in ([9], p. 381).

2. The Theorem is proved without the hypothesis that the potential kernel is proper. Hence it includes all lattice semigroups for which the associated semidynamical system possesses periodical trajectories.

3. The idea of this paper is based on the hypothesis that $\mathbb{P}$ is submarkovian which is often (but not always) verified.

The following example shows that for $\mathcal{R} = \operatorname{Im} V$ it is not necessary that the semigroup is a lattice one.

EXAMPLE 4

Let $X :=]-\infty, 0]$ and $\mathbb{P}$ be the semigroup defined, for every $u \in \mathcal{B}$, $x \in X$, $t \geq 0$ by:

$$P_t u(x) := \begin{cases} u(x-t) & \text{if } x < 0; \\ \exp(-t)\left(u(0) + \int_{-t}^0 u(t)\exp(-r)\,dr\right) & \text{if } x = 0. \end{cases}$$

Then the potential kernel V is proper and we have $\mathcal{R} = \operatorname{Im} V$.

Proof. Let $f \in \mathcal{B}$ and $x < 0$. Then by calculation we obtain $Vf(x) = \int_{-\infty}^x f(y)\,dy$ and $Vf(0) = f(0) + \int_{-\infty}^0 f(y)\,dy$, thus V is proper.

Now let $f = (f_t)$ be an exit law for $\mathbb{P}$. Note that the restriction of $\mathbb{P}$ on $]0,\infty[$ is also a measurable semigroup of kernels on $(]0,\infty[, \mathcal{B}(]0,\infty[))$ which is a lattice. Then the Corollary implies the existence of some $\varphi \in \mathcal{B}(]0,\infty[)$ such that

$$\forall\, x < 0 :\; f_t(x) = P_t\varphi(x) \quad \text{for a.e. } t > 0. \tag{15}$$

Thus we have

$$f_{t+s}(0) = P_t f_s(0)$$
$$= f_s(0)\exp(-t) + \exp(-t)\int_{-t}^0 f_s(r)\exp(-r)\,dr$$

$$= f_s(0)\exp(-t) + \exp(-t)\int_{-t}^{0} \varphi(r-s)\exp(-r)\,dr$$

$$= f_s(0)\exp(-t) + \exp(-(s+t))\int_{-(t+s)}^{-s} \varphi(r)\exp(-r)\,dr.$$

This implies that (with the notation e instead of exp)

$$f_{s+t}(0) - e^{-(s+t)}\int_{-(t+s)}^{0} \varphi(r)e^{-r}\,dr = e^{-t}\left(f_s(0) - e^{-s}\int_{-s}^{0}\varphi(r)e^{-r}\,dr\right).$$

If we set $a(s) := f_s(0) - \exp(-s)\int_{-s}^{0}\varphi(r)\exp(-r)\,dr$ then the preceding relations shows that $a(s+t) = a(s)\exp(-t)$ for every $s, t > 0$. But this implies that $a(t) = \exp(-t)$, for every $t > 0$ because a is a Borel function. From the definition of a, we conclude that

$$\forall\ s > 0 :\ f_s(0) = \exp(-s)\left(1 + \int_{-s}^{0}\varphi(r)\exp(-r)\,dr\right). \tag{16}$$

If we extend the function φ on $]-\infty, 0]$ by putting $\varphi(0) = 1$ then (16) implies that $f_s(0) = P_t\varphi(0)$ and we conclude with (15) that $\mathcal{R} = \operatorname{Im} V$.

Note that in this case $\mathbb{P}$ is not a lattice because the measures $P_t(0,.)$ are not Dirac ones.

4. Concluding remarks

Consider the general situation: Let $\mathbb{P}$ be a measurable semigroup of submarkovian kernels and let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ the associated resolvent.

1. It is an open problem to give necessary and sufficient conditions on $\mathbb{P}$ such that $\mathcal{R} = \operatorname{Im} V$.

2. Taking into account the translation on $\mathbb{R}$ and the Brownian motion on $\mathbb{R}^n$ it is not difficult to construct a semigroup for which $\operatorname{Im} V \neq \mathcal{R} \neq \mathcal{P}$. In this case other problems arise which we have studied in [8].

3. If the potential kernel is not proper, it makes no sense to consider $\mathcal{R}$ to study the exit laws for $\mathbb{P}$. But as in Theorem, let us consider the following:

For every measurable exit law $f = (f_t)_{t>0}$ for $\mathbb{P}$ consider the measurable function $h^f := (u_\alpha^f)_{\alpha>0}$ defined by

$$\forall\ \alpha > 0;\ x \in X\ :\ h^f(\alpha, x) = h_\alpha^f(x) := \int_0^\infty \exp(-\alpha t) f_t(x)\,dt. \tag{17}$$

Then we have

$$\{(\alpha, x) \mapsto V_\alpha\varphi(x)\ :\ \varphi \in \mathcal{F}\} \subset \{(\alpha, x) \mapsto h_\alpha^f(x) :\ f \text{ exit law for } \mathbb{P}\}.$$

The preceding relation appear as the generalization of $\operatorname{Im} V \subset \mathcal{R}$. But we can say more: The function $(\alpha, x) \mapsto h_\alpha^f(x)$ defined by (17) is a solution of the following functional equation (by calculus)

$$\forall\, 0 < \alpha < \beta :\ h_\alpha = h_\beta + (\beta - \alpha) V_\alpha h_\beta. \tag{18}$$

The equation (18) is called the exit law for the resolvent $\mathcal{V}$ (see [3], pp 38-52). Now let $u \in \mathcal{P}$. Then it can be shown (also by calculus) that $h_\alpha := u - \alpha V_\alpha u$ is a solution of (18). This shows that the equation (4) and (18) are not equivalent in general (apply the Theorem for the translation semigroup on $\mathbb{R}$).

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