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On embedding of a linear functional equation

Abstract. It is shown, under assumption of existence of continuous solution given on a real interval, when iterative linear equation is embeddable in equation with "continuous time" argument.

Consider the linear functional equation

$$\phi(f(x)) = g(x)\phi(x) + h(x), \tag{L}$$

where the functions f, g, h are given and ϕ is unknown. Iterating equation (L) and using induction we get (f^n stands here for the n-th iterate of f)

$$\phi(f^n(x)) = g_n(x)\phi(x) + h_n(x), \tag{L_n}$$

where

$$g_n(x) := \prod_{i=0}^{n-1} g(f^i(x))$$

and

$$h_n(x) := g_n(x) \sum_{i=0}^{n-1} \frac{h(f^i(x))}{g_{i+1}(x)}$$

for each x that the above definitions make sence. Moreover, observe that

$$g_{n+m}(x) = g_m(x) g_n(f^m(x))$$

and

$$\begin{aligned} h_{n+m}(x) &= g_{n+m}(x) \sum_{i=0}^{n+m-1} \frac{h(f^i(x))}{g_{i+1}(x)} \\ &= g_{n+m}(x) \left(\sum_{i=0}^{m-1} \frac{h(f^i(x))}{g_{i+1}(x)} + \sum_{i=m}^{n+m-1} \frac{h(f^i(x))}{g_{i+1}(x)} \right) \end{aligned}$$

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$$= g_{n+m}(x)\left(\sum_{i=0}^{m-1}\frac{h(f^i(x))}{g_{i+1}(x)}+\sum_{i=0}^{n-1}\frac{h(f^{i+m}(x))}{g_{i+m+1}(x)}\right)$$
$$= g_{n+m}(x)\left(\sum_{i=0}^{m-1}\frac{h(f^i(x))}{g_{i+1}(x)}+\sum_{i=0}^{n-1}\frac{h(f^i(f^m(x)))}{g_m(x)g_{i+1}(f^m(x))}\right)$$
$$= g_n(f^m(x))g_m(x)\sum_{i=0}^{m-1}\frac{h(f^i(x))}{g_{i+1}(x)}+g_n(f^m(x))\sum_{i=0}^{n-1}\frac{h(f^i(f^m(x)))}{g_{i+1}(f^m(x))}$$
$$= g_n(f^m(x))h_m(x)+h_n(f^m(x)),$$

that is

$$h_{n+m}(x)=g_n(f^m(x))h_m(x)+h_n(f^m(x))$$

for each $n, m \in \mathbb{N}$ and x.

This paper is motivated by a problem of L. Reich posed in 1997 on the 35th International Symposium of Functional Equations in Graz (see [9]). He asked under what possibly weak assumptions on functions f, g, h we can embed equation (L_n) in the equation with "continuous time" argument:

$$\phi(F(t,x)) = G(t,x)\phi(x) + H(t,x). \tag{L_t}$$

More precisely: under what assumptions on f, g, h there exist functions F, G, H from a suitable class of functions fulfilling equations

$$F(t+s,x) = F(t,F(s,x)), \tag{T}$$
$$G(t+s,x) = G(s,x)G(t,F(s,x)), \tag{G}$$
$$H(t+s,x) = G(t,F(s,x))H(s,x) + H(t,F(s,x)) \tag{H}$$

and conditions

$$F(1,\cdot) = f, \tag{T_0}$$
$$G(0,\cdot) = 1 \quad\text{and}\quad G(1,\cdot) = g, \tag{G_0}$$
$$H(0,\cdot) = 0 \quad\text{and}\quad H(1,\cdot) = h, \tag{H_0}$$

such that every solution of (L) from the considered function class satisfies equation (L_t).

Equation (T) is the classical translation equation having a vast literature (cf. [7] for references). Its solutions are called iteration semigroups whenever the "time parameter" t runs over $(0,\infty)$ and iteration groups if $t \in \mathbb{R}$. We say that a function f is embeddable in an iteration (semi)group if and only if there is an F satisfying (T) with condition (T_0). Regarding the problem of embeddability of real functions see [10] by M. C. Zdun.

Equation (G) appears in the theory of abstract automata (cf. [6]) and plays a role in the theory of positive semigroups of operators (cf. [1]). In the theory of dynamical systems it is known as the equation of cocycles (see [2] and [3]).

A first answer to Reich's problem was given by Z. Moszner [8] who treated the general case where none regularity of ϕ, f, g, h as well as F, G, H is required. Recently L. Reich has solved the problem (private communication) in the class of formal power series. In the present paper we study it for continuous mappings defined on a real interval.

Let X be a real interval containing his finite left endpoint ξ and without the right endpoint, and Y be a Banach space over a field $\mathbb{K}$. Let f be a homeomorphism mapping X onto itself and such that $\xi < f(x) < x$ for $x \in X^* := \operatorname{int} X$.

Remark 1

The hypothesis above implies (see [4], Theorem 2.1 and Lemma 5.1) that there exists a homeomorphism α mapping X^* onto $\mathbb{R}$ and satisfying the Abel equation

$$\alpha(f(x)) = \alpha(x) + 1.$$

Then the function $F : \mathbb{R} \times X \to X$, given by

$$F(t,x) := \begin{cases} \alpha^{-1}(\alpha(x) + t)), & \text{if } x \in X^*, \\ \xi, & \text{if } x = \xi, \end{cases} \tag{1}$$

is a continuous iteration group satysfying (T_0). It is clear that for each $x \in X^*$ the function $F(\cdot, x)$ is a homeomorphism mapping $\mathbb{R}$ onto X^*. Moreover, F satisfies the condition

$$F(0, \cdot) = \mathrm{id}\,. \tag{I}$$

In the sequel an iteration group of the form (1) will be called generated by the homeomorphism α.

Further, let $g : X \to \mathbb{K}$ and $h : X \to Y$ be continuous functions and assume that $g(x) \neq 0$ for $x \in X^*$.

It is well known (see [5], Section 3.1A, cf. also [4], Section 2.2) that the number of continuous solutions of equation (L) depends strongly on the behavior of the sequence $(g_n : n \in \mathbb{N})$. Three cases are possible only:

(A) for every $x \in X$ there exists a non-zero limit

$$g_0(x) := \lim_{n \to \infty} g_n(x)$$

and the function $g_0 : X \to \mathbb{K}$ is continuous;

(B) there exists an interval $J \subset X$ such that the sequence $(g_n|_J : n \in \mathbb{N})$ is uniformly convergent to zero;

(C) neither (A) nor (B) occurs.

In what follows F will always denote an arbitrary continuous iteration group on X satisfying (T_0) and (I) (cf. Remark 1).

We say that equation (L) has an embedding with respect to F if there exist solutions $G : \mathbb{R} \times X \to \mathbb{K}$ and $H : \mathbb{R} \times X \to Y$ of (G) and (H), respectively, such that conditions (G_0) and (H_0) are satisfied, for every $t \in \mathbb{R}$ the functions $G(t, \cdot)$ and $H(t, \cdot)$ are continuous and every continuous solution of (L) defined on X satisfies (L_t). If, additionally, G and H are continuous we say that (L) has a continuous embedding with respect to F.

First we shall prove the following lemma.

LEMMA

Assume that equation (L) *has a continuous solution defined on* X. *Then* (L) *has an embedding with respect to* F *if and only if there exists a solution* $G : \mathbb{R} \times X \to \mathbb{K}$ *of* (G) *such that condition* (G_0) *is satisfied, for every* $t \in \mathbb{R}$ *the function* $G(t, \cdot)$ *is continuous and every continuous* Y*-valued solution of the equation*

$$\psi(f(x)) = g(x)\psi(x) \tag{Lh}$$

satisfies also

$$\psi(F(t,x)) = G(t,x)\psi(x). \tag{Lh_t}$$

Equation (L) *has a continuous embedding with respect to* F *if and only if there exists a continuous solution* $G : \mathbb{R} \times X \to \mathbb{K}$ *of* (G) *satisfying condition* (G_0) *and such that every continuous* Y*-valued solution of equation* (Lh) *satisfies also* (Lh_t).

Proof. Let $G : \mathbb{R} \times X \to \mathbb{K}$ and $H : \mathbb{R} \times X \to Y$ be solutions of (G) and (H) respectively, such that conditions (G_0) and (H_0) are satisfied, for every $t \in \mathbb{R}$ the functions $G(t, \cdot)$ and $H(t, \cdot)$ are continuous and each continuous solution of (L) satisfies also (L_t). Fix continuous solutions $\phi : X \to Y$ and $\psi : X \to Y$ of equations (L) and (Lh), respectively. Then $\phi + \psi$ is a continuous solution of equation (L). Thus

$$(\phi + \psi)(F(t,x)) = G(t,x)(\phi + \psi)(x) + H(t,x)$$

and

$$\phi(F(t,x)) = G(t,x)\phi(x) + H(t,x),$$

whence, consequently,

$$\psi(F(t,x)) = G(t,x)\psi(x)$$

for each $t \in \mathbb{R}$ and $x \in X$.

Now assume that there exists a solution $G : \mathbb{R} \times X \to \mathbb{K}$ of equation (G) such that condition (G_0) is satisfied, for every $t \in \mathbb{R}$ the function $G(t,\cdot)$ is continuous, and every continuous Y-valued solution of equation (Lh) satisfies also (Lh_t). Fix a continuous solution $\phi_0 : X \to Y$ of (L) and define $H : \mathbb{R} \times X \to Y$ by

$$H(t,x) = \phi_0(F(t,x)) - G(t,x)\phi_0(x). \tag{2}$$

If $t, s \in \mathbb{R}$ and $x \in X$, by (T) and (G), we have

$$\begin{aligned} H(s,x)G(t,F(s,x)) &+ H(t,F(s,x)) \\ &= (\phi_0(F(s,x)) - G(s,x)\phi_0(x))\, G(t,F(s,x)) \\ &\quad + \phi_0(F(t,F(s,x))) - G(t,F(s,x))\phi_0(F(s,x)) \\ &= \phi_0(F(t+s,x)) - G(s,x)G(t,F(s,x))\phi_0(x) \\ &= \phi_0(F(t+s,x)) - G(t+s,x)\phi_0(x) \\ &= H(t+s,x), \end{aligned}$$

which means that H satisfies equation (H). Moreover, if $x \in X$ then, by (I) and (G_0),

$$H(0,x) = \phi_0(F(0,x)) - G(0,x)\phi_0(x) = \phi_0(x) - \phi_0(x) = 0$$

and, according to (T_0), (G_0) and (L),

$$H(1,x) = \phi_0(F(1,x)) - G(1,x)\phi_0(x) = \phi_0(f(x)) - g(x)\phi_0(x) = h(x).$$

Therefore the function H satisfies condition (H_0). It is clear that for all $t \in \mathbb{R}$ the function $H(t,\cdot)$ is continuous and if, in addition, G is continuous then so is H. Let $\phi : X \to Y$ be a continuous solution of equation (L). Then the function $\psi := \phi - \phi_0$ fulfils (Lh), so, due to the assumption, also (Lh_t). Consequently, if $t \in \mathbb{R}$ and $x \in X$ then by (2) and (Lh_t),

$$\begin{aligned} \phi(F(t,x)) &= \phi_0(F(t,x)) + \psi(F(t,x)) \\ &= (G(t,x)\phi_0(x) + H(t,x)) + G(t,x)\psi(x) \\ &= G(t,x)(\phi_0(x) + \psi(x)) + H(t,x) \\ &= G(t,x)\phi(x) + H(t,x), \end{aligned}$$

which means that ϕ is a solution of (L_t).

THEORE[illegible] 1

Assu[illegible]e that equation (L) *has a continuous solution defined on* X. *Then*

(i) *in [illegible]he case* (A) *equation* (L) *has a continuous embedding with respect to* F,

(ii) *in the case* (B) *equation* (L) *has no embedding with respect to* F,

(iii) *in the case* (C) *equation* (L) *has an embedding with respect to* F *if and only if there exists a solution* $G : \mathbb{R} \times X \to \mathbb{K}$ *of* (G) *such that condition* (G_0) *is satisfied and for any* $t \in \mathbb{R}$ *the function* $G(t, \cdot)$ *is continuous and* (L) *has a continuous embedding with respect to* F *if and only if there exists a continuous solution* $G : \mathbb{R} \times X \to \mathbb{K}$ *of* (G) *satisfying* (G_0).

Proof (i) The function $G : \mathbb{R} \times X \to \mathbb{K}$ defined by

$$G(t,x) := \frac{g_0(x)}{g_0(F(t,x))}$$

is continuous. Fix a point $x \in X$. For each $t, s \in \mathbb{R}$ we have

$$\begin{aligned} G(t+s,x) &= \frac{g_0(x)}{g_0(F(t+s,x))} \\ &= \frac{g_0(x)}{g_0(F(s,x))} \frac{g_0(F(s,x))}{g_0(F(t,F(s,x)))} \\ &= G(s,x)G(t,F(s,x)). \end{aligned}$$

Moreover, by (I),

$$G(0,x) = \frac{g_0(x)}{g_0(F(0,x))} = \frac{g_0(x)}{g_0(x)} = 1$$

and, according to (T_0),

$$\begin{aligned} G(1,x) &= \frac{g_0(x)}{g_0(F(1,x))} = \frac{g_0(x)}{g_0(f(x))} = \frac{\prod_{n=0}^{\infty} g(f^n(x))}{\prod_{n=0}^{\infty} g(f^{n+1}(x))} \\ &= g(x). \end{aligned}$$

Therefore the function G fulfils equation (G) and condition (G_0).

Now fix a continuous solution $\psi : X \to Y$ of equation (Lh). Using [5], Theorem 3.1.2 we infer that there exists an $\eta \in Y$ such that

$$\psi(x) = \frac{1}{g_0(x)}\, \eta \quad \text{for } x \in X,$$

whence for every $t \in \mathbb{R}$ and $x \in X$

$$\psi(F(t,x)) = \frac{1}{g_0(F(t,x))}\,\eta = \frac{g_0(x)}{g_0(F(t,x))}\left(\frac{1}{g_0(x)}\,\eta\right)$$
$$= G(t,x)\psi(x).$$

Thus we have shown that each continuous solution of (Lh) satisfies also equation (Lh_t) which, by the Lemma, completes the proof.

(ii) Suppose that equation (L) has an embedding with respect to F. Then by the Lemma there exists a solution $G : \mathbb{R}\times X \to \mathbb{K}$ of (G) such that condition (G_0) is satisfied, for every $t \in \mathbb{R}$ the function $G(t,\cdot)$ is continuous and every continuous Y-valued solution of equation (Lh) satisfies also (Lh_t).

Let U be (see [5], Lemma 3.1.1) the maximal open (in the topology of X) subset of X such that the sequence $(g_n : n \in \mathbb{N})$ converges to zero uniformly on every compact subset of U. Since we consider the case (B) we have $U \neq \emptyset$. Fix an $x_0 \in U$. Then $(f(x_0), x_0) \cap U$ is a non-empty open set, so putting $X_0 := [f(x_0), x_0]$ we can find $\alpha, \beta \in X_0 \cap U$ and continuous functions $\psi_{0,1}, \psi_{0,2} : X_0 \to Y$ such that

$$\psi_{0,1}(\alpha) = \psi_{0,2}(\alpha), \quad \psi_{0,1}(\beta) \neq \psi_{0,2}(\beta)$$

and, for $i \in \{1,2\}$,

$$\psi_{0,i}(f(x_0)) = g(x_0)\psi_{0,i}(x_0),$$
$$\psi_{0,i}(x) = 0 \quad \text{for } x \in X_0 \setminus U$$

and

$$\lim_{\substack{x \to u \\ x \in X_0 \cap U}} \psi_{0,i}(x) \sup\{|g_n(x)| : n \in \mathbb{N}\} = 0 \ \text{ for } u \in X_0 \cap (\mathrm{cl}\, U \setminus U).$$

By Theorem 3.1.3 from [5] the functions $\psi_{0,1}$, $\psi_{0,2}$ can be extended to continuous solutions $\psi_1, \psi_2 : X \to Y$ of (L). Clearly we have

$$\psi_1(\alpha) = \psi_2(\alpha)$$

and

$$\psi_1(\beta) \neq \psi_2(\beta).$$

It follows from Remark 1 that $\beta = F(t_0, \alpha)$ for a $t_0 \in \mathbb{R}$. Then, since ψ_1 and ψ_2 satisfy equation (Lh_t), we have

$$\begin{aligned}\psi_1(\beta) &= \psi_1(F(t_0,\alpha)) \\ &= G(t_0,\alpha)\psi_1(\alpha) = G(t_0,\alpha)\psi_2(\alpha) \\ &= \psi_2(F(t_0,\alpha)) \\ &= \psi_2(\beta),\end{aligned}$$

which is impossible.

(iii) In the case (C) the zero function is the only continuous solution of (Lh), so the assertion is a simple consequence of the Lemma.

Regarding the existence of a solution $G : \mathbb{R} \times X \to \mathbb{K}$ of (G) satisfying condition (G_0) we have the following two results. In the first of them we follow some ideas of Z. Moszner (cf. [6], Section II, Théorème 4). The second one concerns continuous solutions. First we can make an easy observation.

Remark 2

Let $G : \mathbb{R} \times X \to \mathbb{K}$ be a solution of equation (G), then for each $x \in X$, $G(t,x) \equiv 0$ or $G(t,x) \neq 0$ for all $t \in \mathbb{R}$. In fact, fix $x \in X$ and suppose that there exists $t_0 \in \mathbb{R}$ such that $G(t_0, x) = 0$. Then according to (G) for any $s \in \mathbb{R}$ we have

$$G(t_0 + s, x) = G(t_0, x)G(s, F(t_0, x)) = 0,$$

and it means that $G(t,x) = 0$ for all $t \in \mathbb{R}$. Therefore if a solution of (G) satisfies condition (G_0) it is a non-zero function.

Proposition 1

Assume that the group F is generated by a homeomorphism α. If $G : \mathbb{R} \times X \to \mathbb{K} \setminus \{0\}$ is a solution of equation (G) *satisfying condition* (G_0) *then there exist a solution $\psi : X^* \to \mathbb{K} \setminus \{0\}$ of equation* (Lh) *and a function $m : \mathbb{R} \to \mathbb{K} \setminus \{0\}$ satisfying the Cauchy equation*

$$m(t+s) = m(t)m(s) \tag{3}$$

and the condition

$$m(1) = g(\xi) \tag{4}$$

such that

$$G(t,x) := \begin{cases} \dfrac{\psi(F(t,x))}{\psi(x)}, & \text{for } t \in \mathbb{R} \text{ and } x \in X^* \\ m(t), & \text{for } t \in \mathbb{R} \text{ and } x = \xi. \end{cases} \tag{5}$$

Conversely, if $\psi : X^* \to \mathbb{K} \setminus \{0\}$ is a solution of equation (Lh) *and $m : \mathbb{R} \to \mathbb{K} \setminus \{0\}$ satisfies* (3) *and* (4), *then the function $G : \mathbb{R} \times X \to \mathbb{K} \setminus \{0\}$ given by* (5) *is a solution of equation* (G) *satisfying condition* (G_0).

Proof. Let α be a homeomorphism mapping X^* onto $\mathbb{R}$ such that F is of the form (1).

Let $G : \mathbb{R} \times X \to \mathbb{K} \setminus \{0\}$ be a solution of (G) satisfying (G_0). Put $m := G(\cdot, \xi)$. Then, by (G) and (1),

$$\begin{aligned}m(t+s) &= G(t+s,\xi) = G(s,\xi)G(t,F(s,\xi)) = G(s,\xi)G(t,\xi)\\ &= m(s)m(t)\end{aligned}$$

for each $t, s \in \mathbb{R}$ and, by (G_0),

$$m(1) = G(1,\xi) = g(\xi),$$

that is (3) and (4) are satisfied. Since α maps X^* on $\mathbb{R}$ there is an $x_0 \in X^*$ such that $\alpha(x_0) = 0$. Define $\psi : X^* \to \mathbb{K} \setminus \{0\}$ by

$$\psi(x) := G(\alpha(x), x_0).$$

Then, by (1), for every $x \in X^*$ we have

$$F(\alpha(x), x_0) = \alpha^{-1}(\alpha(x_0) + \alpha(x)) = \alpha^{-1}(\alpha(x)) = x,$$

whence, according to (1) and (G),

$$\begin{aligned}\psi(F(t,x)) &= G(\alpha(F(t,x)), x_0)\\ &= G(\alpha(x) + t, x_0)\\ &= G(\alpha(x), x_0)G(t, F(\alpha(x), x_0))\\ &= \psi(x)G(t,x),\end{aligned}$$

for each $t \in \mathbb{R}$ and $x \in X^*$. Thus (5) holds true and by putting there $t = 1$ and taking into account (T_0) and (G_0) we infer that ψ satisfies (Lh).

Now let $\psi : X^* \to \mathbb{K} \setminus \{0\}$ be a solution of (Lh) and let $m : \mathbb{R} \to \mathbb{K} \setminus \{0\}$ satisfies (3) and (4). Define $G : \mathbb{R} \times X \to \mathbb{K} \setminus \{0\}$ by (5). Then, for every $t, s \in \mathbb{R}$, by (T), we have

$$\begin{aligned}G(t+s,x) &= \frac{\psi(F(t+s,x))}{\psi(x)}\\ &= \frac{\psi(F(t,F(s,x)))}{\psi(x)}\\ &= \frac{\psi(F(s,x))}{\psi(x)}\frac{\psi(F(t,F(s,x)))}{\psi(F(s,x))}\\ &= G(s,x)G(t,F(s,x))\end{aligned}$$

whenever $x \in X^*$ and if $x = \xi$, on account of (3) and (1),

$$\begin{aligned}G(t+s,x) &= G(t+s,\xi) = m(t+s) = m(t)m(s)\\ &= G(t,\xi)G(s,\xi) = G(s,\xi)G(t,F(s,\xi))\\ &= G(s,x)G(t,F(s,x)).\end{aligned}$$

Therefore (G) is satisfied. Moreover, by (I) and (3), we obtain

$$G(0,x) = 1 \quad \text{for } x \in X$$

and, by (T_0), (Lh) and (4), we have

$$G(1,x) = g(x) \quad \text{for } x \in X$$

that is also (G_0) holds true.

PROPOSITION 2

Assume that $\mathbb{K} = \mathbb{R}$, *the function* g *is positive and there exists a continuous function* $\gamma : X \to \mathbb{R}$ *satisfying the condition*

$$\int_0^1 \gamma(F(t,x))\,dt = \log g(x) \quad \textit{for } x \in X. \tag{6}$$

Then the function $G : \mathbb{R} \times X \to \mathbb{R}$ *given by*

$$G(t,x) := \exp \int_0^t \gamma(F(\tau,x))\,d\tau \tag{7}$$

is a continuous solution of equation (G) *satisfying condition* (G_0).

Proof. It is clear that the function G is continuous. By (7) and (6)

$$G(1,x) = g(x) \quad \text{for } x \in X$$

and

$$G(0,x) = \exp(0) = 1 \quad \text{for } x \in X.$$

Therefore condition (G_0) is satisfied. Moreover, for every $t, s \in \mathbb{R}$ and $x \in X$, by (T), we have

$$\begin{aligned}
\log G(t+s,x) &= \int_0^{t+s} \gamma(F(\tau,x))\,d\tau \\
&= \int_0^{s} \gamma(F(\tau,x))\,d\tau + \int_s^{t+s} \gamma(F(\tau,x))\,d\tau \\
&= \int_0^{s} \gamma(F(\tau,x))\,d\tau + \int_0^{t} \gamma(F(\tau+s,x))\,d\tau \\
&= \int_0^{s} \gamma(F(\tau,x))\,d\tau + \int_0^{t} \gamma(F(\tau,F(s,x)))\,d\tau \\
&= \log G(s,x) + \log G(t,F(s,x)),
\end{aligned}$$

which means that G is a solution of (G).

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