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Functions commuting with ternary operations

Abstract. The problem of finding all functions which commute with a given ternary operation leads to a general functional equation. In this note we solve this problem in the case when the ternary operation is given by $[x,y,z] = (x * y - z)^{-1}$, $x,y,z \in F$, $x * y \neq z$ where F is a field having at least four elements and in which $x \in F$, $2x = 0$ imply $x = 0$, and $*$ is any binary operation on F with a right-sided unit element.

1. During the time of The Sixth International Conference on Functional Equations and Inequalities (Muszyna-Złockie, Poland, 1997) we proposed investigations connected with the following type of functional equations: Let X be a set, and $\emptyset \neq D \subset X \times X \times X$. Suppose that there is an element $[x,y,z] \in X$ corresponding to each $(x,y,z) \in D$, i.e., a ternary operation $[\cdot,\cdot,\cdot] : D \to X$ is given. The question is which functions commute with this operation? More exactly: Find all functions $f : X \to X$ for which $(x,y,z) \in D$ implies $(f(x), f(y), f(z)) \in D$, and

$$f([x,y,z]) = [f(x), f(y), f(z)]. \tag{A}$$

This problem is motivated by the following exercise in [1]. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following property: If $x,y,z \in \mathbb{R}$ with $x + y \neq 2z$ then $f(x) + f(y) \neq 2f(z)$ and

$$f\left(\frac{2xy - xz - yz}{x + y - 2z}\right) = \frac{2f(x)f(y) - f(x)f(z) - f(y)f(z)}{f(x) + f(y) - 2f(z)}.$$

Give an "elementary" proof to show that f is a nonconstant linear function.

Obviously, here $X := \mathbb{R}$, $D := \{(x,y,z) \in \mathbb{R}^3 : x+y \neq 2z\}$, and $[x,y,z] := \frac{2xy-xz-yz}{x+y-2z}$.

2. In this note we solve (A) in the following general case: $X := F$, where F is a field with some additional properties, $D := \{(x,y,z) \in F^3 : x * y \neq z\}$

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and $[x,y,z] := (x*y-z)^{-1}$, where $*: F\times F\to F$ is any binary operation with a right–sided unit element $e\in F$.

THEOREM 1

Let F be a field with the following properties: F has at least four elements, and $x\in F$, $2x=0$ imply $x=0$. Let furthermore $(F,*)$ be a groupoid with a right-sided unit element $e\in F$ and $f: F\to F$ be an unknown function for which the functional equation

$$f\left(\frac{1}{x*y-z}\right)(f(x)*f(y)-f(z))=1 \tag{1}$$

holds for all $x,y,z\in F$, $x*y\neq z$. Then the following two cases are possible:

(i) *$f(x)=c$ $(c\in F)$ for all $x\in F$, where $c(c*c-c)=1$;*

or

(ii) *$f(x)=A(x)$ for all $x\in F$, where $A: F\to F$ is an additive function (i.e., $A(x+y)=A(x)+A(y)$ for all $x,y\in F$), $A(x)A(x^{-1})=1$ for all $x\in F\setminus\{0\}$ and $A(x*y)=A(x)*A(y)$ for all $x,y\in F$.*

Proof. Let $y=e$, and $z=0$ in (1). Then

$$f\left(\frac{1}{x}\right)(f(x)*f(e)-f(0))=1,\quad x\neq 0. \tag{2}$$

Therefore (1) implies

$$f(x*y-z)*f(e)-f(0)=f(x)*f(y)-f(z), \tag{3}$$

provided that $x,y,z\in F$, $x*y\neq z$. With the substitution $y=e$ in (3), we have

$$f(x-z)*f(e)-f(0)=f(x)*f(e)-f(z),\quad x\neq z,$$

or, writing $x+z$ instead of x,

$$f(x)*f(e)+f(z)-f(0)=f(x+z)*f(e), \tag{4}$$

for all $x\in F\setminus\{0\}$, $z\in F$. Interchanging x and z in (4) we obtain

$$f(x)*f(e)+f(z)=f(z)*f(e)+f(x),\quad x,z\in F\setminus\{0\}.$$

This implies that there exists a $\gamma\in F$ such that

$$f(x)=f(x)*f(e)+\gamma,\quad x\neq 0. \tag{5}$$

So from (4)

$$f(x)+f(z)-f(0)=f(x+z) \tag{6}$$

follows for all $x, z \in F$, $x \neq 0$, $x + z \neq 0$. If $x = 0$ then (6) obviously holds for all $z \in F$. If $x + z = 0$ then choose a $t \in F$ such that $x + z + t = 0$, and $z + t \neq 0$ (such a $t \in F$ exists because of the assumption on F). Then, by (6),

$$\begin{aligned} f(t) = f(x+z+t) &= f(x) + f(z+t) - f(0) \\ &= f(x) + f(z) + f(t) - 2f(0), \end{aligned}$$

that is,

$$\begin{aligned} f(x+z) + f(0) &= 2f(0) \\ &= f(x) + f(z). \end{aligned}$$

Thus (6) also holds for all $x, z \in F$. Therefore there exists an additive function $A : F \to F$ such that

$$f(x) = A(x) + f(0), \quad x \in F. \tag{7}$$

Equations (2), (5), and (7) imply

$$\left(A\left(\frac{1}{x}\right) + f(0)\right)(A(x) - \gamma) = 1, \quad x \neq 0,$$

that is,

$$A\left(\frac{1}{x}\right)A(x) + f(0)A(x) - \gamma A\left(\frac{1}{x}\right) - \gamma f(0) = 1, \quad x \neq 0. \tag{8}$$

Writing $-x$ instead of x in (8) we have

$$A\left(\frac{1}{x}\right)A(x) - f(0)A(x) + \gamma A\left(\frac{1}{x}\right) - \gamma f(0) = 1, \quad x \neq 0. \tag{9}$$

The difference of (8) and (9) gives

$$2\left(f(0)A(x) - \gamma A\left(\frac{1}{x}\right)\right) = 0, \quad x \neq 0,$$

and because of the assumption on F we have

$$f(0)A(x) - \gamma A\left(\frac{1}{x}\right) = 0, \quad x \neq 0, \tag{10}$$

thus, by (8),

$$A\left(\frac{1}{x}\right)A(x) = 1 + f(0)\gamma, \quad x \neq 0. \tag{11}$$

We consider te following two cases: (i) $f(0) \neq 0$, and (ii) $f(0) = 0$.

(i) Suppose first $A(x) \equiv 0$ ($x \in F$). Then, by (7), we have

$$f(x) = f(0) = c, \quad x \in F,$$

where $c \neq 0$, and by (1) c satisfies the equation $c(c * c - c) = 1$. Then we have case (i) in our theorem. Secondly, suppose $A(x) \not\equiv 0$. We prove that this case is impossible. From (10) and (11) with the notations $\alpha := \frac{\gamma}{f(0)}$, and $\beta := f(0)\gamma$ we have

$$A(x)A\left(\frac{1}{x}\right) = 1 + \beta, \quad \text{and} \quad A(x) = \alpha A\left(\frac{1}{x}\right), \quad x \neq 0, \tag{12}$$

and $A : F \to F$ is an additive function. By (12)

$$A(x)^2 = \alpha(1 + \beta), \quad x \neq 0. \tag{13}$$

Let $x \neq 0$, and $x \neq 1$. Then by the additivity of A and by (13) we have

$$\begin{aligned}\alpha(1 + \beta) = A(x - 1)^2 &= A(x)^2 - 2A(1)A(x) + A(1)^2 \\ &= \alpha(1 + \beta) - 2A(1)A(x) + A(1)^2,\end{aligned}$$

that is,

$$A(1)^2 = 2A(1)A(x), \quad x \neq 0,\ x \neq 1. \tag{14}$$

It follows from (14) that

$$A(1)^2 = 2A(1)A(-x) = -2A(1)A(x), \quad x \neq 0,\ x \neq -1. \tag{15}$$

Since F has at least four elements, there exists an $x_0 \in F$, $x_0 \neq 0$, $x_0 \neq 1$, $x_0 \neq -1$. Putting $x = x_0$ in (14) and (15), we obtain $2A(1)^2 = 0$, i.e., $A(1)^2 = 0$. Therefore, by (13), $A(1)^2 = \alpha(1 + \beta) = 0$, and $A(x)^2 = 0$ for all $x \neq 0$. Since $A(x) \not\equiv 0$, there exists an $x_1 \in F$, $x_1 \neq 0$ such that $A(x_1) \neq 0$, i.e., $A(x_1)^2 \neq 0$, which is a contradiction.

In case (ii) by (11) we have $A(x^{-1})A(x) = 1$ for all $x \in F \setminus \{0\}$. By (7), $f(x) = A(x)$, $(x \in F)$, where $A : F \to F$ is an additive function. From (5) and (3) with $z = 0$

$$\begin{aligned}A(x * y) = f(x * y) &= f(x * y) * f(e) + \gamma = f(x) * f(y) + \gamma \\ &= A(x) * A(y) + \gamma\end{aligned}$$

follows for all $x, y \in F$, $x * y \neq 0$. Here $\gamma = 0$ by (10) and (11) ($\gamma A(x^{-1}) = 0$, $x \neq 0$; $A(x^{-1}) \neq 0$, $x \neq 0$), therefore

$$A(x * y) = A(x) * A(y), \quad x, y \in F,\ x * y \neq 0. \tag{16}$$

Put $x * y = 0$ in (1). Then

$$A\left(\frac{1}{-z}\right)(A(x) * A(y) - A(z)) = 1$$

for all $z \in F \setminus \{0\}$, from which

$$A\left(\frac{1}{z}\right)(A(x) * A(y)) = 0, \quad z \neq 0$$

follows. Since $A(z^{-1}) \neq 0$ $(z \neq 0)$, we have

$$0 = A(0) = A(x * y) = A(x) * A(y),$$

i.e., (16) holds also for $x * y = 0$. This completes the proof of the theorem.

3. Remarks

(a) If there is no $c \in F$ such that $c(c * c - c) = 1$ then case (i) cannot occur. For example, in the case of $F = \mathbb{R}$ the operation $x * y := x \sin y$ has a right-sided unit element $e = \frac{\pi}{2}$. However, $c(c \sin c - c) = 1$ holds only if $c \neq 0$, and $\sin c = 1 + \frac{1}{c^2}$, which is obviously impossible.

(b) Since the function $A(x) = x$ $(x \in F)$ is additive, and $A(x)A(x^{-1}) = 1$ if $x \neq 0$, for any binary operation $* : F \times F \to F$ with a right-sided unit element, $A(x * y) = x * y = A(x) * A(y)$, i.e., $f(x) = x$ $(x \in F)$ is always a solution of the functional equation (1).

(c) The function $A(x) = -x$ $(x \in F)$ is additive, and $A(x)A(x^{-1}) = (-x)(-x^{-1}) = 1$ if $x \neq 0$. If $A(x * y) = -x * y = (-x) * (-y) = A(x) * A(y)$ $(x, y \in F)$ then $f(x) = -x$ $(x \in F)$ is also a solution. For example, the operation $x * y = x \sin y$ $(x, y \in \mathbb{R})$ in Remark (a) does not satisfy the identity $-(x * y) = (-x) * (-y)$, so $f(x) = -x$ $(x \in \mathbb{R})$ is not a solution. However, for the operation $x * y = x \cos y$ $(x, y \in \mathbb{R})$ (with right-sided unit element $e = 0$), the identity $-(x * y) = -x \cos y = (-x)\cos(-y)$ holds, thus $f(x) = -x$ $(x \in \mathbb{R})$ is a solution of the equation.

(d) If $A : F \to F$ is additive, and $A(x)A(x^{-1}) = 1$ for all $x \in F \setminus \{0\}$ then, with the notation $A(1) = \alpha$, we get that $\alpha^2 = 1$. Furthermore if $x \neq 0$, and $x \neq 1$ then

$$A(x) - A(x^2) = A(x(1-x)) = \frac{1}{A\left(\frac{1}{x(1-x)}\right)}$$

$$= \frac{1}{A\left(\frac{1}{x}\right) + A\left(\frac{1}{1-x}\right)} = \frac{1}{\frac{1}{A(x)} + \frac{1}{\alpha - A(x)}}$$

$$= A(x) - \frac{A(x)^2}{\alpha},$$

from which

$$A(x^2) = \frac{A(x)^2}{\alpha}, \quad x \neq 0,\ x \neq 1 \tag{17}$$

follows. On the other hand, if $x = 0$, or $x = 1$ (17) trivially holds. From this we have that for all $x, y \in F$

$$\begin{aligned} 2A(xy) &= A((x+y)^2) - A(x^2) - A(y^2) \\ &= \frac{A(x+y)^2 - A(x)^2 - A(y)^2}{\alpha} \\ &= \frac{2A(x)A(y)}{\alpha}, \end{aligned}$$

from which, by $\alpha^2 = 1$, we have

$$\alpha A(xy) = \alpha A(x) \alpha A(y) \tag{18}$$

for all $x, y \in F$. This implies that the function $a(x) := \alpha A(x)$ $(x \in F)$ is additive and multiplicative, that is, $a : F \to F$ is an endomorphism of F for which $a(1) = \alpha^2 = 1$.

(e) If $F = \mathbb{R}$ then it is known that $a : \mathbb{R} \to \mathbb{R}$, $a(1) = 1$ is an endomorphism if and only if $a(x) = x$ for all $x \in \mathbb{R}$ (see [3]). From this we have $\alpha A(x) = x$ $(\alpha^2 = 1)$, that is, $A(x) = x$, or $A(x) = -x$ for all $x \in \mathbb{R}$. This implies

Corollary 1

If $f : \mathbb{R} \to \mathbb{R}$ is a non-constant solution of the functional equation (1), *where* $(\mathbb{R}, *)$ *is a groupoid with a right-sided unit element then either $f(x) = x$ or $f(x) = -x$ $(x \in \mathbb{R})$. In the first case further assumptions on $*$ are not needed. In the second case, $f(x) = -x$ is a solution if and only if $-x * y = (-x) * (-y)$ holds for all $x, y \in \mathbb{R}$.*

(f) If $F = \mathbb{C}$ then it is known that there are two classes of endomorphisms $a : \mathbb{C} \to \mathbb{C}$, $a(1) = 1$:

(i) the trivial, i.e., continuous ones, which are $f(x) = x$, and $f(x) = \bar{x}$ for all $x \in \mathbb{C}$, where $\bar{x}$ denotes the complex conjugate of $x \in \mathbb{C}$;

(ii) the non-trivial ones, which exist (see [2], [3]). Therefore, in this case, there are discontinuous solutions of the functional equation (1) for special operations $*$ (for example, $x * y = x + y - xy$ if $x, y \in \mathbb{C}$).

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