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On extensibility of some homomorphisms

Abstract. The paper deals with a system of functional equations connected with the determination of some class of homomorphisms of uniquely divisible abelian semigroup $(G,+)$ into the group L^1_s or L^1_∞. We give a necessary form of such homomorphisms into L^1_∞. Moreover, we give a necessary condition for extensibility of homomorphisms into L^1_s to homomorphisms into L^1_{s+1}.

Introduction

In the sequel $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Let $|k,l|$ denote the set of all integers n such that $k \leqslant n \leqslant l$, and let $|k,\infty|$ be the set of all integers $n \geqslant k$. We adhere to the convention that $0^0 = 1$, $\sum_{t\in\emptyset} a_t = 0$, $\sum_{k=m}^{n} a_k = 0$ and $|m,n| = \emptyset$ for $m > n$.

Given a positive integer s we consider a family $\mathcal{J}_0$ of intervals of $\mathbb{R}$ containing 0 and a family $\mathcal{D}_0$ of C^∞-diffeomorphisms defined on an elements of $\mathcal{J}_0$ and mapping 0 to 0. We introduce on $\mathcal{D}_0$ the equivalence relations j^s and j^∞ on the following way

$$(f,g) \in j^s \quad \text{if and only if } (f-g)^{(k)}(0) = 0 \quad \text{for } k \in \{1,\dots,s\},\ f,g \in \mathcal{D}_0,$$
$$(f,g) \in j^\infty \quad \text{if and only if } (f-g)^{(k)}(0) = 0 \quad \text{for } k \in \mathbb{N},\ f,g \in \mathcal{D}_0.$$

On the sets $J_s\mathbb{R}$, $J_\infty\mathbb{R}$ of all the equivalence classes $j^s f$ and $j^\infty f$, respectively, we define binary operations

$$(j^s f)\cdot(j^s g) = (j^s(f\circ g)),$$
$$(j^\infty f)\cdot(j^\infty g) = (j^\infty(f\circ g)).$$

It is known that $L^1_s := (J_s\mathbb{R},\cdot)$ and $L^1_\infty := (J_\infty\mathbb{R},\cdot)$ are groups. These groups appear in the theory of the classification of one dimensional geometric objects which are solutions of the translation equation on L^1_s or on their subgroups.

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The coordinates $(x_1, \ldots, x_s)$ of the point $j^s f$ are the coefficients of the s-th Taylor's expansion of f, that is

$$x_i := f^{(k)}(0) \quad \text{for } k \in \{1, \ldots, s\},$$

where $f^{(k)}$ is the k-th derivative of f. It is known (cf. [5], Chapter III, §7) that if $f, g : \mathbb{R} \to \mathbb{R}$ are s-times differentiable functions, then

$$(f \circ g)^{(n)}(x) = \sum_{k=1}^{n} f^{(k)}(g(x)) \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{u_j} \quad \text{for } n \in |1, s|,$$

where

$$U_{n,k} := \left\{ \bar{u}_n := (u_1, \ldots, u_n) \in |0, k|^n : \sum_{i=1}^{n} u_i = k \ \wedge \ \sum_{i=1}^{n} i u_i = n \right\},$$

$$A_{\bar{u}_n} := \frac{n!}{\prod\limits_{i=1}^{n} (u_i!(i!)^{u_i})}.$$

Therefore the group L_s^1 is isomomorphic to a set

$$Z_s := \{\bar{x}_s := (x_1, \ldots, x_s) \in \mathbb{R}^s : \ x_1 \neq 0\}$$

with the operation

$$\bar{x}_s \cdot \bar{y}_s = \bar{z}_s$$

defined by

$$z_n = \sum_{k=1}^{n} x_k \sum_{u_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} y_j^{u_j} \quad \text{for } n \in |1, s| , \tag{1}$$

where $U_{n,k}$ and $A_{\bar{u}_n}$ are defined as above.

Similarly, the group L_∞^1 is isomorphic to a set

$$Z_\infty := \{\bar{x}_\infty := (x_1, x_2, \ldots) : \ \forall i \in \mathbb{N} : \ x_i \in \mathbb{R} \ \wedge \ x_1 \neq 0\}$$

with the operation

$$\bar{x}_\infty \cdot \bar{y}_\infty = \bar{z}_\infty,$$

where z_n, $U_{n,k}$ and $A_{\bar{u}_n}$ are given above.

In particular cases the formula (1) reads as follows

$$\begin{aligned}
z_1 &= x_1 y_1, \\
z_2 &= x_1 y_2 + x_2 y_1^2, \\
z_3 &= x_1 y_3 + 3x_2 y_1 y_2 + x_3 y_1^3, \\
z_4 &= x_1 y_4 + 4x_2 y_1 y_3 + 3x_2 y_2^2 + 6x_3 y_1^2 y_2 + x_4 y_1^4.
\end{aligned}$$

L. Reich posed the following question (see [1], p. 309): "When does a homomorphism Φ_s of $(\mathbb{R},+)$ into L_s^r (of truncated formal power series transformations in r indeterminates) have an extension $\bar{\Phi}_s$ from $(\mathbb{R},+)$ into L_{s+1}^r?" For $r=1$ the term "extensibility of homomorphisms" one should understand as follows. Given a homomorphism $\Phi_s = (f_1,\ldots,f_s)$ of the group $(\mathbb{R},+)$ into L_s^1, does there exist a function f_{s+1} such that $\bar{\Phi}_s = (f_1,\ldots,f_{s+1})$ is the homomorphism from $(\mathbb{R},+)$ into L_{s+1}^1? If such a function exists, we call $\bar{\Phi}_s$ an extension of Φ_s, and the homomorphism Φ_s-extensible.

In [4] all the homomorphisms of $(\mathbb{R},+)$ into L_s^1 for $s \leqslant 5$ have been determined. The results contained in that work show that for $s = 1,2$ every homomorphism into L_s^1 is extensible. Moreover, for $s = 3,4$ there exist homomorphisms, all in the case $f_1 = 1$, which are not extensible.

Z. Moszner conjectured (see [1], p. 309) that extension of Φ_s is possible if $f_1 \neq 1$, while $\Phi_s = (1,0,\ldots,0,f_{p+2},\ldots,f_s)$ with $f_{p+2} \neq 0$ can be extended iff f_{s-p} is a polynomial in f_{p+2}. We prove that for $f_1 = 1$ the above condition is necessary for extensibility of Φ_s.

We consider here the problem of extensibility of homomorphisms of uniquely divisible abelian semigroup $(G,+)$ into the group L_s^1. Moreover we give a necessary form of homomorphisms of G into L_∞^1.

Auxiliary results

Let us recall some useful results.

(i) If $\bar{u}_n \in U_{n,k}$ $(2 \leqslant k \leqslant n)$, then $u_j = 0$ for all $j \in |n-k+2,n|$.

(ii) If $n \geqslant 3$, $k \in |2,n-1|$, $\bar{u}_n \in U_{n,k}$, then there exists $j \in |2,n-k+1|$ such that $u_j \geqslant 1$.

(iii) For $n \geqslant 2$

$$z_n = x_1 y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-1} y_j^{u_j} + x_n y_1^n. \tag{2}$$

(iv) Let $1 \leqslant p \leqslant q$ be natural numbers and let $r = p+q+1$. If $x_j = 0$ for all $j \in |2,q|$ and $y_j = 0$ for all $j \in |2,p|$, then

1) $z_1 = x_1 y_1$,
2) $z_n = 0$ for $n \in |2,p|$,
3) $z_n = x_1 y_n$ for $n \in |p+1,q|$,
4) $z_n = x_1 y_n + x_n y_1^n$ for $n \in |q+1,p+q|$,
5) $z_r = x_1 y_r + \binom{r}{p+1} x_{q+1} y_1^q y_{p+1} + x_r y_1^r$.

For the proof we refer the reader to [2] and [3].

Notice that in view of (i) we can rewrite (2) as follows

$$z_n = x_1 y_n + \sum_{k=2}^{n-1} x_k \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n-k+1} y_j^{u_j} + x_n y_1^n. \tag{3}$$

For a nonnegative integer $p \leqslant n-1$ we denote

$$U_{n,k}^p := \{\bar{u}_n \in U_{n,k} : \ \forall i \in |2, p+1| : \ u_i = 0\}.$$

Clearly

$$U_{n,k}^{p_2} \subset U_{n,k}^{p_1} \subset U_{n,k}^0 = U_{n,k}$$

for $0 \leqslant p_1 \leqslant p_2 \leqslant n-1$ and arbitrary $1 \leqslant k \leqslant n$.

LEMMA 1

Let $p \geqslant 1$, $n \geqslant 2p+1$, $k \in |n-p, n-1|$. Then $U_{n,k}^p = \emptyset$.

Proof. Suppose that $\bar{u}_n \in U_{n,k}^p$ for $n \geqslant 2p+1$ and $k \in |n-p, n-1|$. Then, by (i), there exists $j \in |2, n-k+1| \subset |2, p+1|$ such that $u_j \geqslant 1$, which with $\bar{u}_n \in U_{n,k}^p$ leads to a contradiction.

From (3) and Lemma 1 we have

LEMMA 2

Let p be a positive integer. If $x_n = y_n = 0$ for all $n \in |2, p+1|$, then for $n \geqslant p+2$

$$z_n = x_1 y_n + \sum_{k=p+2}^{n-p-1} x_k \sum_{\bar{u}_n \in U_{n,k}^p} A_{\bar{u}_n} y_1^{u_1} \prod_{j=p+2}^{n-k+1} y_j^{u_j} + x_n y_1^n.$$

The simply proof of the following lemma is left to the reader.

LEMMA 3

Let p be a non-negative integer and $n \geqslant p+3$. Then

$$U_{n+p+1,n}^p = \{(n-1, 0, \ldots, 0, \overset{p+2}{1}, 0, \ldots, 0)\},$$

$$U_{n+p+1,p+2}^p = \{(p+1, 0, \ldots, 0, \overset{n}{1}, 0, \ldots, 0)\}$$
$$\cup \left\{\bar{u}_{n+p+1} \in U_{n+p+1,p+2}^p : \ \forall j \in |n, n+p+1| : \ u_j = 0\right\}.$$

We denote

$$\tilde{U}_{n+p+1,p+2}^p = \left\{\bar{u}_{n+p+1} \in U_{n+p+1,p+2}^p : \ \forall j \in |n, n+p+1| : \ u_j = 0\right\},$$

$$\tilde{U}_{n+p+1,k}^p = U_{n+p+1,k}^p \quad \text{for } \ k \in |p+3, n-1|.$$

Main results

Let s be a natural number or $s = \infty$ and let $(G, +)$ be an abelian semigroup uniquely divisible by some prime. Consider a function $\Phi_s : G \to Z_s$,

$$\Phi_s = (f_j)_{j \in |1,s|},$$

where $f_1 : G \to \mathbb{R}_0$, $f_n : G \to \mathbb{R}$ for $n \in |2, s|$.

The function Φ_s is a homomorphisms if and only if the functions f_n solve the following system of functional equations

$$f_n(x+y) = \sum_{k=1}^{n} f_k(x) \sum_{\bar{u}_n \in U_{n,k}} A_{\bar{u}_n} \prod_{j=1}^{n} f_j(y)^{u_j} \qquad n \in |1, s|. \tag{4}$$

We consider this system of equations in the case

$$f_1 = 1, \quad f_i = 0 \text{ for } i \in |2, p+1|, \quad f_{p+2} \neq 0.$$

The system (4) reduces then to the following one

$$f_n(x+y) = f_n(x) + \sum_{k=p+2}^{n-p-1} f_k(x) \sum_{\bar{u}_n \in U^p_{n,k}} A_{\bar{u}_n} \prod_{j=p+2}^{n-k+1} f_j(y)^{u_j} + f_n(y), \qquad n \in |p+2, s|. \tag{5}$$

We prove

THEOREM 1

Let functions f_n satisfy the system of equations (5). Then f_{p+2} is a nonzero additive function and for every $n \geqslant p+3$ with $n+p+1 \leqslant s$ there exists a polynomial w_n, such that

$$f_n = w_n(f_{p+2}), \quad w_n(0) = 0, \quad \deg w_n \leqslant \left[\frac{n-1}{p+1}\right].$$

Polynomials w_n are defined as follows

$$w_{p+2}(x) = x,$$

$$w_n(x) = d_n x + \frac{n!\,(p+2)!}{(n+p+1)!\,(n-p-2)} \sum_{k=p+2}^{n-1} \sum_{\bar{u}_{n+p+1} \in \bar{U}^p_{n+p+1,k}} A_{\bar{u}_{n+p+1}} \left[d_k \prod_{j=p+2}^{n+p+2-k} w_j(x)^{u_j} - w_k(x) \prod_{j=p+2}^{n+p+2-k} d_j^{u_j} \right],$$

where d_n are some constants.

Proof. At the beginning we consider the equation $(5.p+2)$, i.e. (compare (iv))

$$f_{p+2}(x+y) = f_{p+2}(x) + f_{p+2}(y).$$

Thus, in view of the assumption, f_{p+2} is a nonzero additive function.

Now our purpose is to prove that there exist polynomials w_n (we determine the form of them) such that $f_n = w_n(f_{p+2})$ whenever $n+p+1 \leqslant s$. The proof is by induction with respect to n. To do this, we denote

$$w_{p+2}(f_{p+2}) = f_{p+2}$$

and assume that for some $n \geqslant p+3$ such that $n+p+1 \leqslant s$ there exist polynomials w_j such that

$$f_j = w_j(f_{p+2}), \quad \deg w_j \leqslant \left[\frac{j-1}{p+1}\right], \quad w_j(0) = 0 \ \text{ for } \ j \in |p+2, n-1|,$$

($[x]$ denotes the integral part of x). We show that then there exists a polynomial w_n such that

$$f_n = w_n(f_{p+2}), \quad \deg w_n \leqslant \left[\frac{n-1}{p+1}\right], \quad w_n(0) = 0.$$

To this end let us consider the equation $(5.r)$ with $r = n+p+1$, that is

$$f_r(x+y) = f_r(x) + \sum_{k=p+2}^{n} f_k(x) \sum_{\bar{u}_r \in U^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(y)^{u_j} + f_r(y). \tag{6}$$

From the symmetry of the left hand side of (6) we get

$$\sum_{k=p+2}^{n} f_k(x) \sum_{\bar{u}_r \in U^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(y)^{u_j} = \sum_{k=p+2}^{n} f_k(y) \sum_{\bar{u}_r \in U^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(x)^{u_j}$$

and next, from Lemma 3, taking into account the adopted notation, we obtain

$$\begin{aligned}
&\binom{r}{p+1} f_{p+2}(x) f_n(y) + \sum_{k=p+2}^{n-1} f_k(x) \sum_{\bar{u}_r \in \tilde{U}^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(y)^{u_j} \\
&\qquad + \binom{r}{p+2} f_n(x) f_{p+2}(y) \\
&\quad = \binom{r}{p+1} f_{p+2}(y) f_n(x) + \sum_{k=p+2}^{n-1} f_k(y) \sum_{\bar{u}_r \in \tilde{U}^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(x)^{u_j} \\
&\qquad + \binom{r}{p+2} f_n(y) f_{p+2}(x).
\end{aligned}$$

Notice that the sum $\sum_{k=p+2}^{n-1} f_k(x) \sum_{\bar{u}_r \in \tilde{U}^p_{r,k}} A_{\bar{u}_r} \prod_{j=p+2}^{r-k+1} f_j(y)^{u_j}$ does not contain the function f_n. From the above equality we thus obtain

$$f_n(x)f_{p+2}(y) = f_{p+2}(x)f_n(y) + \frac{n!\,(p+2)!}{(n+p+1)!\,(n-p-2)} \sum_{k=p+2}^{n-1} \sum_{\bar{u}_r \in \tilde{U}^p_{r,k}} A_{\bar{u}_r} \left[f_k(y) \prod_{j=p+2}^{r-k+1} f_j(x)^{u_j} - f_k(x) \prod_{j=p+2}^{r-k+1} f_j(y)^{u_j} \right]. \tag{7}$$

Since f_{p+2} is a nonzero additive function and G is an abelian semigroup uniquely divisible by some prime, so the set $f(G)$ is dense in $\mathbb{R}$. Thus there exists a sequence $(t_m)_{m\in \mathbb{N}}$ of elements of the group G with $\lim_{m\to\infty} f_{p+2}(t_m) = 1$. Set in (7) $y = t_m$ and take $m \to \infty$. Then we get

$$f_n(x) = d_n\, f_{p+2}(x) + \frac{n!\,(p+2)!}{(n+p+1)!\,(n-p-2)} \sum_{k=p+2}^{n-1} \sum_{\bar{u}_r \in \tilde{U}^p_{r,k}} A_{\bar{u}_r} \left[w_k(1) \prod_{j=p+2}^{r-k+1} w_j\left(f_{p+2}(x)\right)^{u_j} - w_k\left(f_{p+2}(x)\right) \prod_{j=p+2}^{r-k+1} w_j(1)^{u_j} \right],$$

where $d_n = \lim_{m\to\infty} f_n(t_m)$. This limit exists since $f_{p+2}(x) \neq 0$ for some $x \in G$ and there exist both the remaining limits. Thus we can write

$$f_n = w_n\left(f_{p+2}\right).$$

Since $\prod_{j=p+2}^{r-k+1} w_j(0)^{u_j} = 0$ for $k \in |p+2, n-1|$ and $\bar{u}_r \in \tilde{U}^p_{r,k}$, so $w_n(0) = 0$. Moreover

$$\begin{aligned}
\deg\left(\prod_{j=p+2}^{r-k+1} w_j\left(f_{p+2}(x)\right)^{u_j}\right) &\leqslant \sum_{j=p+2}^{r-k+1} \left[\frac{j-1}{p+1}\right] u_j \leqslant \sum_{j=1}^{r} \left[\frac{j-1}{p+1}\right] u_j \\
&\leqslant \sum_{j=1}^{r} \left[\frac{j-1}{p+1} u_j\right] \leqslant \left[\sum_{j=1}^{r} \frac{j-1}{p+1} u_j\right] \\
&= \left[\frac{1}{p+1}\sum_{j=1}^{r} j u_j - \frac{1}{p+1}\sum_{j=1}^{r} u_j\right] = \left[\frac{r-k}{p+1}\right] \\
&\leqslant \left[\frac{n+p+1-(p+2)}{p+1}\right] \\
&= \left[\frac{n-1}{p+1}\right].
\end{aligned}$$

Consequently $\deg w_n \leqslant \left[\frac{n-1}{p+1}\right]$ and this completes the proof.

Thus we have

COROLLARY 1

If a function $\Phi_\infty = (1, 0, \ldots, 0, f_{p+2}, f_{p+3}, \ldots)$ *is a homomorphism of* $(G, +)$ *into* L^1_∞, *then the functions* f_n, $n \in |p+2, \infty|$ *have the form given in Theorem 1.*

From Theorem 1 we deduce

THEOREM 2

Let s be a natural number. If a homomorphism $\Phi_s : G \to Z_s$,

$$\Phi_s = (1, 0, \ldots, 0, f_{p+2}, \ldots, f_s)$$

is extensible, then the function f_{s-p} *is a polynomial in* f_{p+2}.

Proof. Suppose $\Phi_s = (1, 0, \ldots, 0, f_{p+2}, \ldots, f_s)$ is extensible, i.e. there exists a function f_{s+1} such that $\bar{\Phi}_s = (1, 0, \ldots, 0, f_{p+2}, \ldots, f_s, f_{s+1})$ is a homomorphism into L^1_{s+1}. Then the functions f_n satisfy the system (5). From Theorem 1 there exist polynomials w_n such that

$$f_n = w_n(f_{p+2}) \quad \text{whenever } n + p + 1 \in |p+2, s+1|,$$

which completes the proof.

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