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# Takagi's function and its functional equations

**Abstract.** The Takagi function $T : \mathbb{R} \to \mathbb{R}$ is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} d(2^k x),$$

where $d(y)$ denotes the distance from $y$ to the nearest integer.

We give some information on the history and on the analytic properties of $T$. Moreover, we present several functional equations for $T$ and discuss various relationships between them as well as their characterizing properties.

## 1. Introduction

In the year 1903, Takagi [17] introduced a function $T : [0,1] \to \mathbb{R}$, which is defined as follows. For $c_n \in \{0,1\}$,

$$T\left(\sum_{n=1}^{\infty} \frac{1}{2^n} c_n\right) := \sum_{n=1}^{\infty} \frac{1}{2^n} a_n \tag{1}$$

where $a_n$ is the number of digits 1 among $c_1, c_2, \ldots, c_n$ in case $c_n = 0$ and $a_n$ is the number of digits 0 among $c_1, c_2, \ldots, c_n$ in case $c_n = 1$.

Note that the argument $x$ of $T$ is given in its dyadic expansion whereas the value $T(x)$ in general is not. The aim of Takagi was to present a simple example of a continuous nowhere differentiable function. In fact, the *cnd* property for $T$ is much easier established than for other functions of this type considered earlier by Weierstrass.

Since then many other authors discussed various *cnd* functions including Takagi's; some historical information can be found in [1], [6], [9], [10], [16].

*A* note of van der Waerden [18] from the year 1930 became especially well known.

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There it is shown that the function $W : \mathbb{R} \to \mathbb{R}$, given by

$$W(x) := \sum_{k=0}^{\infty} \min \{|x - 10^{-k} \cdot m|; \quad m \in \mathbb{Z}\} \tag{2}$$

has no one-sided derivative at any point.

It is easily seen that, for $a = 10$, $W$ coincides with the function $S_a : \mathbb{R} \to \mathbb{R}$, given by

$$S_a(x) := \sum_{k=0}^{\infty} \frac{1}{a^k} d(a^k x), \tag{3}$$

where $d(y)$ denotes the distance from $y$ to the nearest integer.

It is by far not as obvious that $S_2$, restricted to the interval $[0,1]$, is in fact Takagi's function $T$. This is probably the reason why Takagi's work did not receive due recognition compared to van der Waerden's. For example, Billingsley [2] and Cater [3] denote $S_2$ as "van der Waerden's function", Darsow, Frank and Kairies [4] as "a function of van der Waerden type" without mentioning the name Takagi.

From now on we extend $T$ by 1-periodicity and have for every $x \in \mathbb{R}$

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} d(2^k x). \tag{4}$$

In Section 2 we state geometrical and analytical properties of $T$. However, the main object of this paper are functional equations for $T$. They are discussed in Section 3. Our results are based on previous work by de Rham [15], Darsow, Frank and Kairies [4], Girgensohn [6], [7] and by Kairies [9].

## 2. Geometrical and analytical properties of $T$

The representation (4) shows immediately that $T$ is continuous on $\mathbb{R}$. Now we list some other important properties of $T$ in the following

THEOREM 1

a) *$T$ is nowhere differentiable.*

b) *$T$ satisfies on $[0,1]$ a Lipschitz condition of any order $\alpha \in (0,1)$.*

c) $\max\{T(x);\ x \in [0,1]\} = \frac{2}{3}$; *for any $z \in [0,1]$, $T(z) = \frac{2}{3}$ iff the 4-adic expansion of $z$ contains only the digits 1 or 2.*

d) *The Hausdorff dimension of $\{(x, T(x));\ x \in [0,1]\}$ is one.*

e) *Let $M_n := \int_0^1 x^n T(x)dx$. Then $M_0 = \frac{1}{2}$ and*

$$M_n = \frac{1}{(n+1)\,(n+2)} + \frac{1}{2(2^{n+1}-1)} \sum_{k=0}^{n-1} \binom{n}{k} M_k, \quad n \in \mathbb{N}.$$

*Proof.* a) Takagi [17] proved the *cnd* property of $T$ on the interval $[0,1]$ using his representation (1). Billingsley [2] showed by means of the representation (4) that $T$ is nowhere differentiable and Cater [3], using as well (4), showed that $T$ has no one-sided derivative at any point. This fact can also be deduced from Takagi's paper. A completely different proof using functional equations can be found in the work of Darsow, Frank and Kairies [4].

b) Let $x, y \in [0,1]$ and $|y-x| \geq \frac{1}{2}$. Because of $0 \leq T(z) \leq \frac{2}{3}$, $z \in [0,1]$, we get immediately $|T(y) - T(x)| \leq \frac{4}{3}|y-x|$.

Now let $|y-x| \leq \frac{1}{2}$. It is sufficient to consider the case $x = 0$ and we do so. From the representation (4) we deduce that $T(2^{-k}) = k \cdot 2^{-k}$ for every $k \in \mathbb{N}$ and that $T(y) \leq (k+1)2^{-k}$ whenever $2^{-(k+1)} < y < 2^{-k}$. This implies

$$|T(y) - T(0)| \leq 4|y-0| \log_2 |y-0|^{-1} \leq c(\alpha) \cdot |y-0|^{\alpha}$$

for any $\alpha \in (0,1)$ with a suitable constant $c(\alpha)$.

Clearly $T$ does not belong to $\mathrm{Lip}_1\,[0,1]$ because this would imply the absolute continuity of $T$, contradicting a). Kôno [11] proved the related result

$$\overline{\lim}_{|x-y|\to 0} \quad \frac{T(x) - T(y)}{(x-y) \cdot \log_2\,(1/|x-y|)} = 1,$$

$$\underline{\lim}_{|x-y|\to 0} \quad \frac{T(x) - T(y)}{(x-y) \cdot \log_2\,(1/|x-y|)} = -1.$$

c) The statements have been proved by Martynov [13] and in a more general setting by Dubuc and Elqortobi [5]. Note that the partial sums

$$T_{2m+1}(x) = \sum_{k=0}^{2m+1} 2^{-k} d(2^k x)$$

attain their maximal value $\sum_{k=0}^{m} 2^{-(2k+1)}$ on $2^m$ full intervals, each of length $2^{-(2m+1)}$.

As a consequence, the global maximal value of $T$ is $\sum_{k=0}^{\infty} 2^{-(2k+1)} = 2/3$ and the set $M := \{z \in [0,1];\ T(z) = 2/3\}$ has measure zero. Its minimal element is $1/3$ and its Hausdorff dimension is $1/2$, a fact which has also been proved by Martynov [13].

d) The claim is a particular case of a result due to Mauldin and Williams [14]. These authors proved that $\{(x, S_a(x));\ x \in [0,1]\}$ has Hausdorff dimension one whenever $a > 1$.

e) The moments of $T$ have been obtained by Darsow, Frank and Kairies [4] using neither the representation (1) nor (4) for $T$, but a functional equation which is satisfied by $T$.

REMARKS

a) Let $\psi(x) := 2d(x)$. Then for the $n$-th iterate $\psi^n$ of $\psi$ we have $\psi^n(x) = \psi(2^{n-1}x)$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. As a consequence,

$$T(x) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}\, \psi(2^{k-1}x) = \sum_{k=1}^{\infty} \frac{1}{2^k}\, \psi^k(x). \tag{5}$$

Hata and Yamaguti [8] generalized this third representation (5) for $T$ and discussed the Takagi class $\mathcal{T}$ of functions $F(a_k) : [0,1] \to \mathbb{R}$,

$$F(a_k)\,(x) := \sum_{k=1}^{\infty} a_k \psi^k(x),$$

which are generated by a sequence $(a_k) \in \ell^1$.
They showed for instance that the convergence of $\sum_{k=1}^{\infty} b_k \psi^k(x)$ for every $x \in [0,1]$ implies : $(b_k) \in \ell^1$. They obtained structural properties as Schauder expansions or functional equations for the elements of $\mathcal{T}$ and exhibited a connection of $T$ with de Rham's singular function. The paper of Kôno [11] contains several other interesting statements on $\mathcal{T}$. We give one especially nice example:

$F(a_k)$ is nowhere differentiable iff $\overline{\lim}_{k\to\infty}\, 2^k |a_k| > 0$.

b) Takagi's function $T$ can as well be interpreted as a member of the Knopp class of functions $K : \mathbb{R} \to \mathbb{R}$, given by

$$K(x) := \sum_{k=0}^{\infty} a^k g(b^k x)$$

with $|a| < 1$ and $g$ of period one. This class has been investigated as thoroughly as the Takagi class. Information can be obtained from the references [1], [5], [7], [9], [10], [14], [16].

The graph of Takagi's function $T$ and the graphs of some of the partial sums $T_m$ of the series (4) are sketched at the end of our paper.

## 3. Functional equations for $T$

We first give a list of functional equations which are satisfied by Takagi's $T$. They are all in a single variable and of order one or two according to Kuczma's [12] terminology.

THEOREM 2

*The Takagi function $T$ satisfies the following functional equations* (6)-(12) *for every $x \in \mathbb{R}$ and* (6′)-(8′) *exactly for $x \in [0,1]$.*

$$2f\left(\frac{x}{2}\right) - f(x) = 2d\left(\frac{x}{2}\right) \tag{6}$$

$$2f\left(\frac{x}{2}\right) - f(x) = x \tag{6'}$$

$$2f\left(\frac{x+1}{2}\right) - f(x) = 2d\left(\frac{x+1}{2}\right) \tag{7}$$

$$2f\left(\frac{x+1}{2}\right) - f(x) = 1 - x \tag{7'}$$

$$f\left(\frac{x+1}{2}\right) - f\left(\frac{x}{2}\right) = d\left(\frac{x+1}{2}\right) - d\left(\frac{x}{2}\right) \tag{8}$$

$$f\left(\frac{x+1}{2}\right) - f\left(\frac{x}{2}\right) = \frac{1}{2} - x \tag{8'}$$

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) - f(x) = \frac{1}{2} \tag{9}$$

$$f(2x) - 2f(x) = -2d(x) \tag{10}$$

$$f(x+1) - f(x) = 0 \tag{11}$$

$$f(x) - f(1-x) = 0 \tag{12}$$

*Proof.* By using the representation (4), the verification of these functional equations for $T$ is straightforward. As an example we check the statements about the equations (6) and (6′).

$$\begin{aligned} 2T\left(\frac{x}{2}\right) - T(x) &= \sum_{k=0}^{\infty} 2 \cdot \frac{1}{2^k}\, d\left(2^k \cdot \frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{1}{2^k}\, d(2^k x) \\ &= 2d\left(\frac{x}{2}\right) + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}\, d(2^{k-1}x) - \sum_{k=0}^{\infty} \frac{1}{2^k}\, d(2^k x) \\ &= 2d\left(\frac{x}{2}\right) \end{aligned}$$

for every $x \in \mathbb{R}$, which proves (6). As a consequence, $T$ satisfies (6′) iff $2d(\frac{x}{2}) = x$ and this is exactly the case for $x \in [0,1]$.

Equation (10) was stated by de Rham [15] and used to characterize $T$ on $\mathbb{R}$. The equations (6′)-(8′) and (9)-(12) were discussed by Darsow, Frank and Kairies [4] and used to characterize $T$ on [0,1].

The equations (6) and (7) are a particular case of a system which was examined in great detail by Girgensohn [6], [7]. Equation (9) is of replicativity type, cf. Kairies [9].

We collect the most important facts on functional equations for Takagi's $T$ from the just quoted papers in

THEOREM 3

a) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is bounded and satisfies equation* (10) *for every* $x \in \mathbb{R}$. *Then* $f(x) = T(x)$ *for every* $x \in \mathbb{R}$.

b) *Any solution* $f : [0,1] \to \mathbb{R}$ *of two of the equations* (6′), (7′), (8′), (9) *on* $[0,1]$ *satisfies the other two on* $[0,1]$.

c) *Any solution* $f : [0,1] \to \mathbb{R}$ *of two of the equations* (6′), (7′), (8′), (9) *on* $[0,1]$ *has no one-sided derivative at any point of* $(0,1)$. $f'_+(0)$ *and* $f'_-(1)$ *do not exist (in* $\mathbb{R}$*).*

d) *Assume that* $f : [0,1] \to \mathbb{R}$ *is bounded and satisfies two of the equations* (6′), (7′), (8′), (9) *on* $[0,1]$. *Then* $f(x) = T(x)$ *for every* $x \in [0,1]$.

e) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies equation* (11) *on* $\mathbb{R}$ *and* (9) *on* $[0,1]$. *Then* $f$ *satisfies* (9) *on* $\mathbb{R}$.

f) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies equation* (11) *on* $\mathbb{R}$ *and two of the equations* (6′), (7′), (8′), (9) *on* $[0,1]$. *Then* $f$ *satisfies* (10) *on* $\mathbb{R}$.

*Proof.* Statement a) is due to de Rham and is proved in [15]. All the other statements are due to Darsow, Frank and Kairies and are proved in [4].

REMARKS

a) Each of the functional equations (6)-(12) and (6′)-(8′) has $2^{\mathfrak{c}}$ solutions on $\mathbb{R}$. This can be seen as follows. In case of equations (7), (7′), (8), (8′), (11) and (12), a solution can be arbitrarily prescribed on the interval $(0, \frac{1}{2})$, in case of equations (6), (6′), (9) and (10), a solution can be arbitrarily prescribed on the interval $(\frac{1}{2}, 1)$. In any case, this initial solution can be extended (not necessarily in a unique way) by the corresponding functional equation to a solution on the whole real line. As an example we verify the statement for the equation (9). Start with a prescribed $f$ on the interval $(\frac{1}{2}, 1]$. In the first step, for $x \in (\frac{1}{2}, 1]$, the equation $f(\frac{x}{2}) = f(x) - f(\frac{x+1}{2}) + \frac{1}{2}$ gives a unique extension on $(\frac{1}{4}, 1]$. In the second step, for $x \in (\frac{1}{4}, 1]$, we obtain a unique extension on $(\frac{1}{8}, 1]$.

Proceeding in this manner we get a unique extension of $f$ on $(0, 1]$. Now extend this (0,1]-solution of (9) by 1-periodicity on $\mathbb{R}$. According to Theorem 3 e), this extension satisfies (9) on $\mathbb{R}$.

b) $T$ satisfies equation (6′) exactly on [0,1]. There is a unique extension of (6′) preserving the left hand side which is satisfied by $T$ on $\mathbb{R}$, namely (6). The same relationship is shared by the pairs (7′), (7) and (8′), (8).

c) On $\mathbb{R}$, equation (6) has exactly the same solutions as de Rham's equation (10). It is easy to check that no other pair of the equations (6)-(12), (6′)-(8′) has identical solutions on $\mathbb{R}$.

d) Statement b) of Theorem 3 remains true when any interval [0,1] is replaced by $\mathbb{R}$.

From now on we consider the system (6)-(12) which is satisfied by Takagi's function on the whole real line. The following Lemma is the tool to extend the important statements c) and d) of Theorem 3 on $\mathbb{R}$.

LEMMA

*Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies two of the equations* (6), (7), (8), (9) *on* $\mathbb{R}$. *Then* $f$ *satisfies the other two and equation* (11) *on* $\mathbb{R}$.

*Proof.* Given equations (6), (7), we obtain (8) by subtracting (6) from (7) and (9) by adding up (6) and (7). In a similar fashion we get (in symbolic notation)

| | | |
|---|---|---|
| (7) = (6) + 2 · (8), | (9) = (6) + (8) | for given (6), (8), |
| (7) = − (6) + 2· (9), | (8) = (9) − (6) | for given (6), (9), |
| (6) = (7) − 2 · (8), | (9) = (7) − (8) | for given (7), (8), |
| (6) = − (7) + 2 ·(9), | (8) = (7) − (9) | for given (7), (9), |
| (6) = − (8) + (9), | (7) = (8) + (9) | for given (8), (9). |

Now let (6) and (7) be satisfied for every $x \in \mathbb{R}$. Replacing $x$ in (6) by $x + 1$ gives $f(x+1) = 2f(\frac{x+1}{2}) - 2d(\frac{x+1}{2})$ and by (7) the right hand side is $f(x)$. This proves that any solution of two of the equations (6)-(9) necessarily is of period 1.

THEOREM 4

a) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies two of the equations* (6)-(9) *on* $\mathbb{R}$. *Then* $f$ *has no one-sided derivative at any point* $x \in \mathbb{R}$.

b) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is bounded on* $\{x \in \mathbb{R};\ |x| > a\}$ *for some* $a$ *and let* $f$ *satisfy two of the equations* (6)-(9) *on* $\mathbb{R}$. *Then* $f(x) = T(x)$ *for every* $x \in \mathbb{R}$.

*Proof.* a) By the Lemma, $f$ has period 1. (6), (7), (8) coincide on [0,1] with (6′), (7′), (8′) respectively. Thus the claim is an immediate consequence of Theorem 3 c).

b) Our hypotheses imply that $f$ satisfies equation (11) on $\mathbb{R}$ and (6′), (7′), (8′), (9) on [0,1]. By Theorem 3 f), $f$ satisfies the de Rham equation (10) on $\mathbb{R}$.

Iteration of (10) yields

$$f(x) = \frac{1}{2^n} f(2^n x) + \sum_{k=0}^{n-1} \frac{1}{2^k} d(2^k x) \tag{13}$$

for every $x \in \mathbb{R}$, $n \in \mathbb{N}$. For $x \neq 0$ the boundedness condition on $f$ implies

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{2^k} d(2^k x) = T(x)$$

and $f(0) = T(0) = 0$ follows immediately from (10).

It is somewhat surprising that the characterization of $T$ on $\mathbb{R}$, which is given by Theorem 4 b), can be considerably improved — in contrast to the corresponding characterization of $T$ on the smaller interval [0,1] expressed by Theorem 3 d).

THEOREM 5

*Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is bounded on* $\{x \in \mathbb{R};\ |x| > a\}$ *for some* $a$ *and that* $f$ *satisfies equation* (6) *or* (7) *on* $\mathbb{R}$. *Then* $f(x) = T(x)$ *for every* $x \in \mathbb{R}$.

*Proof.* Equation (6) on $\mathbb{R}$ implies equation (10) on $\mathbb{R}$ (they are in fact equivalent) and the argument used in the proof of Theorem 4 b) gives our assertion.

Now suppose that (7) holds on $\mathbb{R}$. We write (7) in the form

$$2f(g(x)) - f(x) = h(x) \tag{14}$$

where $g(x) = \frac{x+1}{2}$ and $h(x) = 2d(\frac{x+1}{2})$ is a continuous function of period 2.

Iteration of (14) gives

$$f(x) = 2^n f(g^n(x)) - \sum_{k=0}^{n-1} 2^k h(g^k(x)) \tag{15}$$

for every $x \in \mathbb{R}$, $n \in \mathbb{N}$. The $k$-th iterate of $g$ is given by $g^k(x) = \frac{x+2^k-1}{2^k}$. We introduce the new variable $z = (x + 2^n - 1) \cdot 2^{-n}$ and obtain from (15)

$$\begin{aligned}\frac{1}{2^n} f(2^n z - 2^n + 1) &= f(z) - \sum_{k=0}^{n-1} 2^{k-n}\, h\left(\frac{2^n z - 2^n + 2^k}{2^k}\right)\\ &= f(z) - \sum_{k=0}^{n-1} 2^{k-n}\, h(2^{n-k} z + 1)\\ &= f(z) - \sum_{k=1}^{n} 2^{-k}\, h(2^k z + 1)\end{aligned}$$

for every $z \in \mathbb{R}$, $n \in \mathbb{N}$. For $z \neq 1$ and $n \geq n_0(z)$ all the arguments $2^n z - 2^n + 1$ fall in the set on which, by assumption, $f$ is bounded. Consequently, for $z \neq 1$, we have

$$\begin{aligned}f(z) &= \sum_{k=1}^{\infty} \frac{1}{2^k}\, h(2^k z + 1) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot 2d\left(\frac{2^k z + 1 + 1}{2}\right)\\ &= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}\, d(2^{k-1} z)\\ &= T(z).\end{aligned}$$

Now equation (7) for $x = 1$ gives immediately $f(1) = 0 = T(1)$ and that finishes our proof.

We conclude with the observation that Theorem 5 no longer remains true when "equation (6) or (7)" is replaced by "equation (8) or (9)".

Counterexamples are given in form of trigonometric series:

$$d(x) = \frac{1}{4} - \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos 2\pi(2n+1)x$$

represents a bounded solution of (8) on $\mathbb{R}$,

$$\tilde{f}(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi n x$$

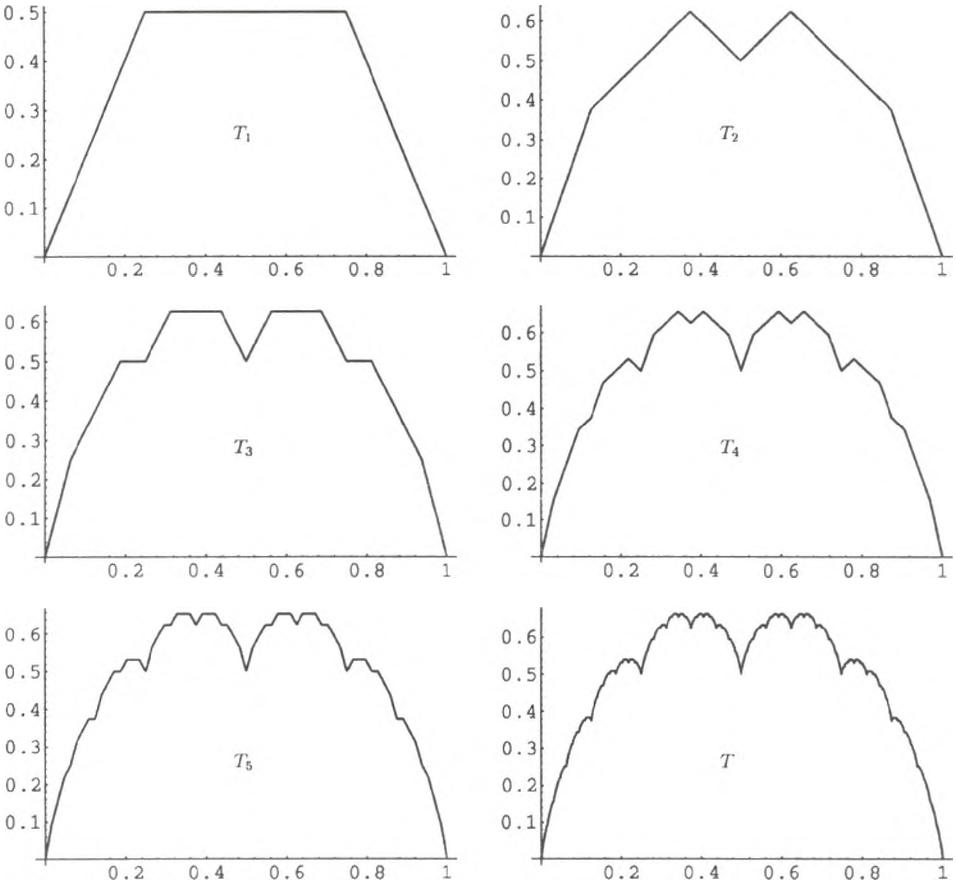
represents a bounded solution of (9) on $\mathbb{R}$.

Obviously both functions are different from Takagi's $T$.

So the first two single equations from (6)-(9) on $\mathbb{R}$ have a much stronger characterizing power with respect to $T$ than the remaining two equations.

**The Takagi function**

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} d(2^k x), \quad T_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} d(2^k x), \quad d(y) = \operatorname{dist}(y, \mathbb{Z}).$$

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