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Oscillation and integration characterizations of bounded a.e. continuous functions

Dedicated to the memory of Imre Makai

Abstract. We show that a bounded vector valued function on a premeasurable set is a.e. continuous if and only if its upper or lower oscillation is zero.

Moreover, a bounded real valued function on a premeasurable set is a.e. continuous if and only if it is strongly Riemann or strongly (weakly) Darboux integrable.

Introduction

By abstracting the most important properties of the right-open intervals in $\mathbb{R}$ and their weighted contents, we introduce the notion of a Lebesgue-Stieltjes type metric premeasure space $(\Omega, \mathcal{S}, \nu)$. And having established the basic properties of the premeasurable system $\mathcal{S}$ and the premeasure ν, we investigate the family $\mathcal{N}_{L_\nu}$ of all Lebesgue ν-negligible subsets of Ω.

We define an ordered triple (X, Y, Z) of normed spaces to be a multiplication system with respect to a given bilinear map $(x, y) \to xy$ of $X \times Y$ into Z if $|xy| \leq |x||y|$ for all $x \in X$ and $y \in Y$. And we consider the space $\mathcal{BC}_{L_\nu}(D, X)$ of all bounded functions f from a subset D of Ω into X which are continuous Lebesgue ν-almost everywhere, and the space $\mathcal{BV}_\nu(\mathcal{S}_D, Y)$ of all finitely additive measures μ from $\mathcal{S}_D = \{A \in \mathcal{S} : \ A \subset D\}$ into Y which satisfy $|\mu(A)| \leq M\nu(A)$ for some $M \geq 0$ and all $A \in \mathcal{S}_D$.

If $A \in \mathcal{S}$, then a finite disjoint family $\sigma = (\sigma_i)_{i \in I}$ in $\mathcal{S}$ is called an $\mathcal{S}$-division of A if $A = \bigcup_{i \in I} \sigma_i$, and the collection of all such divisions is denoted by $\mathcal{D}(A)$. Moreover, a family $\tau = (\tau_i)_{i \in I}$ in A is called a tag for σ, and the collection of all such tags is denoted by $\mathcal{T}(\sigma)$. Whenever $\sigma \in \mathcal{D}(A)$ and $\tau \in \mathcal{T}(\sigma)$, then the

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ordered pair (σ, τ) is called a tagged $\mathcal{S}$-division of A, and the collection of all such tagged divisions is denoted by $\mathcal{DT}(A)$.

If $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$ and $\tau = (\tau_i)_{i\in I} \in \mathcal{T}(\sigma)$, then the extended real number

$$|(\sigma,\tau)|_\nu = \sup\{\operatorname{diam}(\sigma_i \cup \{\tau_i\}) : \quad \nu(\sigma_i) \neq 0\}$$

is called the ν-norm of (σ, τ). And the extended real number

$$d_\nu(\sigma,\tau) = \sup\{d(\sigma_i,\tau_i) : \quad \nu(\sigma_i) \neq 0\}$$

is called the ν-distance of σ and τ. Moreover, a sequence $((\sigma_n, \tau_n))$ in $\mathcal{DT}(A)$ is called ν-normal if $\overline{\lim}_{n\to\infty}|(\sigma_n,\tau_n)|_\nu \leq 0$.

If $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$, and moreover $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$ and $\tau = (\tau_i)_{i\in I} \in \mathcal{T}(\sigma)$, then the vector

$$S(f,\mu,\sigma,\tau) = \sum_{i\in I} f(\tau_i)\mu(\sigma_i)$$

is called the Riemann sum of f with respect to μ corresponding to the tagged division (σ, τ). Moreover, the function f is called strongly Riemann ν-integrable with respect to μ if the limit

$$\nu\text{-}\int_A f\,d\mu = \lim_{n\to\infty} S(f,\mu,\sigma_n,\tau_n)$$

exists for every ν-normal sequence $((\sigma_n, \tau_n))$ in $\mathcal{DT}(A)$.

If $r > 0$, then the relation

$$B_r = \{(t,s) \in \Omega^2 : \quad d(t,s) < r\}$$

is called the r-surrounding of the diagonal Δ_Ω. And if $A \subset \Omega$, then the sets

$$A^{\circ_r} = \{t \in \Omega : \quad B_r(t) \subset A\} \quad \text{and} \quad A^{-_r} = \{t \in \Omega : \quad B_r(t) \cap A \neq \emptyset\}$$

are called the r-interior and r-closure of A. Note that $A^{-_r} = B_r(A)$, and moreover $A^\circ = \bigcup_{r>0} A^{\circ_r}$ and $A^- = \bigcap_{r>0} A^{-_r}$.

If $f \in \mathcal{B}(A, X)$, $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$ and $r > 0$, then the extended real numbers

$$\Omega_{*r}(f,\nu,\sigma) = \sum_{i\in I} \operatorname{diam} f(\sigma_i^{\circ_r})\nu(\sigma_i)$$

and

$$\Omega^{*r}(f,\nu,\sigma) = \sum_{i\in I} \operatorname{diam} f(\sigma_i^{-_r})\nu(\sigma_i)$$

are called the r-size lower and upper ν-oscillations of f on σ. Moreover, the real numbers

$$\Omega_*(f,\nu,\sigma)=\sup_{r>0}\Omega_{*r}(f,\nu,\sigma)\quad\text{and}\quad \Omega^*(f,\nu,\sigma)=\inf_{r>0}\Omega^{*r}(f,\nu,\sigma)$$

are called the lower and upper ν-oscillations of f on σ. And the real numbers

$$\Omega_*(f,\nu,A)=\inf_{\sigma\in\mathcal{D}(A)}\Omega_*(f,\nu,\sigma)\quad\text{and}\quad \Omega^*(f,\nu,A)=\inf_{\sigma\in\mathcal{D}(A)}\Omega^*(f,\nu,\sigma)$$

are called the lower and upper ν-oscillations of f on A.

Similarly, if $f\in\mathcal{B}(A,\mathbb{R})$, $\sigma=(\sigma_i)_{i\in I}\in\mathcal{D}(A)$ and $r>0$, then we define

$$L_{*r}(f,\nu,\sigma)=\sum_{i\in I}\inf f(\sigma_i^{-r})\nu(\sigma_i),\qquad L^{*r}(f,\nu,\sigma)=\sum_{i\in I}\inf f(\sigma_i^{\circ r})\nu(\sigma_i),$$
$$U_{*r}(f,\nu,\sigma)=\sum_{i\in I}\sup f(\sigma_i^{\circ r})\nu(\sigma_i),\qquad U^{*r}(f,\nu,\sigma)=\sum_{i\in I}\sup f(\sigma_i^{-r})\nu(\sigma_i).$$

Moreover, we define

$$L_*(f,\nu,\sigma)=\sup_{r>0}L_{*r}(f,\nu,\sigma),\qquad L^*(f,\nu,\sigma)=\inf_{r>0}L^{*r}(f,\nu,\sigma),$$
$$U_*(f,\nu,\sigma)=\sup_{r>0}U_{*r}(f,\nu,\sigma),\qquad U^*(f,\nu,\sigma)=\inf_{r>0}U^{*r}(f,\nu,\sigma).$$

And we define

$$\int_{-*A}f\,d\nu=\sup_{\sigma\in\mathcal{D}(A)}L_*(f,\nu,\sigma),\qquad \int_{-A}^{*}f\,d\nu=\sup_{\sigma\in\mathcal{D}(A)}L^*(f,\nu,\sigma),$$
$$\int_{*A}^{-}f\,d\nu=\inf_{\sigma\in\mathcal{D}(A)}U_*(f,\nu,\sigma),\qquad \int_{A}^{-*}f\,d\nu=\inf_{\sigma\in\mathcal{D}(A)}U^*(f,\nu,\sigma).$$

It turns out that $\int_{-*A}f\,d\nu\le\int_{-A}^{*}f\,d\nu\le\int_{*A}^{-}f\,d\nu\le\int_{A}^{-*}f\,d\nu$. Therefore, the function f may be called strongly (resp. weakly) Darboux integrable with respect to ν if $\int_{-*A}f\,d\nu=\int_{A}^{-*}f\,d\nu$ (resp. $\int_{-A}^{*}f\,d\nu=\int_{*A}^{-}f\,d\nu$).

Now, the main results of the paper can be briefly summarized in the following three statements:

Theorem 1

If $f\in\mathcal{B}(A,X)$ for some $A\in\mathcal{S}$, then the following assertions are equivalent:

(1) $f\in\mathcal{BC}_{L_\nu}(A,X)$; (2) $\Omega^*(f,\nu,A)=0$; (3) $\Omega_*(f,\nu,A)=0$.

Corollary

If $f\in\mathcal{BC}_{L_\nu}(A,X)$ and $\mu\in\mathcal{BV}_\nu(\mathcal{S}_A,Y)$ for some $A\in\mathcal{S}$, then f is strongly Riemann ν-integrable with respect to μ provided Z is complete.

THEOREM 2

If $f \in \mathcal{B}(A, \mathbb{R})$ for some $A \in \mathcal{S}$, then the following assertions are equivalent:

(1) $f \in \mathcal{BC}_{L_\nu}(A, \mathbb{R})$;

(2) *f is strongly Riemann integrable with respect to ν;*

(3) *f is strongly (weakly) Darboux integrable with respect to ν.*

The proof of the implication (1)$\Longrightarrow$(2) in Theorem 1 is based upon the observation that for each open cover $\mathcal{V}$ of A^- there exist a $(\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and an $r > 0$ such that the family $(\sigma_i^{-r})_{i \in I}$ refines $\mathcal{V}$ in the sense that for each $i \in I$ there exists a $V_i \in \mathcal{V}$ such that $\sigma_i^{-r} \subset V_i$.

While the proof of the above Corollary is based upon the observation that

$$|S(f, \mu, \sigma, \tau) - S(f, \mu, \rho, \omega)| \leq \Omega^{*r}(f, \nu, \rho)\, |\mu|_\nu,$$

whenever (σ, τ) and (ρ, ω) are in $\mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < r$ and $d_\nu(\rho, \omega) < r$.

1. Premeasure spaces

DEFINITION 1.1

Let Ω be a metric space, and assume that $\mathcal{S}$ is a nonvoid family of subsets of Ω and ν is a function from $\mathcal{S}$ into $[0, +\infty[$ such that:

(1) if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;

(2) if $A, B \in \mathcal{S}$ and $B \subset A$, then there exists a disjoint family $(C_i)_{i=1}^n$ in $\mathcal{S}$ such that
$$A \setminus B = \bigcup_{i=1}^n C_i;$$

(3) if $t \in V \subset \Omega$ and V is open, then there exists an $A \in \mathcal{S}$ such that $t \in A^\circ$ and $A \subset V$;

(4) if $A \in \mathcal{S}$, then A^- is compact;

(5) if $A \in \mathcal{S}$ and $(A_i)_{i=1}^n$ is a disjoint family in $\mathcal{S}$ such that $A = \bigcup_{i=1}^n A_i$, then
$$\nu(A) = \sum_{i=1}^n \nu(A_i);$$

(6) if $A \in \mathcal{S}$ and $\varepsilon > 0$, then there exist $B, C \in \mathcal{S}$ such that
$$B^- \subset A^\circ, \qquad A^- \subset C^\circ \quad \text{and} \quad \nu(C) - \nu(B) < \varepsilon.$$

Then the ordered triple $(\Omega, \mathcal{S}, \nu)$ will be called a Lebesgue-Stieltjes type metric premeasure space.

The above definition is mainly motivated by the fact that it is inconvenient to start with a higher dimensional counterpart of the following

EXAMPLE 1.2

If φ is an increasing function on $\mathbb{R}$, C_φ is the set of all continuity points of φ,

$$\mathcal{J}_\varphi = \{[\alpha, \beta[: \quad \alpha, \beta \in C_\varphi, \quad \alpha \leq \beta\},$$

and

$$v_\varphi([\alpha, \beta[) = \varphi(\beta) - \varphi(\alpha) \qquad ([\alpha, \beta[\in \mathcal{J}_\varphi),$$

then $(\mathbb{R}, \mathcal{J}_\varphi, v_\varphi)$ is a Lebesgue-Stieltjes type metric premeasure space.

The necessary verifications are left to the reader. Note that the set D_φ of all discontinuity points of φ is at most countable, and therefore $C_\varphi = \mathbb{R} \setminus D_\varphi$ is dense in $\mathbb{R}$. Moreover, note that a precise proof of the property (5) requires induction on n.

REMARK 1.3

In particular $\mathcal{J} = \mathcal{J}_{\Delta_\mathbb{R}}$ and $v = v_{\Delta_\mathbb{R}}$ will be called the Lebesgue premeasurable system and premeasure on $\mathbb{R}$, respectively.

Moreover, $\mathcal{J}_\varphi$ and v_φ will be called the Lebesgue-Stieltjes premeasurable system and premeasure on $\mathbb{R}$ generated by φ, respectively.

The properties (1)–(6) listed in Definition 1.1 have several useful consequences. The most immediate ones are summarized in the next remarks.

REMARK 1.4

The disjointness property (2), together with the nonvoidness of $\mathcal{S}$, implies that $\emptyset \in \mathcal{S}$.

The additivity property (5), together with the finiteness of $\nu(\emptyset)$, implies that $\nu(\emptyset) = 0$.

REMARK 1.5

The intersection property (1) implies that $\mathcal{S}$ is closed under the formation of finite nonvoid intersections.

The base property (3), together with the regularity of Ω, implies that for each $t \in \Omega$ the family $\{A^-: \ t \in A^\circ, \ A \in \mathcal{S}\}$ is also a neighbourhood base at t.

REMARK 1.6

The properties (2) and (5), together with the nonnegativity of ν, imply that $\nu(B) \leq \nu(A)$ whenever $A, B \in \mathcal{S}$ such that $B \subset A$.

Namely, if A, B and C_i are as in (2), then $A = B \cup \bigcup_{i=1}^n C_i$, and hence by (5) and the nonegativity of ν, we have $\nu(A) = \nu(B) + \sum_{i=1}^n \nu(C_i) \geq \nu(B)$.

REMARK 1.7

Therefore, in the regularity property (6) we actually have $0 \le \nu(A) - \nu(B) < \varepsilon$ and $0 \le \nu(C) - \nu(A) < \varepsilon$.

Moreover, by compactness property (4) it is clear that if A, B and C are as in (6), then there exists an $r > 0$ such that $B^{-r} \subset A^{\circ r}$ and $A^{-r} \subset C^{\circ r}$.

In connection with Definition 1.1, it is also worth considering the following notes.

NOTE 1.8

Example 1.2 would allow us to assume a strengthenig of (2) that for each $A, B \in \mathcal{S}$, with $B \subset A$, there exists an increasing family $(C_i)_{i=0}^n$ in $\mathcal{S}$ such that $C_0 = B$, $C_n = A$ and $C_i \setminus C_{i-1} \in \mathcal{S}$ for all $i = 1, \ldots, n$.

The importance of this latter property seems to lie mainly in the striking observation of Halmos [6, p. 31] that this property, together with the property (1) and the $n = 2$ particular case of (5), implies the property (5).

NOTE 1.9

Example 1.2 would also allow us to assume that for each bounded subset B of Ω there exists an $A \in \mathcal{S}$ such that $B \subset A$.

This boundedness property, together with the compactness property (4), would, in particular, imply the completeness and the separability of Ω.

NOTE 1.10

Finally, we note that the properties (1) and (2) can usually be replaced by the weaker assumption that for each $A, B \in \mathcal{S}$ there exists a disjoint family $(C_i)_{i=1}^n$ in $\mathcal{S}$ such that $A \setminus B = \bigcup_{i=1}^n C_i$.

Namely, according to [12, Corollary 1.11], this latter property already implies that for each $A, B \in \mathcal{S}$ there exists a disjoint family $(D_j)_{j=1}^m$ in $\mathcal{S}$ such that $A \cap B = \bigcup_{j=1}^m D_j$.

2. Division properties

To briefly formulate some less trivial consequences of the properties (1)–(6), it is convenient to introduce the following

DEFINITION 2.1

A disjoint family $(A_i)_{i \in I}$ in $\mathcal{S}$ will be called an $\mathcal{S}$-division of a subset A of Ω if $A = \bigcup_{i \in I} A_i$.

Moreover, a subset A of Ω having at least one finite (countable) $\mathcal{S}$-division will be called finitely (countably) $\mathcal{S}$-divisible.

REMARK 2.2

Whenever $(A_i)_{i\in I}$ is an $\mathcal{S}$-division of a subset of A, then sometimes it is also convenient to say that $(A_i)_{i\in I}$ is an $\mathcal{S}$-division in A.

Moreover, for the above mentioned purposes, it is also convenient to indroduce the following

DEFINITION 2.3

If $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ are families of sets, then we say that:

(1) $(A_i)_{i\in I}$ refines $(B_j)_{j\in J}$ if for each $i\in I$ there exists $j\in J$ such that $A_i\subset B_j$;

(2) $(A_i)_{i\in I}$ divides $(B_j)_{j\in J}$ if for each $j\in J$ there exists $I_j\subset I$ such that $B_j=\bigcup_{i\in I_j} A_i$.

REMARK 2.4

Note that if (1) holds, and moreover $\bigcup_{j\in J} B_j\subset\bigcup_{i\in I} A_i$ and $(B_j)_{j\in J}$ is disjoint, then (2) also holds.

While if (2) holds, and moreover $\bigcup_{i\in I} A_i\subset\bigcup_{j\in J} B_j$ and $(A_i)_{i\in I}$ is disjoint, then (1) also holds, provided that $J\neq\emptyset$.

Now, we can briefly state the following basic theorem which is certainly familiar to the reader.

THEOREM 2.5

(1) *If $A\in\mathcal{S}$ and $(A_i)_{i=1}^n$ is a family in $\mathcal{S}$, then $A\setminus\bigcup_{i=1}^n A_i$ is finitely $\mathcal{S}$-divisible.*

(2) *If $(A_i)_{i=1}^n$ is a family in $\mathcal{S}$, then there exists an $\mathcal{S}$-division $(B_j)_{j=1}^m$ of $\bigcup_{i=1}^n A_i$ such that $(B_j)_{j=1}^m$ divides $(A_i)_{i=1}^n$.*

(3) *If $(A_i)_{i=1}^\infty$ is a family in $\mathcal{S}$, then there exists an $\mathcal{S}$-division $(B_j)_{j=1}^\infty$ of $\bigcup_{i=1}^\infty A_i$ such that $(B_j)_{j=1}^\infty$ refines $(A_i)_{i=1}^\infty$.*

Hint. The assertions (1) and (2) can be proved by induction on n. Note, for instance, that if $(A_i)_{i=1}^n$ and $(B_j)_{j=1}^m$ are as in (2) and $A_{n+1}\in\mathcal{S}$, then

$$\bigcup_{i=1}^{n+1} A_i=\left(\bigcup_{j=1}^m A_{n+1}\cap B_j\right)\cup\left(\bigcup_{j=1}^m B_j\setminus A_{n+1}\right)\cup\left(A_{n+1}\setminus\bigcup_{j=1}^m B_j\right).$$

Moreover, by (1) there exist an $\mathcal{S}$-divison $(C_{jk})_{k=1}^{p_j}$ of $B_j\setminus A_{n+1}$ for every $j=1,\dots,m$, and an $\mathcal{S}$-division $(D_l)_{l=1}^q$ of $A_{n+1}\setminus\bigcup_{j=1}^m B_j$. Therefore, by observing that

$$\mathcal{B}' = \{A_{n+1} \cap B_j\}_{j=1}^{m} \cup \left(\bigcup_{j=1}^{m} \{C_{jk}\}_{k=1}^{p_j}\right) \cup \{D_l\}_{l=1}^{q}$$

is a finite disjoint collection in $\mathcal{S}$ and taking an injection B' of $\{1, \dots, m'\}$ onto $\mathcal{B}'$ for some $m' \in \mathbb{N}$, we can state that $(B_j')_{j=1}^{m'}$, where $B_j' = B'(j)$, is an $\mathcal{S}$-division of $\bigcup_{i=1}^{n+1} A_i$ such that $(B_j')_{j=1}^{m'}$ divides $(A_i)_{i=1}^{n+1}$.

The proof of the assertion (3) relies on the facts that

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup \bigcup_{i=2}^{\infty}\left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)$$

is a disjoint union and for each $i = 2, 3, \dots$ there exists an $\mathcal{S}$-division $(B_{ik})_{k=1}^{n_i}$ of $A_i \setminus \bigcup_{j=1}^{i-1} A_j$.

Remark 2.6

By considering the families $([i^{-1}, 1[)_{i=1}^{\infty}$ and $([0, i^{-1}[)_{i=1}^{\infty}$, it can be easily seen that the assertions (1) and (2) cannot be extended to countable families.

Moreover, it is also worth noticing that if the family $(A_i)_{i=1}^{\infty}$ in the assertion (3) is locally finite at a point of Ω, then the division $(B_j)_{j=1}^{\infty}$ can also be stated to be locally finite there.

Now, as an important aplication of Theorem 2.5, we can also prove

Theorem 2.7

If $A \in \mathcal{S}$ and $\mathcal{V}$ is an open cover of A^-, then there exist an $\mathcal{S}$-division $(A_i)_{i=1}^{n}$ of A and an $r > 0$ such that $(A_i^{-r})_{i=1}^{n}$ refines $\mathcal{V}$.

Proof. Since $\mathcal{V}$ covers A^-, for each $t \in A^-$ there exists a $V_t \in \mathcal{V}$ such that $t \in V_t$. Moreover, since each V_t is open, for each $t \in A^-$ there exists a $\delta_t > 0$ such that $B_{\delta_t}(t) \subset V_t$. Furthermore, if $r_t = 2^{-1}\delta_t$ for all $t \in A^-$, then by the base property of $\mathcal{S}$ for each $t \in A^-$ there exists an $A_t \in \mathcal{S}$ such that $t \in A_t^\circ$ and $A_t \subset B_{r_t}(t)$.

Now, since $(A_t^\circ)_{t \in A^-}$ is an open cover of A^- and A^- is compact, there exists a family $(t_j)_{j=1}^{m}$ in A^- such that $A^- \subset \bigcup_{j=1}^{m} A_{t_j}^\circ$. Therefore, we have $A \subset \bigcup_{j=1}^{m} A_{t_j}$, and hence $A = \bigcup_{j=1}^{m} A \cap A_{t_j}$. Thus, by the second assertions of Theorem 2.5 and Remark 2.4, there exists an $\mathcal{S}$-divison $(A_i)_{i=1}^{n}$ of A such that $(A_i)_{i=1}^{n}$ refines $\left(A_{t_j}\right)_{j=1}^{m}$.

Moreover, if $r = \min\left\{r_{t_j}\right\}_{j=1}^{m}$, then it is clear that $(A_i^{-r})_{i=1}^{n}$ refines $\mathcal{V}$. Namely, for each $i \in \{1, \dots, n\}$ there exists a $j \in \{1, \dots, m\}$ such that $A_i \subset A_{t_j}$, and thus

$$A_i^{-r} = B_r(A_i) \subset B_r(A_{t_j}) \subset B_{r_{t_j}}(B_{r_{t_j}}(t_j)) \subset B_{\delta_{t_j}}(t_j) \subset V_{t_j}.$$

REMARK 2.8

Note that in this case the families $(A_i)_{i=1}^n$ and $(A_i^-)_{i=1}^n$ also refine $\mathcal{V}$, since $A_i \subset A_i^- \subset A_i^{-r}$ holds for all $i = 1, \ldots, n$.

Hence, since $\mathcal{V} = (B_{\varepsilon/3}(t))_{t\in\Omega}$, for each $\varepsilon > 0$, is an open cover of A^- such that $\operatorname{diam}(V) < \varepsilon$ for all $V \in \mathcal{V}$, it is clear that in particular we also have

COROLLARY 2.9

If $A \in \mathcal{S}$ and $\varepsilon > 0$, then there exists an $\mathcal{S}$-division $(A_i)_{i=1}^n$ of A such that $\operatorname{diam}(A_i) < \varepsilon$ *for all* $i = 1, \ldots, n$.

3. Measure properties

The subsequent results are almost standard, therefore the proofs are included here only for the reader's convenience.

LEMMA 3.1

If $A \in \mathcal{S}$, then

(1) $\sum_{i=1}^{n} \nu(A_i) \le \nu(A)$ *whenever $(A_i)_{i=1}^n$ is an $\mathcal{S}$-division in A;*

(2) $\nu(A) \le \sum_{i=1}^{n} \nu(A_i)$ *whenever $(A_i)_{i=1}^n$ is an $\mathcal{S}$-cover of A.*

Proof. The assertion (1) is an immediate consequence of Theorem 2.5(1) and the finite additivity and the nonnegativity of ν.

To prove (2), note that if $(A_i)_{i=1}^n$ is as in (2), then $A = \bigcup_{i=1}^n A_i \cap A$. Therefore, by Theorem 2.5(2) and Remark 2.4, there exists an $\mathcal{S}$-division $(B_j)_{j=1}^m$ of A such that $(B_j)_{j=1}^m$ refines $(A_i \cap A)_{i=1}^n$. Hence, by the finite additivity of ν and the assertion (1), it is clear that

$$\nu(A) = \sum_{j=1}^{m} \nu(B_j) \le \sum_{i=1}^{n} \sum_{B_j \subset A_i} \nu(B_j) \le \sum_{i=1}^{n} \nu(A_i).$$

By using the above covering properties and the regularity property of ν, we can now easily show that the finitely additive premeasure ν is actually countably additive.

Theorem 3.2

If $A \in \mathcal{S}$ and $(A_i)_{i=1}^{\infty}$ is an $\mathcal{S}$-division of A, then

$$\nu(A) = \sum_{i=1}^{\infty} \nu(A_i).$$

Proof. Because of the regularity property of ν, for each $\varepsilon > 0$ there exists a $B \in \mathcal{S}$ such that

$$B^- \subset A \qquad \text{and} \qquad \nu(A) - \nu(B) < \varepsilon.$$

Moreover, for each $i \in \mathbb{N}$, there exists a $C_i \in \mathcal{S}$ such

$$A_i \subset C_i^\circ \qquad \text{and} \qquad \nu(C_i) - \nu(A_i) < \varepsilon/2^i.$$

Now, since $B^- \subset \bigcup_{i=1}^{\infty} C_i^\circ$ and B^- is compact, there exists an injective family $(i_k)_{k=1}^{n}$ in $\mathbb{N}$ such that $B^- \subset \bigcup_{k=1}^{n} C_{i_k}^\circ$, and hence $B \subset \bigcup_{k=1}^{n} C_{i_k}$. Therefore, by Lemma 3.1(2)

$$\nu(A) - \varepsilon \leq \nu(B) \leq \sum_{k=1}^{n} \nu(C_{i_k}) \leq \sum_{i=1}^{\infty} \nu(C_i) \leq \sum_{i=1}^{\infty} \nu(A_i) + \varepsilon.$$

And hence, by letting $\varepsilon \to 0$, we can infer that $\nu(A) \leq \sum_{i=1}^{\infty} \nu(A_i)$.

The converse inequality is immediate from Lemma 3.1(1).

Now, as a useful consequence of Lemma 3.1 and Theorem 3.2, we can also prove

Corollary 3.3

If $(A_i)_{i=1}^{\infty}$ is an $\mathcal{S}$-division in Ω and $(B_j)_{j=1}^{\infty}$ is an $\mathcal{S}$-cover of $\bigcup_{i=1}^{\infty} A_i$, then

$$\sum_{i=1}^{\infty} \nu(A_i) \leq \sum_{j=1}^{\infty} \nu(B_j).$$

Proof. Note that now we have $A_i \subset \bigcup_{j=1}^{\infty} A_i \cap B_j$ for all $i \in \mathbb{N}$. Hence, quite similarly as in the proof of Lemma 3.1(2), but using Theorems 2.5(3) and 3.2 instead of Theorem 2.5(2) and the finite additivity of ν, we can infer that $\nu(A_i) \leq \sum_{j=1}^{\infty} \nu(A_i \cap B_j)$ for all $i \in \mathbb{N}$.

Moreover, note that now we also have $\bigcup_{i=1}^{\infty} A_i \cap B_j \subset B_j$ for all $j \in \mathbb{N}$. Hence, by using Lemma 3.1(1), we can infer that $\sum_{i=1}^{\infty} \nu(A_i \cap B_j) \leq \nu(B_j)$ for all $j \in \mathbb{N}$. Therefore, we have

$$\sum_{i=1}^{\infty} \nu(A_i) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \nu(A_i \cap B_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \nu(A_i \cap B_j) \leq \sum_{j=1}^{\infty} \nu(B_j).$$

Note that Corollary 3.3 is a substantial generalization of not only Lemma 3.1, but also of Theorem 3.2. Namely, from this corollary, we can at once get

COROLLARY 3.4

If $(A_i)_{i=1}^{\infty}$ *and* $(B_j)_{j=1}^{\infty}$ *are* $\mathcal{S}$*-divisions of the same subset of* Ω *then*

$$\sum_{i=1}^{\infty} \nu(A_i) = \sum_{j=1}^{\infty} \nu(B_j).$$

REMARK 3.5

This corollary would in particular allow of an easy extension of ν to the countably $\mathcal{S}$-divisible subsets of Ω.

Finally, we note that, by using Corollary 3.4, one can also easily prove the following monotone continuity property of ν.

THEOREM 3.6

If $A \in \mathcal{S}$ *and* $(A_i)_{i=1}^{\infty}$ *is an increasing (resp. decreasing) sequence in* $\mathcal{S}$ *such that* $A = \bigcup_{i=1}^{\infty} A_i$ *(resp.* $A = \bigcap_{i=1}^{\infty} A_i$*), then*

$$\nu(A) = \lim_{i \to \infty} \nu(A_i).$$

Hint. Note that if $(A_i)_{i=1}^{\infty}$ decreasing and $A = \bigcap_{i=1}^{\infty} A_i$, then

$$A_1 \setminus A = \bigcup_{i=1}^{\infty} (A_i \setminus A_{i+1})$$

is a disjoint union. Moreover, by the disjointness property of $\mathcal{S}$, there exist an $\mathcal{S}$-division $(B_j)_{j=1}^{p}$ of $A_1 \setminus A$ and an $\mathcal{S}$-division $(C_{ik})_{k=1}^{q_i}$ of $A_i \setminus A_{i+1}$ for every $i \in \mathbb{N}$. Therefore, by Corollary 3.4, we have

$$\begin{aligned} \nu(A_1) - \nu(A) = \sum_{j=1}^{p} \nu(B_j) &= \sum_{i=1}^{\infty} \sum_{k=1}^{q_i} \nu(C_{ik}) \\ &= \sum_{i=1}^{\infty} (\nu(A_i) - \nu(A_{i+1})) = \nu(A_1) - \lim_{n \to \infty} \nu(A_n), \end{aligned}$$

and hence $\nu(A) = \lim_{n \to \infty} \nu(A_n)$.

NOTE 3.7

Note that the above results actually depend only on some of the properties (1), (2), (5) and (6) of $\mathcal{S}$ and ν.

Moreover, it is also worth noticing that only one half of the regularity property of ν has been needed to prove the σ-additivity of ν.

The other half of the regularity property of ν will be needed to show only that the members of $\mathcal{S}$ have negligible boundaries.

4. Negligible sets

DEFINITION 4.1

A point t of Ω will be called ν-negligible if for each $\varepsilon > 0$ there exists an $A \in \mathcal{S}$ such that $t \in A$ and $\nu(A) < \varepsilon$.

And the family of all ν-negligible points of ν will be denoted by N_ν.

REMARK 4.2

The ν-negligible points of Ω may now also be called the continuity points of ν.

Namely, because of the corresponding properties of ν and $\mathcal{S}$, for any $t \in \Omega$, we now have $t \in N_\nu$ if and only if for each $\varepsilon > 0$ there exists a neighbourhood V of t such that $\nu(A) < \varepsilon$ whenever $A \in \mathcal{S}$ such that $A \subset V$.

DEFINITION 4.3

A subset A of Ω will be called Lebesgue ν-negligible if for each $\varepsilon > 0$ there exists an $\mathcal{S}$-cover $(A_i)_{i=1}^{\infty}$ of A such that $\sum_{i=1}^{\infty} \nu(A_i) < \varepsilon$.

And the family of all Lebesgue ν-negligible sets will be denoted by $\mathcal{N}_{L_\nu}$.

REMARK 4.4

A subset A of Ω may be called Jordan ν-negligible if for each $\varepsilon > 0$ there exists an $\mathcal{S}$-cover $(A_i)_{i=1}^{n}$ of A such that $\sum_{i=1}^{n} \nu(A_i) < \varepsilon$. And the family of all Jordan ν-negligible sets may be denoted by $\mathcal{N}_{J_\nu}$.

Because of $\emptyset \in \mathcal{S}$ and $\nu(\emptyset) = 0$, it is clear that $\mathcal{N}_{J_\nu} \subset \mathcal{N}_{L_\nu}$. Moreover, it can be easily seen that $\mathbb{N} \in \mathcal{N}_{L_\nu}$, but $\mathbb{N} \notin \mathcal{N}_{J_\nu}$. Therefore, the converse inclusion need not be true.

The basic properties of the Lebesgue ν-negligible sets are listed in the following

THEOREM 4.5

(1) $\{t\} \in \mathcal{N}_{L_\nu} \iff t \in N_\nu$;

(2) *if* $A \in \mathcal{N}_{L_\nu}$ *and* $B \subset A$, *then* $B \in \mathcal{N}_{L_\nu}$;

(3) *if* $(A_i)_{i=1}^{\infty}$ *is a family in* $\mathcal{N}_\nu$, *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}_{L_\nu}$.

Hint. Note that if $(A_i)_{i=1}^{\infty}$ is in $\mathcal{N}_{L_\nu}$, then for each $\varepsilon > 0$ and $i \in \mathbb{N}$ there exists an $\mathcal{S}$-cover $(A_{(i,j)})_{j=1}^{\infty}$ of A_i such that $\sum_{j=1}^{\infty} \nu(A_{(i,j)}) < \varepsilon/2^i$. Hence, by taking an injection φ of $\mathbb{N}$ onto $\mathbb{N}^2$, we can get an $\mathcal{S}$-cover $(A_{\varphi(k)})_{k=1}^{\infty}$

of $\bigcup_{i=1}^{\infty} A_i$ such that $\sum_{k=1}^{\infty} \nu(A_{\varphi(k)}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \nu(A_{(i,j)}) < \varepsilon$. Therefore, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}_{L_\nu}$ is also true.

Now, as a trivial consequence of the above theorem, we can also state

COROLLARY 4.6
If A is a countable subset of N_ν then $A \in \mathcal{N}_{L_\nu}$.

REMARK 4.7
Note that in the assertions (1) and (2) of Theorem 4.5, we may write $\mathcal{N}_{J_\nu}$ in place of $\mathcal{N}_{L_\nu}$.

But the family $\mathcal{N}_{J_\nu}$ is, in general, closed only under finite unions. Therefore, we can only state that finite subsets of N_ν are in $\mathcal{N}_{J_\nu}$.

To have some larger Lebesgue ν-negligible sets than the countable ones, we can also prove

THEOREM 4.8
If $A \in \mathcal{S}$, then $A \in \mathcal{N}_{L_\nu}$ if and only if $\nu(A) = 0$.

Hint. Note that if $A \in \mathcal{N}_{L_\nu}$, then for each $\varepsilon > 0$ there exists an $\mathcal{S}$-cover $(A_i)_{i=1}^{\infty}$ of A such that $\sum_{i=1}^{\infty} \nu(A_i) < \varepsilon$. Therefore, by Corollary 3.3, we also have $\nu(A) < \varepsilon$. And this implies that $\nu(A) = 0$.

THEOREM 4.9
If $A \in \mathcal{S}$ and $\varepsilon > 0$, then there exist an $r > 0$ and an $\mathcal{S}$-cover $(A_i)_{i=1}^{n}$ of $\partial_r A$ such that $\sum_{i=1}^{n} \nu(A_i) < \varepsilon$.

Proof. By Remark 1.7, there exist $B, C \in \mathcal{S}$ and $r > 0$ such that $B \subset A^{\circ_r}$, $A^{-r} \subset C$ and $\nu(C) - \nu(B) < \varepsilon$.

Moreover, by the disjointness property of $\mathcal{S}$, there exists an $\mathcal{S}$-division $(A_i)_{i=1}^{n}$ of $C \setminus B$. Hence, by the finite additivity of ν, it is clear that $\sum_{i=1}^{n} \nu(A_i) = \nu(C) - \nu(B)$.

Now, since $\partial_r A = A^{-r} \setminus A^{\circ_r}$, it is clear that $\partial_r A \subset C \setminus B = \bigcup_{i=1}^{n} A_i$. Moreover, it is clear that $\sum_{i=1}^{n} \nu(A_i) < \varepsilon$.

Hence, since $\partial A = \bigcap_{r>0} \partial_r A$, it is clear that in particular we also have

THEOREM 4.10
If $A \in \mathcal{S}$, then $\partial A \in \mathcal{N}_{L_\nu}$.

REMARK 4.11
Note that in Theorems 4.8 and 4.10, we may write $\mathcal{N}_{J_\nu}$ in place of $\mathcal{N}_{L_\nu}$.

Now, as a useful consequence of Theorems 4.8 and 4.10, we can also state

THEOREM 4.12

If $A \in \mathcal{S}$ such that $\nu(A) \neq 0$, then there exists an $\delta > 0$ such that $A^{\circ r} \notin \mathcal{N}_{L_\nu}$, and hence $A^{\circ r} \neq \emptyset$ for all $r \in]0, \delta]$.

Proof. In this case, by Theorem 4.8, $A \notin \mathcal{N}_{L_\nu}$. Therefore, since $A \subset A^\circ \cup \partial A$, by Theorems 4.10 and 4.5, $A^\circ \notin \mathcal{N}_{L_\nu}$. And hence, since $A^\circ = \bigcup_{n=1}^{\infty} A^{\circ 1/n}$, by Theorem 4.5, $A^{\circ\delta} \notin \mathcal{N}_{L_\nu}$ for some $\delta = 1/n$ with $n \in \mathbb{N}$. Moreover, it is clear that $A^{\circ\delta} \subset A^{\circ r}$ for all $r \in]0, \delta]$. And thus, again by Theorem 4.5, $A^{\circ r} \notin \mathcal{N}_{L_\nu}$ for all $r \in]0, \delta]$.

Moreover, as some useful characterectizations of Lebesgue ν-negligible sets, we can also prove

THEOREM 4.13

If $A \subset \Omega$, then $A \in \mathcal{N}_{L_\nu}$ if and only if for each $\varepsilon > 0$ there exists an interior (closure) $\mathcal{S}$-cover $(C_i)_{i=1}^{\infty}$ of A such that $\sum_{i=1}^{\infty} \nu(C_i) < \varepsilon$.

Hint. Note that if $A \in \mathcal{N}_{L_\nu}$, then for each $\varepsilon > 0$ there exists an $\mathcal{S}$-cover $(A_i)_{i=1}^{\infty}$ of A such that $\sum_{i=1}^{\infty} \nu(A_i) < \varepsilon/2$. Moreover, because of the regularity property of ν, for each $i \in \mathbb{N}$ there exists a $C_i \in \mathcal{S}$ such that $A_i \subset C_i^\circ$ and $\nu(C_i) - \nu(A_i) < \varepsilon/2^{i+1}$. And hence, it is clear that $A \subset \bigcup_{i=1}^{\infty} C_i^\circ$ and $\sum_{i=1}^{\infty} \nu(C_i) < \varepsilon$.

Now, as an immediate consequence of Theorem 4.13 and Remark 4.4, we can also state

COROLLARY 4.14

If A is a compact subset of Ω, then $A \in \mathcal{N}_{L_\nu}$ if and only if $A \in \mathcal{N}_{J_\nu}$.

Moreover, by using Theorem 2.5(3) and Corollary 3.3, we can also easily prove

THEOREM 4.15

If $A \subset \Omega$, then $A \in \mathcal{N}_{L_\nu}$ if and only if for each $\varepsilon > 0$ there exists a disjoint $\mathcal{S}$-cover $(D_i)_{i=1}^{\infty}$ of A such that $\sum_{i=1}^{\infty} \nu(D_i) < \varepsilon$.

NOTE 4.16

Whenever the boundedness property of $\mathcal{S}$ is also assumed, then the family $(D_i)_{i=1}^{\infty}$ can also be stated to be locally finite at each point of the set $\bigcup_{i=1}^{\infty} D_i^-$.

Moreover, in this case as some extension of Corollary 4.14, we can also prove that $A \in \mathcal{N}_{J_\nu}$ if and only if $A^- \in \mathcal{N}_{L_\nu}$ and A is bounded.

Or even more generally, $A^- \in \mathcal{N}_{L_\nu}$ if and only if there exists a locally finite (disjoint) family $(A_i)_{i=1}^{\infty}$ in $\mathcal{N}_{J_\nu}$ such that $A = \bigcup_{i=1}^{\infty} A_i$.

5. A.e. continuous functions and dominated premeasures

DEFINITION 5.1

If X, Y and Z are normed spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $(x, y) \to xy$ is a bilinear map of $X \times Y$ into Z such that

$$|xy| \leq |x||y|$$

for all $x \in X$ and $y \in Y$, then the ordered triple (X, Y, Z) will be called a multiplication system of normed spaces with respect to the above bilinear map.

REMARK 5.2

Multiplication systems play an important role in advanced calculus. (See, for instance, Lang [7, pp. 135, 372 and 455].)

The above definition can be well motivated by the next useful theorem and the several important examples listed in [14].

THEOREM 5.3

If X, Y and Z are normed spaces over $\mathbb{K}$ and u is a bilinear map of $X \times Y$ into Z, then u is continuous if and only if there exists a $c > 0$ such that $|u(x, y)| \leq c|x||y|$ for all $x \in X$ and $y \in Y$.

REMARK 5.4

Note that if the latter inequality holds, then by considering the new norm $c|\ |$ on X or Y, or the new bilinear map $c^{-1}u$, the triple (X, Y, Z) becomes a multiplication system.

DEFINITION 5.5

A function f from a subset D of Ω into X will be called continuous Lebesgue ν-almost everywhere if the set D_f of all discontinuity points of f is in $\mathcal{N}_{L_\nu}$.

The family of all such functions will be denoted by $\mathcal{C}_{L_\nu}(D, X)$. And the family of all bounded members of $\mathcal{C}_{L_\nu}(D, X)$ will be denoted by $\mathcal{BC}_{L_\nu}(D, X)$.

REMARK 5.6

Quite similarly f may be called continuous Jordan ν-almost everywhere if $D_f \in \mathcal{N}_{J_\nu}$. Moreover, the family of all such functions may be denoted by $\mathcal{C}_{J_\nu}(D, X)$, and the family of all bounded members of $\mathcal{C}_{J_\nu}(D, X)$ may be denoted by $\mathcal{BC}_{J_\nu}(D, X)$. Clearly, $\mathcal{C}_{J_\nu}(D, X) \subset \mathcal{C}_{L_\nu}(D, X)$, and hence $\mathcal{BC}_{J_\nu}(D, X) \subset \mathcal{BC}_{L_\nu}(D, X)$, but the converse inclusions are not true in general.

Among the several useful properties of a.e. continuous functions established in [14], we shall only mention here the following

THEOREM 5.7

The family $\mathcal{BC}_{L_\nu}(D,X)$, with the pointwise linear operations and the uniform norm, is a normed space such that $\mathcal{BC}_{L_\nu}(D,X)$ is complete whenever X is complete.

Hint. Let $\mathcal{B}(D,X)$ be the family of all bounded functions from D into X. Moreover, for each $t \in D$, let $\mathcal{C}_t(D,X)$ be the family of all functions from D into X which are continuous at t, and let $\mathcal{BC}_t(D,X)$ be the family of all bounded members of $\mathcal{C}_t(D,X)$.

Assume that $f \in \mathcal{B}(D,X)$ and (f_n) is a sequence in $\mathcal{BC}_{L_\nu}(D,X)$ such that

$$\lim_{n\to\infty} |f_n - f|_u = 0,$$

and define

$$C_f = \{t \in D: \ f \in \mathcal{C}_t(D,X)\} \quad \text{and} \quad D_f = D \setminus C_f.$$

Then, by the closedness of $\mathcal{BC}_t(D,X)$ in $\mathcal{B}(D,X)$,

$$\bigcap_{n=1}^{\infty} C_{f_n} \subset C_f, \quad \text{and hence} \quad D_f \subset \bigcup_{n=1}^{\infty} D_{f_n}.$$

Moreover, we have $D_{f_n} \in \mathcal{N}_{L_\nu}$ for all $n \in \mathbb{N}$, and hence by Theorem 4.5, $D_f \in \mathcal{N}_{L_\nu}$. Therefore, $f \in \mathcal{BC}_{L_\nu}(D,X)$. Consequently, $\mathcal{BC}_{L_\nu}(D,X)$ is also closed in $\mathcal{B}(D,X)$.

REMARK 5.8

Note that the normed space $\mathcal{BC}_{J_\nu}(D,X)$ need not be complete even if X is complete.

DEFINITION 5.9

If $D \subset \Omega$ and μ is a function from

$$\mathcal{S}_D = \{A \in \mathcal{S}: \ A \subset D\}$$

into Y such that:

(1) if $A \in \mathcal{S}_D$ and $(A_i)_{i=1}^{n}$ is an $\mathcal{S}$-division of A, then

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i);$$

(2) there exists an $M \geq 0$ such that for all $A \in \mathcal{S}_D$

$$|\mu(A)| \leq M\nu(A);$$

then μ will be called an Y-valued ν-dominated premeasure on $\mathcal{S}_D$.

And the family of all such premeasures will be denoted by $\mathcal{BV}_\nu(\mathcal{S}_D, Y)$.

EXAMPLE 5.10

Take $D = [\alpha, \beta[\in \mathcal{J}$, and let g be a function of bounded variation from D^- into Y. And define $\varphi_g(t) = 0$ for $-\infty < t < \alpha$,

$$\varphi_g(t) = \overset{t}{\underset{\alpha}{V}}(g) \quad \text{for} \quad \alpha \leq t \leq \beta$$

and $\varphi_g(t) = \overset{\beta}{\underset{\alpha}{V}}(g)$ for $\beta < t < +\infty$. Moreover, define

$$\mu_g([t, s[) = g(s) - g(t) \qquad ([t, s[\in (\mathcal{J}_{\varphi_g})_D),$$

where $\mathcal{J}_{\varphi_g}$ is as in Example 1.2. Then $\mu_g \in \mathcal{BV}_{v_{\varphi_g}}((\mathcal{J}_{\varphi_g})_D, Y)$.

In particular, it is clear that if g is a Lipschitz function from D^- into Y, then $\mu_g \in \mathcal{BV}_v(\mathcal{J}_D, Y)$.

REMARK 5.11

Note that if $\alpha \leq t \leq s \leq \beta$, then

$$|g(s) - g(t)| \leq \overset{s}{\underset{t}{V}}(g) = \varphi_g(s) - \varphi_g(t).$$

Therefore, $C_{\varphi_g} \cap [\alpha, \beta] \subset C_g$. Moreover, by [7, p. 225], the converse inclusion is also true.

DEFINITION 5.12

If $\mu \in \mathcal{BV}_\nu(\mathcal{S}_D, Y)$, then the number

$$|\mu|_\nu = \inf\{M \geq 0 : \ |\mu(A)| \leq M\nu(A) \text{ for all } A \in \mathcal{S}_D\}$$

will be called the ν-uniform norm of μ.

REMARK 5.13

If $\mu \in \mathcal{BV}_\nu(\mathcal{S}_D, Y)$, then $|\mu|_\nu$ is the smallest nonnegative number such that

$$|\mu(A)| \leq |\mu|_\nu \nu(A)$$

for all $A \in \mathcal{S}_D$.

Namely, by the above definition, we have $|\mu(A)| \leq (|\mu|_\nu + \varepsilon)\nu(A)$ for all $A \in \mathcal{S}_D$ and $\varepsilon > 0$, whence by letting $\varepsilon \to 0$, the stated inequality follows.

Now, because of the complete analogy of the space $\mathcal{BV}_\nu(\mathcal{S}_D, Y)$ to the space $\mathcal{L}(X, Y)$ of all bounded linear maps from X into Y [7, p. 359], it is clear that the following theorem is also true.

THEOREM 5.14

The family $\mathcal{BV}_\nu(\mathcal{S}_D, Y)$, with the pointwise linear operations and the ν-uniform norm, is a normed space such that $\mathcal{BV}_\nu(\mathcal{S}_D, Y)$ is complete whenever Y is complete.

REMARK 5.15

Moreover, it is also worth mentioning that the σ-additivity, and the continuity and regularity properties of ν are inherited by the ν-dominated premeasures [14].

6. Oscillations and approximating sums

DEFINITION 6.1

If $f : D \subset \Omega \to X$, $E \subset \Omega$ and $t \in \Omega$, then the extended real numbers

$$\omega_f(E) = \operatorname{diam}(f(E)) \qquad \text{and} \qquad \omega_f(t) = \inf_{r>0} \omega_f(B_r(t))$$

will be called the oscillations of f on E and at t, respectively.

REMARK 6.2

Note that $\omega_f(E) = -\infty$ if $D \cap E = \emptyset$; $0 \leq \omega_f(E) \leq +\infty$ if $D \cap E \neq \emptyset$; and $\omega_f(E) \leq 2|f|_u < +\infty$ if f is bounded.

The importance of the oscillations taken at points lies mainly in the next simple

PROPOSITION 6.3

If $f : D \subset \Omega \to X$ and $t \in D$, then f is continous at t if and only if $\omega_f(t) = 0$.

DEFINITION 6.4

If $A \in \mathcal{S}$, then the collection of all finite $\mathcal{S}$-divisions of A will be denoted by $\mathcal{D}(A)$.

Moreover, if $f \in \mathcal{B}(A, X)$ and $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$, then the real numbers

$$\Omega(f, \nu, \sigma) = \sum_{i \in I} \omega_f(\sigma_i)\nu(\sigma_i) \qquad \text{and} \qquad \Omega(f, \nu, A) = \inf_{\sigma \in \mathcal{D}(A)} \Omega(f, \nu, \sigma)$$

will be called the ν-oscillations of f on σ and A, respectively.

REMARK 6.5

Note that $0 \leq \omega_f(\sigma_i) \leq 2|f|_u$, whenever $\sigma_i \neq \emptyset$. Therefore, because of $\nu \geq 0$ and $\nu(\emptyset) = 0$, we have

$$0 \leq \Omega(f, \nu, A) \leq \Omega(f, \nu, \sigma) \leq 2|f|_u \nu(A).$$

Now we may also naturally introduce the following more complicated

DEFINITION 6.6

If $f \in \mathcal{B}(A, X)$, $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and $r > 0$, then the extended real numbers

$$\Omega_{*r}(f,\nu,\sigma) = \sum_{i\in I} \omega_f(\sigma_i^{\circ r})\nu(\sigma_i) \quad \text{and} \quad \Omega^{*r}(f,\nu,\sigma) = \sum_{i\in I} \omega_f(\sigma_i^{-r})\nu(\sigma_i)$$

will be called the r-size lower and upper ν-oscillations of f on σ.

Moreover, the real numbers

$$\Omega_*(f,\nu,\sigma) = \sup_{r>0} \Omega_{*r}(f,\nu,\sigma) \quad \text{and} \quad \Omega^*(f,\nu,\sigma) = \inf_{r>0} \Omega^{*r}(f,\nu,\sigma)$$

will be called the lower and upper ν-oscillations of f on σ.

And the real numbers

$$\Omega_*(f,\nu,A) = \inf_{\sigma\in\mathcal{D}(A)} \Omega_*(f,\nu,\sigma) \quad \text{and} \quad \Omega^*(f,\nu,A) = \inf_{\sigma\in\mathcal{D}(A)} \Omega^*(f,\nu,\sigma)$$

will be called the lower and upper ν-oscillations of f on A.

REMARK 6.7

Note that

$$0 \le \omega_f(\sigma_i^{\circ r}) \le \omega_f(\sigma_i) \le \omega_f(\sigma_i^{-r}) \le 2|f|_u$$

whenever $\sigma_i^{\circ r} \neq \emptyset$. Therefore, by Theorem 4.12, we have

$$0 \le \Omega_{*r}(f,\nu,\sigma) \le \Omega(f,\nu,\sigma) \le \Omega^{*r}(f,\nu,\sigma) \le 2|f|_u\nu(A)$$

for all sufficiently small $r > 0$. And hence, we can now easily infer that

$$0 \le \Omega_*(f,\nu,\sigma) \le \Omega(f,\nu,\sigma) \le \Omega^*(f,\nu,\sigma) \le 2|f|_u\nu(A),$$

and thus

$$0 \le \Omega_*(f,\nu,A) \le \Omega(f,\nu,A) \le \Omega^*(f,\nu,A) \le 2|f|_u\nu(A).$$

DEFINITION 6.8

If $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$, then the collection of all families $\tau = (\tau_i)_{i\in I}$ in A will be denoted by $\mathcal{T}(\sigma)$. And the collection of all families $\tau = (\tau_i)_{i\in I}$ in A such that $\tau_i \in \sigma_i$, whenever $\nu(\sigma_i) \neq 0$, will be denoted by $\mathcal{C}_\nu(\sigma)$.

Moreover, the collection of all ordered pairs (σ,τ) such that $\sigma \in \mathcal{D}(A)$ and $\tau \in \mathcal{T}(\sigma)$ will be denoted by $\mathcal{DT}(A)$. And the collection of all ordered pairs (σ,τ) such that $\sigma \in \mathcal{D}(A)$ and $\tau \in \mathcal{C}_\nu(\sigma)$ will be denoted by $\mathcal{DC}_\nu(A)$.

REMARK 6.9

If $\tau \in \mathcal{T}(\sigma)$, then we shall say that τ is a tag for σ, and (σ,τ) is a tagged $\mathcal{S}$-division of A. While if $\tau \in \mathcal{C}_\nu(\sigma)$, then we shall say that τ is a ν-choice for σ, and (σ,τ) is a ν-choiced $\mathcal{S}$-division of A.

Now, by making use of the multiplication system (X,Y,Z), we can also introduce the following

DEFINITION 6.10

If $A \in \mathcal{S}$, $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$, and moreover $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and $\tau = (\tau_i)_{i \in I} \in \mathcal{T}(\sigma)$, then the vector

$$S(f, \mu, \sigma, \tau) = \sum_{i \in I} f(\tau_i)\mu(\sigma_i)$$

will be called the Riemann sum of f with respect to μ corresponding to the tagged division (σ, τ) of A.

To define limits of the above approximating sums, we shall also need

DEFINITION 6.11

If $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and $\tau = (\tau_i)_{i \in I} \in \mathcal{T}(\sigma)$, then the extended real numbers

$$|\sigma|_\nu = \sup\{\operatorname{diam}(\sigma_i) : \quad \nu(\sigma_i) \neq 0\}$$

and

$$|(\sigma, \tau)|_\nu = \sup\{\operatorname{diam}(\sigma_i \cup \{\tau_i\}) : \quad \nu(\sigma_i) \neq 0\}$$

will be called the ν-norms of σ and (σ, τ). And the extended real number

$$d_\nu(\sigma, \tau) = \sup\{d(\sigma_i, \tau_i) : \quad \nu(\sigma_i) \neq 0\}$$

will be called the ν-distance of σ and τ.

REMARK 6.12

Note that $-\infty \leq |\sigma|_\nu \leq |(\sigma, \tau)|_\nu < +\infty$ and

$$-\infty \leq d_\nu(\sigma, \tau) \leq |(\sigma, \tau)|_\nu \leq |\sigma|_\nu + d_\nu(\sigma, \tau) < +\infty$$

such that $|\sigma|_\nu$, $|(\sigma, \tau)|_\nu$ or $d_\nu(\sigma, \tau)$ equals $-\infty$ if and only if $\nu(A) = 0$, and $d_\nu(\sigma, \tau) \leq 0$ if $\tau \in \mathcal{C}_\nu(\sigma)$.

The above unusual definitions are mainly motivated by the next useful

THEOREM 6.13

If $A \in \mathcal{S}$, $f \in \mathcal{B}(A, X)$ and $\mu \in \mathcal{BV}_\nu(\mathcal{S}_A, Y)$, and moreover $(\sigma, \tau) \in \mathcal{DT}(A)$ and $(\rho, \omega) \in \mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < r$ and $d_\nu(\rho, \omega) < r$, then

$$|S(f, \mu, \sigma, \tau) - S(f, \mu, \rho, \omega)| \leq \Omega^{*r}(f, \nu, \rho)\, |\mu|_\nu.$$

Proof. If $\sigma = (\sigma_i)_{i \in I}$, $\tau = (\tau_i)_{i \in I}$, $\rho = (\rho_j)_{j \in J}$ and $\omega = (\omega_j)_{j \in J}$, then it is clear that, for each $i \in I$ and $j \in J$,

$$\nu(\sigma_i \cap \rho_j) \neq 0 \Longrightarrow \tau_i, \omega_j \in B_r(\rho_j).$$

Namely, if $\nu(\sigma_i \cap \rho_j) \neq 0$, then $\nu(\sigma_i) \neq 0$ and $\nu(\rho_j) \neq 0$, and thus $\operatorname{diam}(\sigma_i \cup \{\tau_i\}) < r$ and $d(\rho_j, \omega_j) < r$. Moreover, $\sigma_i \cap \rho_j \neq \emptyset$, and therefore $d(\rho_j, \tau_i) < r$ also holds.

On the other hand, it is clear that, for each $i \in I$ and $j \in J$,

$$(\sigma_i \cap \rho_j)_{j \in J} \in \mathcal{D}(\sigma_i) \quad \text{and} \quad (\sigma_i \cap \rho_j)_{i \in I} \in \mathcal{D}(\rho_j),$$

and thus

$$\mu(\sigma_i) = \sum_{j \in J} \mu(\sigma_i \cap \rho_j) \quad \text{and} \quad \mu(\rho_j) = \sum_{i \in I} \mu(\sigma_i \cap \rho_j).$$

Therefore, we have

$$\begin{aligned}
|S(f,\mu,\sigma,\tau) - S(f,\mu,\rho,\omega)| &= \left| \sum_{i \in I} f(\tau_i)\mu(\sigma_i) - \sum_{j \in J} f(\omega_j)\mu(\rho_j) \right| \\
&= \left| \sum_{i \in I} \sum_{j \in J} (f(\tau_i) - f(\omega_j))\, \mu(\sigma_i \cap \rho_j) \right| \\
&\leq \sum_{i \in I} \sum_{j \in J} |f(\tau_i) - f(\omega_j)|\, |\mu(\sigma_i \cap \rho_j)| \\
&\leq \sum_{i \in I} \sum_{j \in J} |f(\tau_i) - f(\omega_j)|\, |\mu|_\nu \nu(\sigma_i \cap \rho_j) \\
&\leq \sum_{i \in I} \sum_{j \in J} \omega_f\,(B_r(\rho_j))\, |\mu|_\nu \nu(\sigma_i \cap \rho_j) \\
&= \sum_{j \in J} \omega_f\,(B_r(\rho_j))\, |\mu|_\nu \nu(\rho_j) \\
&= \Omega^{*r}\,(f,\nu,\rho)\, |\mu|_\nu.
\end{aligned}$$

7. Oscilation characterizations of bounded a.e. continuous functions

The following theorem is of fundamental importance for our integration procedure.

THEOREM 7.1
If $f \in \mathcal{B}C_{L_v}(A,X)$ for some $A \in \mathcal{S}$, then $\Omega^*(f,\nu,A) = 0$.

Proof. Define

$$C = \{t \in A : \ f \in \mathcal{C}_t(A,X)\} \quad \text{and} \quad D = A \setminus C,$$

where $\mathcal{C}_t(A,X)$ is the family of all functions from A into X which are continuous at t. Then, by Theorems 4.10, 4.5(3) and 4.13, for each $\varepsilon > 0$, there exists an interior $\mathcal{S}$-cover $(A_j)_{j=1}^{\infty}$ of $D \cup \partial A$ such that

$$\sum_{j=1}^{\infty} \nu(A_j) < \varepsilon.$$

Moreover, by Proposition 6.3, for each $t \in C$, there exists a $\delta_t > 0$ such that

$$\omega_f(B_{\delta_t}(t)) < \varepsilon.$$

Now, it is clear that

$$\mathcal{C} = \{A_j^\circ\}_{j=1}^{\infty} \cup \{B_{\delta_t}(t)\}_{t \in C}$$

is an open cover of A^-. Thus, by Theorem 2.7, there exist a division $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and an $r > 0$ such that $(\sigma_i^{-r})_{i \in I}$ refines $\mathcal{C}$. Define

$$I_1 = \{i \in I : \quad \exists\, j : \quad \sigma_i \subset A_j\} \qquad \text{and} \qquad I_2 = I \setminus I_1.$$

Then, by Lemma 3.1(1), it is clear that

$$\sum_{i \in I_1} \nu(\sigma_i) \leq \sum_{j=1}^{\infty} \sum_{\sigma_i \subset A_j} \nu(\sigma_i) \leq \sum_{j=1}^{\infty} \nu(A_j) < \varepsilon.$$

Moreover, since $(\sigma_i^{-r})_{i \in I}$ refines $\mathcal{C}$, for each $i \in I_2$ there exists a $t_i \in C$ such that $\sigma_i^{-r} \subset B_{\delta_{t_i}}(t_i)$. Therefore,

$$\omega_f(\sigma_i^{-r}) \leq \omega_f(B_{\delta_{t_i}}(t_i)) < \varepsilon$$

for all $i \in I_2$.

Now, since $\omega_f(\sigma_i^{-r}) \leq 2|f|_u$ and $\sum_{i \in I} \nu(\sigma_i) = \nu(A)$, it is clear that

$$\begin{aligned} \Omega^{*r}(f, \nu, \sigma) &= \sum_{i \in I_1} \omega_f(\sigma_i^{-r})\nu(\sigma_i) + \sum_{i \in I_2} \omega_f(\sigma_i^{-r})\nu(\sigma_i) \\ &\leq \sum_{i \in I_1} 2|f|_u \nu(\sigma_i) + \sum_{i \in I_2} \varepsilon\nu(\sigma_i) \leq 2|f|_u\varepsilon + \varepsilon\nu(A). \end{aligned}$$

Therefore,

$$\Omega^*(f, \nu, A) \leq 2(|f|_u + \nu(A))\varepsilon,$$

and hence $\Omega^*(f, \nu, A) = 0$.

Moreover, as a certain converse to Theorem 7.1, we can also prove the following less important

THEOREM 7.2

If $f \in \mathcal{B}(A, X)$ *for some* $A \in \mathcal{S}$ *such that* $\Omega_*(f, \nu, A) = 0$, *then* $f \in \mathcal{BC}_{L_\nu}(A, X)$.

Proof. Define again

$$C = \{t \in A : \quad f \in \mathcal{C}_t(A, X)\} \qquad \text{and} \qquad D = A \setminus C,$$

and moreover

$$D_n = \{t \in A : \quad \omega_f(t) > n^{-1}\}$$

for all $n \in \mathbb{N}$. Then, by Proposition 6.3, it is clear that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Therefore, by Theorem 4.5(3), it is enough to prove only that $D_n \in \mathcal{N}_{L_\nu}$ for all $n \in \mathbb{N}$.

Since $\Omega_*(f, \nu, A) = 0$, for each $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ such that for all $r > 0$

$$\sum_{i \in I} \omega_f(\sigma_i^{\circ r})\nu(\sigma_i) < (2n)^{-1}\varepsilon.$$

Now, since

$$A = \bigcup_{i \in I} \sigma_i \subset \bigcup_{i \in I} \sigma_i^{-r} = \bigcup_{i \in I} (\sigma_i^{\circ r} \cup \partial_r \sigma_i),$$

it is clear that

$$D_n \subset \left(\bigcup_{i \in I} \sigma_i^{\circ r} \cap D_n\right) \cup \left(\bigcup_{i \in I} \partial_r \sigma_i \cap D_n\right) \subset \bigcup_{i \in J_r} \sigma_i \cup \bigcup_{i \in I} \partial_r \sigma_i,$$

where

$$J_r = \{i \in I : \quad \sigma_i^{\circ r} \cap D_n \neq \emptyset\}.$$

Moreover, by Theorem 4.9, it is clear that there exist an $r > 0$ and an $\mathcal{S}$-cover $(A_k)_{k=1}^m$ of $\bigcup_{i \in I} \partial_r \sigma_i$ such that $\sum_{k=1}^m \nu(A_k) < 2^{-1}\varepsilon$. Therefore, it is enough to show only that $\sum_{i \in J_r} \nu(\sigma_i) < 2^{-1}\varepsilon$.

For this, note that if $i \in J_r$, then there exists a $t_i \in \Omega$ such that $t_i \in \sigma_i^{\circ r}$ and $t_i \in D_n$, and hence

$$B_r(t_i) \subset \sigma_i \qquad \text{and} \qquad \omega_f(t_i) > n^{-1}.$$

Now, by taking $s = 2^{-1}r$, it is easy to see that $B_s(t_i) \subset \sigma_i^{\circ s}$, and thus

$$n^{-1} < \omega_f(t_i) \leq \omega_f(B_s(t_i)) \leq \omega_f(\sigma_i^{\circ s}).$$

Therefore, we also have

$$\sum_{i \in J_r} \nu(\sigma_i) = n \sum_{i \in J_r} n^{-1}\nu(\sigma_i) \leq n \sum_{i \in I} \omega_f(\sigma_i^{\circ s})\nu(\sigma_i) < 2^{-1}\varepsilon.$$

Now, as an immediate consequence of Theorems 7.1 and 7.2 and Remark 6.7, we can also state

COROLLARY 7.3

If $f \in \mathcal{B}(A, X)$ for some $A \in \mathcal{S}$, then the following assertions are equivalent:

(1) $f \in \mathcal{B}\mathcal{C}_{L_\nu}(A, X)$; (2) $\Omega^*(f, \nu, A) = 0$;

(3) $\Omega(f, \nu, A) = 0$; (4) $\Omega_*(f, \nu, A) = 0$.

8. A strong Riemann integral

DEFINITION 8.1

A sequence (σ_n) in $\mathcal{D}(A)$ will be called ν-normal if

$$\overline{\lim}_{n\to\infty} |\sigma_n|_\nu \leq 0.$$

Likewise, a sequence $((\sigma_n, \tau_n))$ in $\mathcal{DT}(A)$ will be called ν-normal if

$$\overline{\lim}_{n\to\infty} |(\sigma_n, \tau_n)|_\nu \leq 0.$$

REMARK 8.2

Note that, by Corollary 2.9, such normal sequences exist for every $A \in \mathcal{S}$.

DEFINITION 8.3

If $A \in \mathcal{S}$, $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$ such that for each ν-normal sequence $((\sigma_n, \tau_n))$ in $\mathcal{DT}(A)$ the sequence $(S(f, \mu, \sigma_n, \tau_n))$ converges in Z, then f will be called strongly Riemann ν-integrable with respect to μ over A.

To define the corresponding integral, we must prove

LEMMA 8.4

If $A \in \mathcal{S}$, $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$ such that f is strongly Riemann ν-integrable with respect to μ, and moreover $((\sigma_n, \tau_n))$ and $((\rho_n, \omega_n))$ are ν-normal sequences in $\mathcal{DT}(A)$, then

$$\lim_{n\to\infty} S(f, \mu, \sigma_n, \tau_n) = \lim_{n\to\infty} S(f, \mu, \rho_n, \omega_n).$$

Proof. For each $n \in \mathbb{N}$, define

$$(\delta_n, \xi_n) = \begin{cases} (\sigma_{\frac{n+1}{2}}, \tau_{\frac{n+1}{2}}) & \text{if } n \text{ is odd,} \\ (\rho_{\frac{n}{2}}, \omega_{\frac{n}{2}}) & \text{if } n \text{ is even.} \end{cases}$$

Then $((\delta_n, \xi_n))$ is also a ν-normal sequence in $\mathcal{DT}(A)$. And thus, the sequence $(S(f, \mu, \delta_n, \xi_n))$ is also convegent. Now, since

$$S(f, \mu, \sigma_n, \tau_n) = S(f, \mu, \delta_{2n-1}, \xi_{2n-1})$$

and

$$S(f, \mu, \rho_n, \omega_n) = S(f, \mu, \delta_{2n}, \xi_{2n})$$

for all n, it is clear that the corresponding sequences have the same limit.

Now we may also have the following straightforward

DEFINITION 8.5

If $A \in \mathcal{S}$, $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$ such that f is strongly Riemann ν-integrable with respect to μ, then the limit

$$\nu\text{-}\int_A f d\mu = \lim_{n \to \infty} S(f, \mu, \sigma_n, \tau_n),$$

where $((\sigma_n, \tau_n))$ is a ν-normal sequence in $\mathcal{DT}(A)$, will be called the strong Riemann ν-integral of f with respect to μ over A.

REMARK 8.6

Now if $A = [\alpha, \beta[\in \mathcal{J}$, $f : A \to X$ and $g : A^- \to Y$, then we may also define

$$\int_\alpha^\beta f dg = \upsilon\text{-}\int_A f d\mu_g,$$

whenever the latter integral, where

$$\mu_g([t, s[) = g(s) - g(t) \qquad ([t, s[\in \mathcal{J}_A),$$

exists in the sense of the above definition.

A simple reformulation of Definition 8.5 yields

THEOREM 8.7

If $A \in \mathcal{S}$, $f : A \to X$, $\mu : \mathcal{S}_A \to Y$ and $S \in Z$, then the following assertions are equivalent:

(1) $S = \nu\text{-}\int_A f d\mu$;

(2) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|S(f, \mu, \sigma, \tau) - S| < \varepsilon$$

whenever $(\sigma, \tau) \in \mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < \delta$.

Moreover, by supposing that the normed space Z in the multiplication system (X, Y, Z) is complete, we can also easily prove

THEOREM 8.8

If $A \in \mathcal{S}$, $f : A \to X$ and $\mu : \mathcal{S}_A \to Y$, then the following assertions are equivalent:

(1) *f is strongly Riemann ν-integrable with respect to μ;*

(2) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|S(f, \mu, \sigma, \tau) - S(f, \mu, \rho, \omega)| < \varepsilon$$

whenever (σ, τ) and (ρ, ω) are in $\mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < \delta$ and $|(\rho, \omega)|_\nu < \delta$.

Hint. If the assertion (2) does not hold, then there exists an $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exist (σ_n, τ_n) and (ρ_n, ω_n) in $\mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < 1/n$ and $|(\rho, \omega)|_\nu < 1/n$ and

$$|S(f, \mu, \sigma_n, \tau_n) - S(f, \mu, \rho_n, \omega_n)| \geq \varepsilon.$$

And hence, if (1) holds, then by letting $n \to \infty$, we can immediately arrive at the contradiction $0 \geq \varepsilon$. Therefore, the implication (1)$\Longrightarrow$(2) is true.

Remark 8.9

By using Theorem 8.8, we can easily prove that if $A \in \mathcal{S}$ and $f : A \to X$ such that f is strongly Riemann integrable with respect to ν, then there exists an $r > 0$ such that f is bounded on the set $B^{-r} \cap A$ for all $B \in \mathcal{S}_A$ with $\operatorname{diam}(B) < r$ and $\nu(B) \neq 0$.

Moreover, by using Theorems 7.1, 6.13 and 8.8, we can now also easily prove the next fundamental

Theorem 8.10

If $f \in \mathcal{BC}_{L_\nu}(A, X)$ and $\mu \in \mathcal{BV}_\nu(\mathcal{S}_A, Y)$ for some $A \in \mathcal{S}$, then f is strongly Riemann ν-integrable with respect to μ.

Proof. Now, by Theorem 7.1, we have $\Omega^*(f, \nu, A) = 0$. Therefore, by Definition 6.6, for each $\varepsilon > 0$ there exists a $\rho \in \mathcal{D}(A)$ and an $r > 0$ such that

$$\Omega^{*r}(f, \nu, \rho) < \left(2(|\mu|_\nu + 1)\right)^{-1} \varepsilon.$$

Hence, by Theorem 6.13, it is clear that if $\omega \in \mathcal{C}_\nu(\rho)$, then

$$|S(f, \mu, \sigma, \tau) - S(f, \mu, \rho, \omega)| < 2^{-1} \varepsilon$$

whenever $(\sigma, \tau) \in \mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < r$. Therefore,

$$\begin{aligned} |S(f, \mu, \sigma, \tau) - S(f, \mu, \delta, \xi)| &\leq |S(f, \mu, \sigma, \tau) - S(f, \mu, \rho, \omega)| \\ &\quad + |S(f, \mu, \rho, \omega) - S(f, \mu, \delta, \xi)| < \varepsilon, \end{aligned}$$

whenever (σ, τ) and (δ, ξ) are in $\mathcal{DT}(A)$ such that $|(\sigma, \tau)|_\nu < r$ and $|(\delta, \xi)|_\nu < r$. And thus Theorem 8.8 can be applied.

REMARK 8.11

Now if $A \in \mathcal{S}$ and f is a bounded function from A into $\mathbb{R}$ such that

$$g(t) = \nu\left(f^{-1}\left([\alpha, t[\right)\right)$$

exists for all $t \in [\alpha, \beta]$, where $-\infty < \alpha < \inf f(A)$ and $\sup f(A) < \beta < +\infty$, then we may also define

$$\int_A f d\nu = \beta\nu(A) - \int_\alpha^\beta g(t)dt,$$

which is certainly the usual Lebesgue integral of f with respect to ν whenever $\mathcal{S}$ is closed under countable unions.

9. Lower and upper aproximating sums

For real valued functions, we may also consider the lower and upper approximating sums.

DEFINITION 9.1

If $f \in \mathcal{B}(A, \mathbb{R})$ and $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$, then the real numbers

$$L(f,\nu,\sigma) = \sum_{i\in I} \inf f(\sigma_i)\nu(\sigma_i) \quad \text{and} \quad U(f,\nu,\sigma) = \sum_{i\in I} \sup f(\sigma_i)\nu(\sigma_i)$$

will be called the lower and upper Darboux sums of f with respect to ν corresponding to σ.

Moreover, the real numbers

$$\int_{-A} f d\nu = \sup_{\sigma\in\mathcal{D}(A)} L(f,\nu,\sigma) \quad \text{and} \quad \int_A^{-} f d\nu = \inf_{\sigma\in\mathcal{D}(A)} U(f,\nu,\sigma)$$

will be called the lower and upper Darboux integrals of f on A with respect to ν.

REMARK 9.2

It is clear that

$$L(f,\nu,\sigma) \leq S(f,\nu,\sigma,\tau) \leq U(f,\nu,\sigma)$$

for all $(\sigma,\tau) \in \mathcal{DC}_\nu(A)$. Moreover, it can be easily seen that

$$L(f,\nu,\sigma) \leq L(f,\nu,\rho) \quad \text{and} \quad U(f,\nu,\rho) \leq U(f,\nu,\sigma)$$

whenever $\sigma, \rho \in \mathcal{D}(A)$ such that ρ divides (refines) σ.

Therefore, if $\sigma = (\sigma_i)_{i\in I}$ and $\rho = (\rho_j)_{j\in J}$ are arbitrary members of $\mathcal{D}(A)$ and $\delta = \sigma \wedge \rho = (\sigma_i \cap \rho_j)_{(i,j)\in I\times J}$, then we also have

$$L(f,\nu,\sigma) \leq L(f,\nu,\delta) \leq U(f,\nu,\delta) \leq U(f,\nu,\rho).$$

And hence it is clear that

$$\int_{-A} f\,d\nu \leq \int_A^{-} f\,d\nu.$$

Now, we may also naturally introduce the following more complicated

DEFINITION 9.3

If $f \in \mathcal{B}(A, \mathbb{R})$, $\sigma = (\sigma_i)_{i \in I} \in \mathcal{D}(A)$ and $r > 0$, then we define

$$L_{*r}(f, \nu, \sigma) = \sum_{i \in I} \inf f(\sigma_i^{-r})\nu(\sigma_i), \qquad L^{*r}(f, \nu, \sigma) = \sum_{i \in I} \inf f(\sigma_i^{\circ r})\nu(\sigma_i),$$

$$U_{*r}(f, \nu, \sigma) = \sum_{i \in I} \sup f(\sigma_i^{\circ r})\nu(\sigma_i), \qquad U^{*r}(f, \nu, \sigma) = \sum_{i \in I} \sup f(\sigma_i^{-r})\nu(\sigma_i).$$

Moreover, we define

$$L_*(f, \nu, \sigma) = \sup_{r>0} L_{*r}(f, \nu, \sigma), \qquad L^*(f, \nu, \sigma) = \inf_{r>0} L^{*r}(f, \nu, \sigma),$$

$$U_*(f, \nu, \sigma) = \sup_{r>0} U_{*r}(f, \nu, \sigma), \qquad U^*(f, \nu, \sigma) = \inf_{r>0} U^{*r}(f, \nu, \sigma).$$

And we define

$$\int_{-*A} f\,d\nu = \sup_{\sigma \in \mathcal{D}(A)} L_*(f, \nu, \sigma), \qquad \int_{-A}^{*} f\,d\nu = \sup_{\sigma \in \mathcal{D}(A)} L^*(f, \nu, \sigma),$$

$$\int_{*A}^{-} f\,d\nu = \inf_{\sigma \in \mathcal{D}(A)} U_*(f, \nu, \sigma), \qquad \int_A^{-*} f\,d\nu = \inf_{\sigma \in \mathcal{D}(A)} U^*(f, \nu, \sigma).$$

REMARK 9.4

Note that

$$\inf f(\sigma_i^{-r}) \leq \inf f(\sigma_i) \leq \inf f(\sigma_i^{\circ r}) \leq \sup f(\sigma_i^{\circ r}) \leq \sup f(\sigma_i) \leq \sup f(\sigma_i^{-r})$$

whenever $\sigma_i^{\circ r} \neq \emptyset$. Therefore, by Theorem 4.12, we have

$$L_{*r}(f, \nu, \sigma) \leq L(f, \nu, \sigma) \leq L^{*r}(f, \nu, \sigma) \leq U_{*r}(f, \nu, \sigma) \leq U(f, \nu, \sigma) \leq U^{*r}(f, \nu, \sigma),$$

for all sufficently small $r > 0$. And hence it is clear that

$$L_*(f, \nu, \sigma) \leq L(f, \nu, \sigma) \leq L^*(f, \nu, \sigma) \leq U_*(f, \nu, \sigma) \leq U(f, \nu, \sigma) \leq U^*(f, \nu, \sigma).$$

On the other hand, it can be easily seen that

$$L^{*r}(f, \nu, \sigma) \leq L^{*r}(f, \nu, \rho) \quad \text{and} \quad U_{*r}(f, \nu, \rho) \leq U_{*r}(f, \nu, \sigma)$$

for all $r > 0$, and hence

$$L^*(f,\nu,\sigma) \leq L^*(f,\nu,\rho) \quad \text{and} \quad U_*(f,\nu,\rho) \leq U_*(f,\nu,\sigma)$$

whenever $\sigma, \rho \in \mathcal{D}(A)$ such that ρ divides (refines) σ.

Therefore, if $\sigma, \rho \in \mathcal{D}(A)$ and $\delta = \sigma \wedge \rho$, then we also have

$$L^*(f,\nu,\sigma) \leq L^*(f,\nu,\delta) \leq U_*(f,\nu,\delta) \leq U_*(f,\nu,\rho).$$

And now it is clear that

$$\int_{-*A} f\,d\nu \leq \int_{-A} f\,d\nu \leq \int_{-A}^{*} f\,d\nu \leq \int_{*A}^{-} f\,d\nu \leq \int_{A}^{-} f\,d\nu \leq \int_{A}^{-*} f\,d\nu.$$

10. Integral characterizations of a.e. continuous functions

THEOREM 10.1

If $f \in \mathcal{B}(A,\mathbb{R})$, for some $A \in \mathcal{S}$, such that f is strongly Riemann integrable with respect to ν, and (σ_n) is a ν-normal sequence in $\mathcal{D}(A)$, then

$$\int_A f\,d\nu = \lim_{n\to\infty} L_*(f,\nu,\sigma_n) \quad \text{and} \quad \int_A f\,d\nu = \lim_{n\to\infty} U^*(f,\nu,\sigma_n).$$

Proof. By Definition 9.3, it is clear that for each $n \in \mathbb{N}$ there exists an $r_n \in]0, n^{-1}]$ such that

$$0 \leq L_*(f,\nu,\sigma_n) - L_{*r_n}(f,\nu,\sigma_n) < n^{-1}.$$

Namely $L_{*r}(f,\nu,\sigma)$ is a decreasing function of r.

Moreover, it is clear that for each $\sigma_n = (\sigma_{ni})_{i\in I_n}$ there exists a family $\tau_n = (\tau_{ni})_{i\in I_n}$ in A such that

$$\tau_{ni} \in \sigma_{ni}^{-r_n} \quad \text{and} \quad 0 \leq f(\tau_{ni}) - \inf f(\sigma_{ni}^{-r_n}) < n^{-1}$$

whenever $\sigma_{ni} \neq \emptyset$. Therefore

$$0 \leq S(f,\nu,\sigma_n,\tau_n) - L_{*r_n}(f,\nu,\sigma_n) \leq n^{-1}\nu(A)$$

for all $n \in \mathbb{N}$.

Consequently, we have

$$|S(f,\nu,\sigma_n,\tau_n) - L_*(f,\nu,\sigma_n)| < n^{-1}(1+\nu(A))$$

for all $n \in \mathbb{N}$. And hence, by letting $n \to \infty$, we can at once get the first statement of the theorem. Namely, $((\sigma_n,\tau_n))$ is a ν-normal sequence in $\mathcal{DT}(A)$.

The proof of the second statement is quite similar.

Now, as an immediate consequence of Theorem 10.1, we can also state

THEOREM 10.2

If $f \in \mathcal{B}(A,\mathbb{R})$, for some $A \in \mathcal{S}$, such that f is strongly Riemann integrable with respect to ν, then

$$\int_A f d\nu = \int_{-*A} f d\nu \qquad \text{and} \qquad \int_A f d\nu = \int_A^{-*} f d\nu.$$

Proof. If (σ_n) is a ν-normal sequence in $\mathcal{D}(A)$, then by Definition 9.3 and Remark 9.4 it is clear that

$$L_*(f,\nu,\sigma_n) \le \int_{-*A} f d\nu \le \int_A^{-*} f d\nu \le U^*(f,\nu,\sigma_n)$$

for all $n \in \mathbb{N}$. And hence, by using Theorem 10.1, we can immediately get the required equalities.

REMARK 10.3

Therefore, if $f \in \mathcal{B}(A,\mathbb{R})$ is strongly Riemann integrable with respect to ν, then f is also strongly Darboux integrable with respect to ν in the sense that

$$\int_{-*A} f d\nu = \int_A^{-*} f d\nu.$$

Moreover, as a certain converse to Theorem 8.10, we can also prove

THEOREM 10.4

If $f \in \mathcal{B}(A,\mathbb{R})$, for some $A \in \mathcal{S}$, such that f is weakly Darboux integrable with respect to ν in the sense that

$$\int_{-A}^{*} f d\nu = \int_{*A}^{-} f d\nu,$$

then $\Omega_*(f,\nu,A) = 0$.

Proof. If I denotes the common value of the above integrals, then by Definition 9.3 and Remark 9.4, for any $\varepsilon > 0$, there exists a $\sigma = (\sigma_i)_{i\in I} \in \mathcal{D}(A)$ such that

$$I - \varepsilon < L^*(f,\nu,\sigma) \qquad \text{and} \qquad U_*(f,\nu,\sigma) < I + \varepsilon.$$

Hence, again by Definition 9.3, it is clear that

$$I - \varepsilon < L^{*r}(f,\nu,\sigma) \qquad \text{and} \qquad U_{*r}(f,\nu,\sigma) < I + \varepsilon$$

for all $r > 0$. Therefore, under the notations

$$m_{ir} = \inf f(\sigma_i^{\circ r}) \qquad \text{and} \qquad M_{ir} = \sup f(\sigma_i^{\circ r}),$$

we have

$$\sum_{i\in I} (M_{ir} - m_{ir})\nu(\sigma_i) = U_{*r}(f,\nu,\sigma) - L^{*r}(f,\nu,\sigma) < \varepsilon$$

for all $r > 0$. Hence, since

$$\omega_f(\sigma_i^{\circ r}) = \sup_{t,s \in \sigma_i^{\circ r}} |f(t) - f(s)| \leq M_{ir} - m_{ir},$$

it is clear that

$$\Omega_{*r}(f, \nu, \sigma) = \sum_{i \in I} \omega_f(\sigma_i^{\circ r}) \nu(\sigma_i) < \varepsilon$$

for all $r > 0$. Therefore $\Omega_*(f, \nu, \sigma) \leq \varepsilon$, and hence $\Omega_*(f, \nu, A) \leq \varepsilon$. Consequently, $\Omega_*(f, \nu, A) = 0$.

Now, as an immediate consequence of Theorems 8.10, 10.2 and 10.3, we can also state

COROLLARY 10.5

If $f \in \mathcal{B}(A, \mathbb{R})$ for some $A \in \mathcal{S}$, then the following assertions are equivalent:

(1) $f \in \mathcal{B}C_{L_\nu}(A, \mathbb{R})$;

(2) *f is strongly Riemann integrable with respect to ν;*

(3) *f is strongly (weakly) Darboux integrable with respect to ν.*

REMARK 10.6

The implication (2)$\Longrightarrow$(1) is certainly not, in general, true for X-valued functions.

Namely, according to Graves [5] there exists a Riemann integrable function from $[0, 1]$ into $\mathcal{B}([0, 1], \mathbb{R})$ which is everywhere discontinuous.

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