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Superadditive solutions of a functional equation

Abstract. Let $\{A^s: s \geq 0\}$ be a given iteration semigroup of additive set-valued functions. In this paper we study superadditive set-valued solutions Φ of the functional equation

$$\Phi(t+s) = A^s[\Phi(t)] + \Phi(s)$$

for $t, s \geq 0$.

1. We start with some definitions.

A set-valued function F defined on a convex cone S in a real vector space X into the set $n(Y)$ of all non-empty subsets of a real vector space Y is said to be *additive* iff

$$F(x+y) = F(x) + F(x)$$

for all $x, y \in S$.

A set-valued function F defined on a convex cone S in a real vector space X into the set $n(Y)$ is said to be *superadditive* iff

$$F(x) + F(x) \subset F(x+y)$$

for all $x, y \in S$.

A family

$$\{F^t: t \geq 0\}$$

of set-valued functions $F^t: S \to n(S)$ is said to be an *iteration semigroup* iff

$$F^t \circ F^s = F^{t+s},$$

in S, where $(F^t \circ F^s)(x) = \bigcup\{F^t(y): y \in F^s(x)\}$, for every $t, s \geq 0$ and $x \in S$.

In this paper we study superadditive solutions $\Phi: [0, \infty) \to n(S)$ of the functional equation

$$\Phi(t+s) = A^s[\Phi(t)] + \Phi(s) \tag{1}$$

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where $\{A^s : s \geq 0\}$ is a given iteration semigroup of additive set-valued functions $A^s : S \to n(S)$. This equation arises in a study of iteration semigroups of Jensen set-valued functions.

2. If Y is a topological vector space, we denote by $cc(Y)$ the familiy of all compact convex members of $n(Y)$.

We need the following lemma

LEMMA 1 (cf. [8])

Let A, B and C be subsets of a topological vector space such that

$$A + C \subset B + C.$$

If B is convex closed and C is non-empty bounded, then

$$A \subset B.$$

Let X be a Banach space and let $[a, b] \subset [0, \infty)$ be a closed interval. Suppose that a set-valued function $G : [a, b] \to cc(X)$ is continuous with respect to the Hausdorff distance. Then there exists the Hukuhara version of the Riemann integral

$$\int_a^b G(t)\, dt$$

(see [4]).

The following four lemmas describe some important properties of this integral (see [4]).

LEMMA 2

If $C \in cc(X)$ and $F(t) = C$ for every $t \in [a, b]$ then

$$\int_a^b F(t)\, dt = (b - a)C.$$

LEMMA 3

For every continuous set-valued functions F, G we have

$$d\left(\int_a^b F(t)\, dt, \int_a^b G(t)\, dt\right) \leq \int_a^b d(F(t), G(t))\, dt,$$

where d is the Hausdorff distance connected with the norm in X.

LEMMA 4

Let X and Y be two Banach spaces and let S be an open convex cone in X. If $F : [a, b] \to cc(S)$ is a continuous set-valued function and $A : S \to cc(Y)$ is a continuous additive set-valued function, then

$$\int_a^b A(F(t))\,dt = A\left(\int_a^b F(t)\,dt\right).$$

LEMMA 5

If $F : [a,b] \to X$ is continuous and $a < c < b$, then

$$\int_a^b F(u)\,du = \int_a^c F(u)\,du + \int_c^b F(u)\,du.$$

3. We start this section with the following auxiliary result

LEMMA 6 (see [7])

Let X and Y be two real separable Banach spaces. Assume that S is an open convex cone in X. Moreover, let $A : S \to cc(Y)$ be a lower semicontinuous additive set-valued function. Then there exists a constant $M \in (0,\infty)$ such that

$$d(A(x), A(y)) \le M||x-y||$$

for $x, y \in S$.

Let Y be a metric space. An *iteration semigroup* $\{F^t : t \ge 0\}$ of set-valued functions $F^t : Y \to cc(Y)$ is said to be *continuous* iff the function $t \mapsto F^t(y)$ is continuous for every $y \in Y$.

Now we can prove the main result of the paper.

THEOREM

Let X be a real separable Banach space and let S be an open convex cone in X. Suppose that $\{A^t : t \ge 0\}$ is a continuous iteration semigroup of continuous additive set-valued functions $A^t : \operatorname{cl} S \to cc(\operatorname{cl} S)$ such that

$$A^0 = \lim_{t\to 0+} A^t(x) = \{x\} \tag{2}$$

for every $x \in S$. A set-valued function $\Phi : [0,\infty) \to cc(\operatorname{cl} S)$ is a continuous superadditive solution of (1) if and only if there exists a set $D \in cc(\operatorname{cl} S)$ such that

$$\Phi(s) = \int_0^s A^u(D)\,du \tag{3}$$

and

$$D \subset A^s(D) \tag{4}$$

for every nonnegative s.

Proof. I. Suppose that $\Phi : [0,\infty) \to cc(\operatorname{cl} S)$ is an upper semicontinuous superadditive solution of equation (1). The graph of Φ is closed (see Theorem 6 of Chapter VI in [1]). According to Proposition 3 in [10] the function

$$t \mapsto \frac{\Phi(t)}{t}$$

is increasing in $(0, \infty)$. It is easy to see that the set

$$D = \bigcap_{t>0} \frac{\Phi(t)}{t}$$

is an element of $cc(S)$.

Let (t_n), $t_n \in (0,1)$, be an arbitrary sequence tending to zero. Take an arbitrary subsequence (t_{n_k}) of it. The family of compact subsets of compact metric space $\Phi(1)$ is compact in Hausdorff metric (see Chapter XVI of [5]) therefore there exists a compact set $E \subset \Phi(1)$ and a subsequence $\left(t_{n_{k_l}}^{-1} \Phi\left(t_{n_{k_l}}\right)\right)$ of $(t_{n_k}^{-1} \Phi(t_{n_k}))$ such that

$$t_{n_{k_l}}^{-1} \Phi\left(t_{n_{k_l}}\right) \to E.$$

Without loss of generality we can assume that the sequence $\left(t_{n_{k_l}}\right)$ is strictly decreasing. Consequently we have

$$E = \bigcap_{l=1}^{\infty} \mathrm{cl}\left(\bigcup_{p \geq l} \left(t_{n_{k_p}}\right)^{-1} \Phi\left(t_{n_{k_p}}\right)\right) = \bigcap_{l=1}^{\infty} \left(t_{n_{k_l}}\right)^{-1} \Phi\left(t_{n_{k_l}}\right) = D$$

(see Theorem II-2 in [2]). Therefore

$$(t_n)^{-1}\Phi(t_n) \to D,$$

thus the formula

$$D = \lim_{t \to 0} \frac{\Phi(t)}{t} \tag{5}$$

holds.

II. The set

$$A^{[0,1]}(x) = \bigcup\{A^t(x) : \ t \in [0,1]\}$$

being the image of $[0,1]$ by the continuous set-valued mapping $t \mapsto A^t(x)$ with compact values, is compact for every $x \in S$ (see [1], Chapter VI, Theorem 3). Let $\mathcal{F}$ be the family of all continuous additive maps $f : X \to X$ such that $f(x) \in A^{[0,1]}(x)$ for every $x \in S$. Fix a $z \in X$. Since S is an open cone there are $x, y \in S$ such that $z = x - y$. We have $\|f(z)\| \leq \left\|A^{[0,1]}(x)\right\| + \left\|A^{[0,1]}(y)\right\| < \infty$ for arbitrary $f \in \mathcal{F}$. Thus there exists a positive constant such that $\|f\| \leq M$ for every $f \in \mathcal{F}$.

Now, we are going to prove that the convergence in (2) is uniform on the set D. Suppose that it is not true. Then there exist $\varepsilon > 0$, $t_n \in \left(0, \frac{1}{n}\right)$ and $x_n \in D$ such that

$$d_n := \sup\left\{\|y - x_n\| : \ y \in A^{t_n}(x_n)\right\} \geq \varepsilon. \tag{6}$$

By the compactness of sets $A^{t_n}(x_n)$ there exist $y_n \in A^{t_n}(x_n)$ fulfilling the equalities

$$d_n = \|y_n - x_n\|. \tag{7}$$

According to Theorem 2 in [9], for every positive integer n, there exists an additive selection f_n of A^{t_n} such that $f_n(x_n) = y_n$. Each f_n has an extension belonging to $\mathcal{F}$. There exist a subsequence (x_{n_k}) of (x_n) and an $x_0 \in D$ such that

$$x_{n_k} \to x_0. \tag{8}$$

Since

$$f_{n_k}(x_0) - x_0 = (f_{n_k}(x_{n_k}) - x_{n_k}) - (f_{n_k}(x_{n_k}) - f_{n_k}(x_0)) - (x_0 - x_{n_k})$$

and by (6) and (7) we have

$$\|f_{n_k}(x_0) - x_0\| \geq \varepsilon - M\|x_{n_k} - x_0\| - \|x_0 - x_{n_k}\|. \tag{9}$$

By (2) the left hand side of (9) tends to zero and the right one tends to ε. This contradiction proves that the convergence in (2) is uniform on D.

Take a positive ε. There exists a positive number δ such that

$$d\left(A^t(x), \{x\}\right) < \varepsilon$$

whenever $0 < t < \delta$ and $x \in D$. We have

$$A^t(x) \subset x + \varepsilon B \subset D + \varepsilon B$$

for every $x \in D$, where B denotes the unit closed ball, and therefore

$$A^t(D) = \bigcup_{x \in D} A^t(x) \subset D + \varepsilon B. \tag{10}$$

Moreover

$$x \in A^t(x) + \varepsilon B \subset A^t(D) + \varepsilon B$$

for every $x \in D$ and, consequently,

$$D \subset A^t(D) + \varepsilon B. \tag{11}$$

Conditions (10) and (11) imply that

$$d(A^t(D), D) < \varepsilon,$$

whenever $t \in (0, \delta)$. Thus

$$\lim_{t \to 0+} A^t(D) = D. \tag{12}$$

III. Let s be a nonnegative number. According to Lemma 6 there exists the smallest real number $M(s)$ such that

$$d(A^s(x), A^s(y)) \leq M(s)\|x - y\|$$

for every $x, y \in S$.

It is obvious that the function M is nonnegative. It is also measurable since

$$M(s) = \sup_{n \neq m} \frac{d(A^s(x_n), A^s(x_m))}{\|x_n - x_m\|}$$

where $\{x_n : n = 1, 2, \ldots\}$ is a dense subset of S. Moreover, this function is submultiplicative. Now, we can define the function $m(s) := \log(M(s) + 1)$. It is a finite measurable subadditive function, whence by Theorem 7.4.1 in [3] it is bounded above on any compact subset of $(0, \infty)$. Thus for any positive s there exists a positive number K such that

$$M(u) \leq K$$

whenever $u \in [\frac{s}{2}, s]$. Now, let s be a positive number and let $t \in (0, \frac{s}{2})$. Then

$$d(A^{s+t}(D), A^s(D)) = d(A^s(A^t(D)), A^s(D)) \leq M(s) d(A^t(D), D)$$

and

$$\begin{aligned} d(A^s(D), A^{s-t}(D)) &= d(A^{s-t}(A^t(D)), A^{s-t}(D)) \\ &\leq M(s-t) d(A^t(D), D) \\ &\leq K d(A^t(D), D). \end{aligned}$$

These inequalities and (12) imply that the function $t \mapsto A^t(D)$ is continuous.

IV. Now, we can define

$$\Psi(t) = \int_0^t A^u(D)\, du$$

and

$$h(t) = d(\Phi(t), \Psi(t))$$

for nonnegative t. By Lemmas 5 and 4 we have

$$\Psi(t + s) = \int_0^{t+s} A^u(D)\, du = \Psi(s) + \int_0^t A^s(A^u(D))\, du = \Psi(s) + A^s(\Psi(t))$$

which shows that the set-valued function Ψ is a solution of (1).

For an arbitrary positive ε there exists a positive number δ such that

$$d(A^u(D), D) < \varepsilon$$

whenever $u \in (0, \delta)$. Lemmas 2 and 3 imply that

$$d\left(\frac{\Psi(t)}{t}, D\right) = d\left(\frac{1}{t}\int_0^t A^u(D)\,du, D\right) \le \frac{1}{t}\int_0^t d\,(A^u(D), D)\,du < \varepsilon,$$

whence

$$\lim_{t\to 0+} \frac{\Psi(t)}{t} = D. \tag{13}$$

This implies that $\lim_{t\to 0+} \Psi(t) = \{0\} = \Psi(0)$ and, consequently, Ψ is right hand side continuous at 0.

Now, let s be a positive number and $t \in (0, \frac{s}{2})$. Then

$$\begin{aligned} d(\Psi(s+t), \Psi(s)) &= d(A^s(\Psi(t)), \{0\}) \\ &\le d(A^s(\Psi(t)), A^s(tD)) + \|A^s(tD)\| \\ &\le M(s)\, t\, d\left(\frac{\Psi(t)}{t}, D\right) + t\|A^s(D)\| \end{aligned}$$

and

$$\begin{aligned} d(\Psi(s), \Psi(s-t)) &= d(A^{s-t}(\Psi(t)), \{0\}) \\ &\le d(A^{s-t}(\Psi(t)), A^{s-t}(tD)) + d(A^{s-t}(tD), \{0\}) \\ &\le M(s-t) d(\Psi(t), tD) + t\left\|A^{s-t}\right\| \\ &\le K\, t\, d\left(\frac{\Psi(t)}{t}, D\right) + t\left\|A^{s-t}\right\|. \end{aligned}$$

These inequalities and (13) imply that the funcion Ψ is continuous. It follows that so is h. Moreover,

$$\begin{aligned} D^+h(s) &= \limsup_{t\to 0} \frac{h(s+t) - h(s)}{t} \\ &= \limsup_{t\to 0} \frac{d(\Psi(s+t), \Phi(s+t)) - d(\Psi(s), \Phi(s))}{t} \\ &\le \limsup_{t\to 0} \frac{d(A^s(\Psi(t)), A^s(\Phi(t)))}{t} \\ &\le \lim_{t\to 0} M(s)\, d\left(\frac{\Psi(t)}{t}, \frac{\Phi(t)}{t}\right) \\ &= 0 \end{aligned}$$

Since h is a continuous function with $D_+h(s) \le 0$, according to Zygmund's Lemma (see [6]) the function h is non-increasing. Therefore $h(s) \le h(0) = 0$ for $s \ge 0$. This means that $\Psi = \Phi$.

The superadditivity of Φ and (1) imply that

$$\Phi(t) + \Phi(s) \subset \Phi(t+s) = A^s(\Phi(t)) + \Phi(s)$$

and by Lemma 1 we have (4), the monotonicity of the function $\frac{\Phi(t)}{t}$, and (5).

We know that formula (3) defines a solution of (1). According to condition (4) we have

$$\Phi(t) = \int_0^t A^u(D)\,du \subset \int_0^t A^u(A^s(D))\,du = \int_s^{t+s} A^u(D)\,du$$

so

$$\Phi(t) + \Phi(s) \subset \int_0^{t+s} A^u(D)\,du = \Phi(t+s).$$

This completes the proof.

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