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On a generalization of a functional equation associated with Simpson's rule

Abstract. The general solution of the functional equation

$$f(x) - g(x) = (x - y)[h(sx + ty) + \phi(x) + \psi(y)]$$

for all $x, y \in \mathbb{R}$ (the set of reals) with s and t being *a priori* known parameters is determined without any regularity assumptions (differentiability, continuity, measurability, etc.) imposed on the real functions f, g, h, ϕ and ψ. The motivation for studying this equation came from Simpson's rule for evaluating definite integrals. Special cases of this equation include functional equations studied by Aczél [1] and Haruki [2].

1. Introduction

Let $\mathbb{R}$ be the set of real numbers. A function $A : \mathbb{R} \to \mathbb{R}$ is said to be *additive* if and only if $A(x + y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$. For a comprehensive account on additive functions the interested reader should refer to [5]. In connection with the Simpson's rule for evaluating definite integrals, we came across the following functional equation

$$f(x) - f(y) = \frac{(x - y)}{6} \left[g(x) + 4g\left(\frac{x + y}{2}\right) + g(y)\right] \tag{1}$$

where f is an antiderivative of g. This equation is a special case of

$$f(x) - g(y) = (x - y)\left[h(x + y) + \psi(x) + \phi(y)\right] \tag{2}$$

where $f, g, h, \psi, \phi : \mathbb{R} \to \mathbb{R}$ are unknown functions. The equation (2) was treated in [3]. The middle term of the equation (1), that is $4g\left(\frac{x+y}{2}\right)$ is due to the fact that in Simpson rule one partitions the interval into subintervals of equal lenghts. However, there is no reason why one should be restricted to

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such an equal partition. If we allow unequal partition, then the middle term is no longer of the form $4g\left(\frac{x+y}{2}\right)$ but rather it is of the form $\alpha g(sx+ty)$, where α, s, t are constants. Taking this into account we have a generalization of (1) as

$$f(x) - f(y) = (x-y)\left[h(sx+ty) + g(x) + g(y)\right] \tag{3}$$

for all $x, y \in \mathbb{R}$ with s and t being *a priori* chosen parameters.

Our main objective in this paper is to determine the general solution of the functional equation (3) without any regularity assumptions (differentiability, continuity, measurability, etc.) imposed on the unknown functions. Further, utilizing the solution of this equation we determine the general solution of the functional equation

$$f(x) - g(y) = (x-y)\left[h(sx+ty) + \psi(x) + \phi(y)\right] \tag{4}$$

for all $x, y \in \mathbb{R}$ where s and t are *a priori* chosen parameters. Special cases of this equation include functional equations studied by Aczél, Haruki, Kannappan, Sahoo, Jacobson, and Riedel (cf. [1], [2], [4], [3]).

2. Some auxiliary results

In this section, we prepare some auxiliary results to be used in determining the general solutions of the functional equations (3) and (4). The first result is due to Haruki [2].

LEMMA 1

The functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$f(x) - f(y) = (x-y)\left(\frac{g(x)+g(y)}{2}\right) \tag{5}$$

for all $x, y \in \mathbb{R}$ if and only if

$$f(x) = ax^2 + bx + c \quad \text{and} \quad g(x) = 2ax + b \tag{6}$$

where a, b and c are arbitrary real costants.

The following result will be used in determining the general solution of the functional equation (4). This result is due to Kannappan, Riedel and Sahoo [3].

LEMMA 2

The functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$xf(y) - yf(x) = (x-y)[g(x+y) - g(x) - g(y)] \tag{7}$$

for all $x, y \in \mathbb{R}$ if and only if

$$f(x) = 3ax^3 + 2bx^2 + cx + d \quad \text{and} \quad g(x) = -ax^3 - bx^2 - A(x) - d, \tag{8}$$

where $A : \mathbb{R} \to \mathbb{R}$ *is additive and* a, b, c, d *are arbitrary real costants.*

3. Solution of the functional equation (3)

In this section we determine the most general solution of the functional equation (3) with no regularity assumptions (differentiability, continuity, measurability, etc.) imposed on f, g and h. The solution of this equation will be used in the next section to determine the general solution of the functional equation (4).

THEOREM 1

Let s *and* t *be real parameters. The functions* $f, g, h : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (3) *for all* $x, y \in \mathbb{R}$ *if and only if*

$$f(x) = \begin{cases} ax^2 + (b+d)x + c & \text{if } s = 0 = t \\ ax^2 + (b+d)x + c & \text{if } s = 0,\ t \neq 0 \\ ax^2 + (b+d)x + c & \text{if } s \neq 0,\ t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha & \text{if } s = t \neq 0 \\ 2ax^3 + cx^2 + 2\beta x - A(x) + \alpha & \text{if } s = -t \neq 0 \\ ax^2 + (b+d)x + c & \text{if } 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$g(x) = \begin{cases} ax + \dfrac{b}{2} & \text{if } s = 0 = t \\ ax + \dfrac{b}{2} & \text{if } s = 0,\ t \neq 0 \\ ax + \dfrac{b}{2} & \text{if } s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta & \text{if } s = t \neq 0 \\ 3ax^2 + cx + \beta & \text{if } s = -t \neq 0 \\ ax + \dfrac{b}{2} & \text{if } 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$h(x) = \begin{cases} \text{arbitrary with } h(0) = d & \text{if } s = 0 = t \\ d & \text{if } s = 0,\ t \neq 0 \\ d & \text{if } s \neq 0,\ t = 0 \\ a\left(\dfrac{x}{t}\right)^3 + b\left(\dfrac{x}{t}\right)^2 + A\left(\dfrac{x}{t}\right) + d & \text{if } s = t \neq 0 \\ -a\left(\dfrac{x}{t}\right)^2 - \dfrac{t}{x}A\left(\dfrac{x}{t}\right),\ x \neq 0 & \text{if } s = -t \neq 0 \\ d & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and a, b, c, d, α, β are arbitrary real constants.

Proof. To prove the theorem, we consider several cases depending on the parameters s and t.

Case 1. Suppose $s = 0 = t$. Then (3) reduces to

$$f(x) - f(y) = (x - y)[d + g(x) + g(y)], \tag{9}$$

where $d = h(0)$. Defining

$$F(x) := f(x) - dx \quad \text{and} \quad G(x) := 2g(x), \tag{10}$$

and using (10) in (9), we obtain

$$F(x) - F(y) = (x - y)\left[\frac{G(x) + G(x)}{2}\right] \tag{11}$$

for all $x, y \in \mathbb{R}$. The general solution of (11) can be obtained from Lemma 1 as

$$F(x) = ax^2 + bx + c \quad \text{and} \quad G(x) = 2ax + b, \tag{12}$$

where a, b, c are arbitrary constants. Hence from (10) and (12), we have

$$\left.\begin{aligned} f(x) &= ax^2 + (b + d)x + c \\ g(x) &= ax + \frac{b}{2} \\ h(x) &\quad \text{arbitrary with } h(0) = d, \end{aligned}\right\} \tag{13}$$

where a, b, c, d are arbitrary constants.

Case 2. Suppose $s = 0$ and $t \neq 0$. (The case $s \neq 0$ and $t = 0$ can be handled in a similar manner.) Then (3) reduces to

$$f(x) - f(y) = (x - y)[h(ty) + g(x) + g(y)]. \tag{14}$$

Letting $y = 0$ in (14), we obtain

$$f(x) = f(0) + x[h(0) + g(x) + g(0)]. \tag{15}$$

Using (15) in (14), we get

$$xg(x) - yg(y) = (x - y)[h(ty) + g(x) + g(y) - g(0) - h(0)]. \tag{16}$$

Interchanging x and y in (16), we obtain

$$yg(y) - xg(x) = (y - x)[h(tx) + g(y) + g(x) - g(0) - h(0)]. \tag{17}$$

Adding (16) to (17), we see that

$$h(tx) = h(ty) \tag{18}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. Hence from (18), we have

$$h(x) = d \quad \text{for all } x \in \mathbb{R}, \tag{19}$$

where d is an arbitrary constant. Inserting (19) into (14), we obtain

$$f(x) - f(y) = (x - y)[d + g(x) + g(y)], \tag{20}$$

which is (9). Thus, by case 1, (19) and (13), we get

$$\left.\begin{aligned} f(x) &= ax^2 + (b+d)x + c \\ g(x) &= ax + \frac{b}{2} \\ h(x) &= d, \end{aligned}\right\} \tag{21}$$

where a, b, c, d are arbitrary constants.

Case 3. Next suppose $s \neq 0$ and $t \neq 0$. Letting $y = 0$ and $x = 0$, separately in (3), we obtain

$$f(x) = f(0) + x[h(sx) + g(x) + g(0)] \tag{22}$$

and

$$f(y) = f(0) + y[h(ty) + g(y) + g(0)], \tag{23}$$

respectively. Comparing f in (22) and (23), we have

$$h(sx) = h(tx) \tag{24}$$

for all $x \in \mathbb{R} \setminus \{0\}$. Letting (22) and (23) into (3) and rearranging terms, we get

$$\begin{aligned} y[h(sx) + g(x) - g(0)] - x[h(ty) + g(y) - g(0)] \\ = (x - y)[h(sx + ty) - h(sx) - h(ty)] \end{aligned} \tag{25}$$

for all $x, y \in \mathbb{R}$.

Now consider several subcases

Subcase 3.1. Suppose $s = t$. Then (25) yields

$$x\phi(y) - y\phi(x) = (x - y)[\psi(x + y) - \psi(x) - \psi(y)] \tag{26}$$

for all $x, y \in \mathbb{R}$, where

$$\phi(x) := h(tx) + g(x) - g(0) \quad \text{and} \quad \psi(x) := -h(tx). \tag{27}$$

The solution of the functional equation (26) can be obtained from Lemma 2 as

$$\left.\begin{aligned} \phi(x) &= 3ax^3 + 2bx^2 + cx + d \\ \psi(x) &= -ax^3 - bx^2 - A(x) - d, \end{aligned}\right\} \tag{28}$$

where $A : \mathbb{R} \to \mathbb{R}$ is additive and a, b, c, d are constants. From (28), (27) and (22), we have the asserted solution

$$\left.\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha \\ g(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta \\ h(x) &= a\left(\frac{x}{t}\right)^3 + b\left(\frac{x}{t}\right)^2 + A\left(\frac{x}{t}\right) + d, \end{aligned}\right\} \tag{29}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive map and a, b, c, d, α, β are arbitrary constants.

Subcase 3.2. Next, suppose $s = -t$. Then from (24), we have $h(tx) = h(-tx)$ for all $x \in \mathbb{R} \setminus \{0\}$. That is, h is an even function in $\mathbb{R}$. Now with $s = -t$ and using the evenness of h, from (25), we have

$$\begin{aligned} y[h(tx) + g(x) - g(0)] - x[h(ty) + g(y) - g(0)] \\ = (x - y)[h(tx - ty) - h(tx) - h(ty)] \end{aligned} \tag{30}$$

for all $x, y \in \mathbb{R}$. Defining

$$G(x) := h(tx) + g(x) - g(0) \quad \text{and} \quad H(x) := -h(tx) \tag{31}$$

we have from (30)

$$xG(y) - yG(x) = (x - y)[H(x - y) - H(x) - H(y)]. \tag{32}$$

Note that H is also an even function in view of (31). Replacing y by $-y$ in (32), we get

$$xG(-y) + yG(x) = (x + y)[H(x + y) - H(x) - H(y)]. \tag{33}$$

Letting $x = y$ in (33), we get

$$G(-x) + G(x) = 2[H(2x) - 2H(x)] \tag{34}$$

for all $x \neq 0$. By (31), (34) holds for $x = 0$ also. Adding (32) and (33) and using (34), we have

$$(x + y)H(x + y) + (x - y)H(x - y) = 2xH(x) + 2x[H(2y) - H(y)]. \tag{35}$$

Interchanging x with y in (35), we get

$$(x + y)H(x + y) + (y - x)H(x - y) = 2yH(y) + 2y[H(2x) - H(x)]. \tag{36}$$

Adding (35) to (36) and rearranging terms, we obtain

$$(x+y)H(x+y) - xH(x) - yH(y) = y[H(2x) - H(x)] + x[H(2y) - H(y)]. \tag{37}$$

The equation (37) yields

$$\phi(x+y) - \phi(x) - \phi(y) = y\psi(x) + x\psi(y), \tag{38}$$

where

$$\phi(x) := xH(x) \quad \text{and} \quad \psi(x) := H(2x) - H(x). \tag{39}$$

Note from (39) since H is even, ϕ is odd and ψ is even. Replace x by $x - y$ and y by $-y$ separately in (38) to get

$$\phi(x) - \phi(x-y) - \phi(y) = y\psi(x-y) + (x-y)\psi(y) \tag{40}$$

and

$$\phi(x-y) - \phi(x) - \phi(-y) = -y\psi(x) + x\psi(-y). \tag{41}$$

Adding (40) to (41) and using the fact that ϕ is odd and ψ is even, we obtain

$$y[\psi(x-y) - \psi(x) - \psi(y)] = -2x\psi(y). \tag{42}$$

Replacing y by $-y$ in (42), we have

$$y[\psi(x+y) - \psi(x) - \psi(y)] = 2x\psi(y),$$

that is

$$xy[\psi(x+y) - \psi(x) - \psi(y)] = 2x^2\psi(y) \tag{43}$$

for $x \neq 0$. Interchanging x and y in (43), we have

$$xy[\psi(x+y) - \psi(x) - \psi(y)] = 2y^2\psi(x). \tag{44}$$

Hence, from (43) and (44), we see that

$$2x^2\psi(y) = 2y^2\psi(x)$$

for all $x, y \in \mathbb{R} \setminus \{0\}$. Thus we have

$$\psi(x) = 3ax^2 \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \tag{45}$$

where a is a constant. By (39), (45) holds for $x = 0$ also. Letting (45) into (38), we get

$$\phi(x+y) - \phi(x) - \phi(y) = 3ax^2y + 3axy^2 \tag{46}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$. This in turns gives a Cauchy equation

$$\phi(x+y) - a(x+y)^3 = \phi(x) - ax^3 + \phi(y) - ay^3 \tag{47}$$

and hence

$$\phi(x) = ax^3 + A(x) \tag{48}$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function. From (48) and (39), we get

$$xH(x) = ax^3 + A(x). \tag{49}$$

Using (49) and (32), we obtain

$$x\left[G(y) + \frac{A(y)}{y} - 2ay^2\right] = y\left[G(x) + \frac{A(x)}{x} - 2ax^2\right] \tag{50}$$

for all $x, y \in \mathbb{R}\setminus\{0\}$ with $x \neq y$. Thus

$$G(x) = 2ax^2 + cx - \frac{A(x)}{x}, \quad x \neq 0, \tag{51}$$

where c is a constant. From (22), (31), (49) and (51), we have the asserted solution

$$\left.\begin{aligned} f(x) &= 2ax^3 + cx^2 + 2\beta x - A(x) + \alpha \\ g(x) &= 3ax^2 + cx + \beta \\ h(x) &= -a\left(\frac{x}{t}\right)^2 - \frac{t}{x}A\left(\frac{x}{t}\right), \ x \neq 0, \end{aligned}\right\} \tag{52}$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive map and a, c, α, β are arbitrary constants.

Subcase 3.3. Suppose $s^2 \neq t^2$, that is $\det\begin{pmatrix} s & t \\ t & s\end{pmatrix} \neq 0$. Note that if x and y are linearly independent, so also $u = sx + ty$ and $v = sy + tx$. Suppose not, then for some constants a and b (not both zero), we have $0 = au + bv = (as + bt)x + (at + bs)y$. Since x and y are linearly independent, we have

$$\begin{pmatrix} s & t \\ t & s\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}.$$

Since determinant of the matrix $\begin{pmatrix} s & t \\ t & s\end{pmatrix}$ is nonzero, this yields that both a and b are zero which is a contradiction.

Now we return to equation (25). Using (24) in (25), we have

$$\begin{aligned} y[h(sx) + g(x) - g(0)] - x[h(sy) + g(y) - g(0)] \\ = (x-y)[h(sx+ty) - h(sx) - h(sy)] \end{aligned} \tag{53}$$

for all $x, y \in \mathbb{R}$. Interchanging x and y in (53) we have

$$\begin{aligned} x[h(sy) + g(y) - g(0)] - y[h(sx) + g(x) - g(0)] \\ = (y-x)[h(sy+tx) - h(sy) - h(sx)]. \end{aligned} \tag{54}$$

Adding (53) to (54), we obtain

$$h(sx + ty) = h(sy + tx) \tag{55}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$ with $x \neq y$. Hence

$$h(x) = d \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \tag{56}$$

where d is a constant. Using (56) in (3), we get

$$f(x) - f(y) = (x - y)[d + g(x) + g(y)], \tag{57}$$

which is (9). Thus, by case 1, (56) and (13), we obtain the asserted solution

$$\left.\begin{aligned} f(x) &= ax^2 + (b + d)x + c \\ g(x) &= ax + \frac{b}{2} \\ h(x) &= d, \end{aligned}\right\} \tag{58}$$

where a, b, c, d are arbitrary constants. Since no more cases are left, the proof of the theorem is now complete.

The following result is obvious from Theorem 1 and it was established in [4] to answer a problem posed by Walter Rudin [6].

Corollary 1

Let s and t be real parameters. The functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$f(x) - g(y) = (x - y)h(sx + ty)$$

for all $x, y \in \mathbb{R}$ if and only if $g(x) = f(x)$ and

$$f(x) = \begin{cases} dx + c & \text{if } s = 0 = t \\ dx + c & \text{if } s = 0,\ t \neq 0 \\ dx + c & \text{if } s \neq 0,\ t = 0 \\ cx^2 + dx + \alpha & \text{if } s = t \neq 0 \\ \alpha - A(x) & \text{if } s = -t \neq 0 \\ dx + c & \text{if } 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$h(x)=\begin{cases} \text{arbitrary with } h(0)=d & \text{if } s=0=t\\ d & \text{if } s=0,\ t\neq 0\\ d & \text{if } s\neq 0,\ t=0\\ \dfrac{cx}{s}+d & \text{if } s=t\neq 0\\ -\dfrac{t}{x}A\left(\dfrac{x}{t}\right),\ x\neq 0 & \text{if } s=-t\neq 0\\ d & \text{if } 0\neq s^2\neq t^2\neq 0, \end{cases}$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and c, d, α are arbitrary real constants.

4. Solution of the functional equation (4)

Now we proceed to determine the most general solution of the equation (4).

THEOREM 2

Let s and t be real parameters. The functions $f, g, h, \phi, \psi : \mathbb{R}\to\mathbb{R}$ satisfy the functional equation (4) for all $x, y \in \mathbb{R}$ if and only if $g(x) = f(x)$ and

$$f(x)=\begin{cases} ax^2+(b+d)x+c & \text{if } s=0=t\\ ax^2+bx+c & \text{if } s=0,\ t\neq 0\\ ax^2+bx+c & \text{if } s\neq 0,\ t=0\\ 3ax^4+2bx^3+cx^2+(d+2\beta)x+\alpha & \text{if } s=t\neq 0\\ 2ax^3+cx^2+(2\beta-d)x-A(x)+\alpha & \text{if } s=-t\neq 0\\ -2bstx^3+\beta x^2+(2\gamma+\alpha-d)x+\delta & \text{if } 0\neq s^2\neq t^2\neq 0 \end{cases}$$

$$\phi(x)=\begin{cases} ax+\dfrac{b-\delta}{2} & \text{if } s=0=t\\ ax+\dfrac{b+\delta}{2} & \text{if } s=0,\ t\neq 0\\ ax+\dfrac{b+\delta}{2} & \text{if } s\neq 0,\ t=0\\ 2ax^3+bx^2+cx-A(x)+\beta+\dfrac{\delta}{2} & \text{if } s=t\neq 0\\ 3ax^2+cx-\dfrac{1}{2}A_0(x)+\beta & \text{if } s=-t\neq 0\\ bs(s-2t)x^2+\beta x+A(sx)+\gamma+\alpha & \text{if } 0\neq s^2\neq t^2\neq 0 \end{cases}$$

$$\psi(x) = \begin{cases} ax + \dfrac{b+\delta}{2} & \text{if } s = 0 = t \\ ax + \dfrac{b-\delta}{2} - h(tx) & \text{if } s = 0,\ t \neq 0 \\ ax + \dfrac{b-\delta}{2} - h(sx) & \text{if } s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta - \dfrac{\delta}{2} & \text{if } s = t \neq 0 \\ 3ax^2 + cx + \dfrac{1}{2}A_0(x) + \beta - d & \text{if } s = -t \neq 0 \\ bt(t-2s)x^2 + \beta x + A(tx) + \gamma & \text{if } 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$h(x) = \begin{cases} \text{arbitrary with } h(0) = d & \text{if } s = 0 = t \\ \text{arbitrary} & \text{if } s = 0,\ t \neq 0 \\ \text{arbitrary} & \text{if } s \neq 0,\ t = 0 \\ a\left(\dfrac{x}{t}\right)^3 + b\left(\dfrac{x}{t}\right)^2 + A\left(\dfrac{x}{t}\right) + d & \text{if } s = t \neq 0 \\ -a\left(\dfrac{x}{t}\right)^2 - \dfrac{t}{x}A\left(\dfrac{x}{t}\right) + \dfrac{1}{2}A_0\left(\dfrac{x}{t}\right),\ x \neq 0 & \text{if } s = -t \neq 0 \\ -bx^2 - A(x) - d & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

where $A_0, A : \mathbb{R} \to \mathbb{R}$ *are additive functions and* a, b, c, d, α, β, γ, δ *are arbitrary real constants.*

Proof. Letting $x = y$ in (4), we see that

$$f(x) = g(x) \tag{59}$$

for all $x \in \mathbb{R}$. Hence (59) in (4) yields

$$f(x) - f(y) = (x - y)[h(sx + ty) + \phi(x) + \psi(y)]. \tag{60}$$

Interchanging x and y in (60) and adding the resulting equation to (60), we have

$$h(sx + ty) + \phi(x) + \psi(y) = h(sy + tx) + \phi(y) + \psi(x) \tag{61}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. But (61) holds even for $x = y$.

Now we consider several cases.

Case 1. Suppose $s = 0 = t$. Then (61) yields

$$\phi(x) = \psi(x) - \delta, \tag{62}$$

where δ is a constant. Letting (62) into (60) we have

$$f(x) - f(y) = (x - y)[h(sx + ty) + \psi(x) + \psi(y) - \delta]. \tag{63}$$

Hence by Lemma 1, (59) and (62) we obtain the asserted solution

$$\left.\begin{aligned} f(x) &= ax^2 + (b + d)x + c \\ g(x) &= f(x) \\ \phi(x) &= ax + \frac{b - \delta}{2} \\ \psi(x) &= ax + \frac{b + \delta}{2} \\ h(x) &\quad \text{arbitrary with } h(0) = d, \end{aligned}\right\}$$

where a, b, c, d, δ are arbitrary constants.

Case 2. Suppose $s = 0$ and $t \neq 0$. (The case $s \neq 0$ and $t = 0$ can be handled in a similar manner.) Then for this case, from (61), we have

$$h(ty) + \psi(y) - \phi(y) = h(tx) + \psi(x) - \phi(x) \tag{64}$$

for all $x, y \in \mathbb{R}$. Thus

$$\psi(x) = \phi(x) - h(tx) - \delta \tag{65}$$

where δ is a constant. Letting (65) in (60) with $s = 0$, we see that

$$f(x) - f(y) = (x - y)[\phi(x) + \phi(y) - \delta]. \tag{66}$$

By Lemma 1, (59) and (65), we have the asserted solution

$$\left.\begin{aligned} f(x) &= ax^2 + bx + c \\ g(x) &= f(x) \\ \phi(x) &= ax + \frac{b + \delta}{2} \\ \psi(x) &= ax + \frac{b - \delta}{2} - h(tx) \\ h(x) &\quad \text{arbitrary}, \end{aligned}\right\}$$

where a, b, c, d, δ are arbitrary constants.

Case 3. Suppose $s \neq 0 \neq t$. Next, we consider several subcases.

Subcase 3.1. Suppose $s = t$. Then from (61), we get

$$h(tx + ty) + \phi(x) + \psi(y) = h(ty + tx) + \phi(y) + \psi(x). \tag{67}$$

Hence, we have

$$\phi(x) = \psi(x) - \delta, \tag{68}$$

where δ is a constant. Letting (68) into (60), we obtain

$$f(x) - f(y) = (x - y)[h(tx + ty) + \psi(x) + \psi(y) - \delta]. \tag{69}$$

From Theorem 1, (68) and (59), we obtain

$$\left.\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha \\ g(x) &= f(x) \\ \phi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta + \frac{\delta}{2} \\ \psi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta - \frac{\delta}{2} \\ h(x) &= a\left(\frac{x}{t}\right)^3 + b\left(\frac{x}{t}\right)^2 + A\left(\frac{x}{t}\right) + d, \end{aligned}\right\}$$

where a, b, c, d, α, β, δ are arbitrary constants and $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

Subcase 3.2. Suppose $s = -t$. From (61), we have

$$h(ty - tx) + \phi(x) + \psi(y) = h(tx - ty) + \phi(y) + \psi(x) \tag{70}$$

for all $x, y \in \mathbb{R}$. This in turn yields

$$h(tx - ty) - h(ty - tx) = H(x) - H(y), \tag{71}$$

where $H(x) := \phi(x) - \psi(x)$. Letting $x = 0$ in (71), we observe that

$$h(-ty) - h(ty) = d - H(y), \tag{72}$$

where $d = H(0)$. Using (72) in (71), we have

$$H(x - y) + d = H(x) + d - H(y) - d,$$

that is $H(x) + d$ is additive on the set of reals. Hence

$$\psi(x) = \phi(x) + A_0(x) - d, \tag{73}$$

where $A_0 : \mathbb{R} \to \mathbb{R}$ is an additive map. Substituting (73) into (60), we get

$$f(x) - f(y) = (x - y)[h(ty - tx) + \phi(x) + \phi(y) + A_0(y) - d] \tag{74}$$

which is

$$F(x) - F(y) = (x - y)[K(tx - ty) + \Phi(x) + \Phi(y)] \tag{75}$$

where

$$\left.\begin{aligned} F(x) &= f(x) + dx \\ K(x) &= h(-tx) - \frac{1}{2}A_0\left(\frac{x}{t}\right) \\ \Phi(x) &= \phi(x) + \frac{1}{2}A_0(x). \end{aligned}\right\} \tag{76}$$

Thus from Theorem 1, (59), (73) and (76), we again have the asserted solution

$$\left.\begin{aligned} f(x) &= 2ax^3 + cx^2 + (2\beta - d)x - A(x) + \alpha \\ g(x) &= f(x) \\ \phi(x) &= 3ax^2 + cx - \frac{1}{2}A_0(x) + \beta \\ \psi(x) &= 3ax^2 + cx + \frac{1}{2}A_0(x) + \beta - d \\ h(x) &= -a\left(\frac{x}{t}\right)^2 - \frac{t}{x}A\left(\frac{x}{t}\right) + \frac{1}{2}A_0\left(\frac{x}{t}\right), \quad x \neq 0, \end{aligned}\right\}$$

where a, b, c, d, α, β are arbitrary constants and $A, A_0 : \mathbb{R} \to \mathbb{R}$ are additive functions.

Subcase 3.3. Suppose $s^2 \neq t^2$. Letting $y = 0$ in (61), we get

$$h(sx) + \phi(x) + \psi(0) = h(tx) + \phi(0) + \psi(x). \tag{77}$$

Letting (77) in (61) and simplifying, we have

$$h(sx + ty) - h(sx) - h(ty) = h(sy + tx) - h(tx) - h(sy). \tag{78}$$

Replacing x by $\frac{x}{s}$ and y by $\frac{y}{t}$ in (60), we obtain

$$f\left(\frac{x}{s}\right) - f\left(\frac{y}{t}\right) = \left(\frac{xt - ys}{st}\right)\left[h(x + y) + \phi\left(\frac{x}{s}\right) + \psi\left(\frac{y}{t}\right)\right]. \tag{79}$$

Defining

$$\left.\begin{aligned} F(x) &= stf\left(\frac{x}{s}\right) \\ \Phi(x) &= \phi\left(\frac{x}{s}\right) \\ \Psi(x) &= \psi\left(\frac{x}{t}\right) \end{aligned}\right\} \tag{80}$$

and using (80) in (79), we have

$$F(x) - F\left(\frac{sy}{t}\right) = (xt - ys)[h(x + y) + \Phi(x) + \Psi(y)]. \tag{81}$$

Letting $y = 0$ and $x = 0$ separately in (81), we have

$$F(x) = F(0) + xt[h(x) + \Phi(x) + \Psi(0)] \tag{82}$$

and

$$F\left(\frac{sy}{t}\right) = F(0) + ys[h(y) + \Phi(0) + \Psi(y)], \tag{83}$$

respectively. Letting (82) and (83) into (81), we obtain (after some simplifications)

$$\begin{aligned} &xt[\Psi(0) - \Psi(y) - h(y)] - ys[\Phi(0) - \Phi(x) - h(x)] \\ &\quad = (xt - ys)[h(x+y) - h(x) - h(y)]. \end{aligned} \tag{84}$$

Interchanging x with y in (84), we obtain

$$\begin{aligned} &yt[\Psi(0) - \Psi(x) - h(x)] - xs[\Phi(0) - \Phi(y) - h(y)] \\ &\quad = (yt - xs)[h(x+y) - h(x) - h(y)]. \end{aligned} \tag{85}$$

Subtracting (85) from (84), we have

$$xP(y) - yP(x) = (x - y)(s+t)[h(x+y) - h(x) - h(y)], \tag{86}$$

where

$$P(x) = t[\Phi(0) - \Phi(x) - h(x)] + s[\Psi(0) - \Psi(x) - h(x)]. \tag{87}$$

The general solution of (86) can be obtained from Lemma 2 as

$$\left.\begin{aligned} P(x) &= 3ax^3 + 2bx^2 + cx + d \\ (s+t)h(x) &= -ax^3 - bx^2 - A(x) - d, \end{aligned}\right\} \tag{88}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and a, b, c, d are arbitrary constants. Letting the form of $h(x)$ from (88) in (78), we obtain

$$3a\, st\, xy(s - t)(x - y) = 0$$

for all $x, y \in \mathbb{R}$. Hence $a = 0$ as $0 \neq s^2 \neq t^2 \neq 0$. Thus, we have

$$(s+t)h(x) = -bx^2 - A(x) - d. \tag{89}$$

From (89) and (84), we have

$$\begin{aligned} &xt\left[\Psi(0) - \Psi(y) + \frac{by^2}{s+t} + \frac{A(y)}{s+t} + \frac{d}{s+t}\right] \\ &-ys\left[\Phi(0) - \Phi(y) + \frac{bx^2}{s+t} + \frac{A(x)}{s+t} + \frac{d}{s+t}\right] \\ &\quad = (xt - ys)\left[-\frac{2bxy}{s+t} + \frac{d}{s+t}\right] \\ &\quad = -\frac{2bt}{s+t}\frac{1}{s}syx^2 + \frac{2bs}{s+t}\frac{1}{t}txy^2 + (xt - ys)\frac{d}{s+t}, \end{aligned}$$

that is

$$
\begin{aligned}
xt &\left[\Psi(0) - \Psi(y) + \frac{by^2}{s+t} + \frac{A(y)}{s+t} - \frac{2bsy^2}{t(s+t)}\right] \\
&= ys\left[\Phi(0) - \Phi(y) + \frac{bx^2}{s+t} + \frac{A(x)}{s+t} - \frac{2btx^2}{s(s+t)}\right].
\end{aligned} \tag{90}
$$

Hence, we have

$$
\Phi(x) = \frac{bx^2}{s+t} - \frac{2btx^2}{s(s+t)} + \frac{A(x)}{s+t} + \frac{\beta x}{s} + \Phi(0), \tag{91}
$$

and

$$
\Psi(x) = \frac{bx^2}{s+t} - \frac{2bsx^2}{t(s+t)} + \frac{A(x)}{s+t} + \frac{\beta x}{t} + \Psi(0), \tag{92}
$$

where β is constant. From (89), (91), (92) and (80), we have

$$
\left.
\begin{aligned}
\phi(x) &= \frac{bs(s-2t)x^2}{s+t} + \beta x + \frac{A(sx)}{s+t} + \gamma + \alpha \\
\psi(x) &= \frac{bt(t-2s)x^2}{s+t} + \beta x + \frac{A(tx)}{s+t} + \gamma \\
h(x) &= -\frac{bx^2}{s+t} - \frac{A(x)}{s+t} - \frac{d}{s+t}.
\end{aligned}
\right\} \tag{93}
$$

From (80), (82), (89) and (93), we obtain

$$
f(x) = -\frac{2b\,st\,x^3}{s+t} + \beta x^2 + \left(2\gamma + \alpha - \frac{d}{s+t}\right)x + \delta. \tag{94}
$$

Renaming the constants $\frac{b}{s+t}$ as b, $\frac{d}{s+t}$ as d, and the additive function $\frac{A(x)}{s+t}$ as $A(x)$, we have from (93) and (94) the asserted solution

$$
\left.
\begin{aligned}
f(x) &= -2bstx^3 + \beta x^2 + (2\gamma + \alpha - d)x + \delta \\
\phi(x) &= bs(s-2t)x^2 + \beta x + A(sx) + \gamma + \alpha \\
\psi(x) &= bt(t-2s)x^2 + \beta x + A(tx) + \gamma \\
h(x) &= -bx^2 - A(x) - d.
\end{aligned}
\right\} \tag{95}
$$

Since no more cases are left, the proof of the theorem is now complete.

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