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On sets for which the difference set is the whole space

Abstract. We give a sufficient condition for a set $A \subset \mathbb{R}^p$ to get $A - A = \mathbb{R}^p$.

The well-known theorem of Steinhaus [4] says that if $A \subset \mathbb{R}$ (the set of all reals) has a positive Lebesgue measure then $A - A := \{a - b : \ a, b \in A\}$ has a non-empty interior. Usually in the proof of this theorem the property of having a density point of the set A is used. Steinhaus theorem says nothing on the "size" of the intervals contained in $A - A$. However Boardmann [1] and Świątkowski [5] obtained some results of this type. On the other hand, it is known that there exists a subset $A \subset \mathbb{R}$ of Lebesgue measure zero such that $A - A = \mathbb{R}$. An example of such set is $A = \mathbb{Z} + C$ where $\mathbb{Z}$ denotes the set of all integers and C is a classical Cantor set (a nice geometrically proof of this fact can be find in a paper of Utz [6]). The aim of this note is to prove some sufficient condition for a set $A \subset \mathbb{R}^p$ to have $A - A = \mathbb{R}^p$. In the sequel by m_p we will denote p-dimensional Lebesgue measure in $\mathbb{R}^p$, the symbol $K_p(0, r)$ means the p-dimensional ball with center at zero and with the radius r (if $p = 1$ the index 1 will be omitted). Our main result reads as follows

Theorem 1

Let $A \subset \mathbb{R}^p$ be a Lebesgue measurable set. If

$$\limsup_{r \to \infty} \frac{m_p[A \cap K_p(0,r)]}{m_p[K_p(0,r)]} =: \lambda > \frac{1}{2}, \tag{1}$$

then $A - A = \mathbb{R}^p$.

Proof. In the first step we assume that $p = 1$. Choose an $\varepsilon > 0$ and a sequence $(r_n)_{n \in \mathbb{N}}$ of positive numbers tending to infinity such that

$$\lambda > 1 - \frac{1}{2^{1+\varepsilon}} \quad \text{and} \quad m[A \cap K(0, r_n)] > \left(1 - \frac{1}{2^{1+\varepsilon}}\right) 2r_n\,. \tag{2}$$

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Put

$$\varphi(x) := \left(1 - \frac{1}{2^\varepsilon}\right) x, \quad x > 0,$$

and note that

$$\lim_{x\to\infty} \varphi(x) = \infty.$$

To prove our theorem in this case it is enough to show that

$$K(0, \varphi(r_n)) \subset A - A, \quad n \in \mathbb{N}. \tag{3}$$

In the sequel we will write r instead of r_n ($n \in \mathbb{N}$ is arbitrary and fixed). Fix an arbitrary $k \in K(0, \varphi(r))$. Then

$$k + [A \cap K(0, r - \varphi(r))] \subset (k + A) \cap K(0, r) \tag{4}$$

and

$$A \cap K(0,r) = [A \cap K(0, r - \varphi(r))] \cup [A \cap (K(0,r) \setminus K(0, r - \varphi(r)))]. \tag{5}$$

We shall show that

$$m\,[k + (A \cap K(0, r - \varphi(r)))] > \frac{r}{2^\varepsilon}. \tag{6}$$

In fact, by virtue of (5), we obtain

$$\begin{aligned} m[k + (A \cap K(0, r - \varphi(r)))] &= m[A \cap K(0, r - \varphi(r)))] \\ &= m[A \cap K(0,r)] \\ &\quad - m[A \cap (K(0,r) \setminus K(0, r - \varphi(r)))] \\ &> \left(1 - \frac{1}{2^{1+\varepsilon}}\right) 2r - m[K(0,r) \setminus K(0, r - \varphi(r))] \\ &= -\frac{2r}{2^{1+\varepsilon}} + 2(r - \varphi(r)) \\ &= 2r\left[\left(1 - \frac{\varphi(r)}{r}\right) - \frac{1}{2^{1+\varepsilon}}\right] = \frac{2r}{2^{1+\varepsilon}} \\ &= \frac{r}{2^\varepsilon}. \end{aligned}$$

To show (3) it is enough to prove

$$A \cap K(0,r) \cap [k + (A \cap K(0, r - \varphi(r)))] \neq \emptyset. \tag{7}$$

Assume that the set on the left-hand side of (7) is empty. Hence and by (4) we get

$$k + [A \cap K(0, r - \varphi(r))] \subset K(0,r) \setminus A$$

and applying (2)

$$
\begin{aligned}
m[k + [A \cap K(0, r - \varphi(r))]] &\leqslant m[K(0,r) \setminus A] \\
&= m[K(0,r) \setminus (K(0,r) \cap A)] \\
&< 2r - \left(1 - \frac{1}{2^{1+\varepsilon}}\right) 2r \\
&= \frac{r}{2^{\varepsilon}}.
\end{aligned}
$$

This contradicts (6) and in this way proves (7). Consequently, (3) holds true for every positive integer n and it ends the proof of Theorem 1 in the case $p = 1$.

Now we assume that $p \geqslant 2$. Recall that $m_p[(K_p(0,r)] = \gamma_p r^p$, where γ_p is a constant depending only on p. Choose an arbitrary $y \in \mathbb{R}^p \setminus \{0\}$ and let $\mathbb{R}_y$ be one-dimensional subspace of $\mathbb{R}^p$ generated by y i.e. $\mathbb{R}_y := \{ty : \ t \in \mathbb{R}\}$. Let $(\mathbb{R}_y)^{\perp}$ be a $(p-1)$-dimensional orthogonal complement of $\mathbb{R}_y$ to the space $\mathbb{R}^p$. Take an $\varepsilon > 0$ and a positive number r such that

$$
m_p(A \cap K_p(0,r)) > \left(\frac{1}{2} + \frac{\gamma_p + 2\gamma_{p-1}}{\gamma_p}\varepsilon\right)\gamma_p r^p \tag{8}
$$

and

$$
\|y\| < r\varepsilon \sqrt{1 - \sqrt[p-1]{(1-\varepsilon)^2}}\,. \tag{9}
$$

For an arbitrary $x \in (\mathbb{R}_y)^{\perp}$ with $\|x\| \leqslant r$ we define a set A_x and a function h in the following manner:

$$
A_x := \left\{t \in \mathbb{R} : \ x + t\frac{y}{\|y\|} \in A\right\}, \quad h(x) := \sqrt{r^2 - \|x\|^2}
$$

(the symbol $\|x\|$ denotes here the norm in the space $(\mathbb{R}_y)^{\perp}$). Let us denote

$$
B := \left\{x \in (\mathbb{R}_y)^{\perp} : \ m(A_x \cap [-h(x), h(x)]) > \left(\frac{1}{2} + \varepsilon\right) 2h(x)\right\}.
$$

We shall prove that

$$
m_{p-1}(B) > \gamma_{p-1} r^{p-1}\varepsilon. \tag{10}
$$

For this assume that $m_{p-1}(B) \leqslant \gamma_{p-1} r^{p-1}\varepsilon$. Let $K_{p-1}^{\perp}(0,\delta)$ denotes an orthogonal projection of $K_p(0,\delta)$ to the space $(\mathbb{R}_y)^{\perp}$. Then

$$
\begin{aligned}
m_p(A \cap K_p(0,r)) \leqslant &\int_{K_{p-1}^{\perp}(0,r)\cap B} m(A_x \cap [-h(x), h(x)])\,dx \\
&+ \int_{K_{p-1}^{\perp}(0,r)\setminus B} m(A_x \cap [-h(x), h(x)])\,dx
\end{aligned}
$$

$$\begin{aligned}
&\leqslant m_{p-1}(B)2r + \int_{K_{p-1}^{\perp}(0,r)\setminus B} \left(\frac{1}{2}+\varepsilon\right) 2h(x)\,dx \\
&\leqslant 2\varepsilon\gamma_{p-1}r^p + \left(\frac{1}{2}+\varepsilon\right)\gamma_p r^p \\
&= \gamma_p r^p \left(\frac{1}{2} + \left(1 + \frac{2\gamma_{p-1}}{\gamma_p}\right)\varepsilon\right) \\
&= \gamma_p r^p \left(\frac{1}{2} + \frac{\gamma_p + 2\gamma_{p-1}}{\gamma_p}\varepsilon\right),
\end{aligned}$$

which contradicts (8) and proves (10). Since

$$\begin{aligned}
m_{p-1}\left(K_{p-1}^{\perp}\left(0, \sqrt[p-1]{1-\varepsilon}\; r\right)\right) + m_{p-1}(B) &> \gamma_{p-1}(1-\varepsilon)r^{p-1} + \gamma_{p-1}r^{p-1}\varepsilon \\
&= m_{p-1}(K_{p-1}^{\perp}(0,r))
\end{aligned}$$

and

$$K_{p-1}^{\perp}\left(0, \sqrt[p-1]{1-\varepsilon}\; r\right) \cup B \subset K_{p-1}^{\perp}(0,r),$$

then there exists an $x_0 \in K_{p-1}^{\perp}\left(0, \sqrt[p-1]{1-\varepsilon}\; r\right) \cap B$. Therefore,

$$\begin{aligned}
h(x_0) &= \sqrt{r^2 - \|x_0\|^2} \\
&\geqslant \sqrt{r^2 - \sqrt[p-1]{(1-\varepsilon)^2}\; r^2} \\
&= r\sqrt{1 - \sqrt[p-1]{(1-\varepsilon)^2}}.
\end{aligned}$$

Setting

$$E_{x_0} := A_{x_0} \cap [-h(x_0), h(x_0)]$$

and using (9) we get

$$E_{x_0} \cup (E_{x_0} + \|y\|) \subset [-h(x_0) - \varepsilon h(x_0), h(x_0) + \varepsilon h(x_0)].$$

Consequently,

$$m(E_{x_0} \cup (E_{x_0} + \|y\|)) \leqslant 2h(x_0)(1+\varepsilon).$$

On the other hand the condition $x_0 \in B$ implies that

$$m(E_{x_0}) = m(E_{x_0} + \|y\|) > \left(\frac{1}{2}+\varepsilon\right) 2h(x_0).$$

This proves that

$$E_{x_0} \cap (E_{x_0} + \|y\|) \neq \emptyset.$$

Assume that $t_0 \in E_{x_0}$ and $t_0 \in E_{x_0} + \|y\|$. Then $x_0 + t_0\frac{y}{\|y\|} \in A$ and $x_0 + (t_0 + \|y\|)\frac{y}{\|y\|} \in A$. This means that $y \in A - A$. The proof of Theorem 1 is completed.

REMARK 1

The condition (1) is the best one in a sense. There exists a set $T \subset \mathbb{R}^p$ such that

$$\limsup_{r\to\infty} \frac{m_p[T \cap K_p(0,r)]}{m_p[K_p(0,r)]} = \frac{1}{2}$$

and simultanously $T - T \neq \mathbb{R}^p$.

For, define

$$T := \bigcup_{n\in\mathbb{Z}} [2n, 2n+1) \times \mathbb{R}^{p-1}$$

where $\mathbb{Z}$ denotes the set of all integers. It is easily seen that $(T - T) \cap ((2\mathbb{Z}+1) \times \mathbb{R}^{p-1}) = \emptyset$ which implies that $T - T \neq \mathbb{R}^p$ and on the other hand

$$\lim_{n\to\infty} \frac{m_p[T \cap K_p(0,n)]}{m_p[K(0,n)]} = \frac{1}{2}.$$

The following two theorems concern the set of all distances of elements of a given set A contained in a real normed space.

THEOREM 2

Let A be an arbitrary subset of a real normed space X and assume that $\dim X \geqslant 2$. *Then*

$$D(A) := \{\, \|x-y\| : \ x, y \in A \,\} = [0, \infty)$$

or

$$D(X \setminus A) := \{\, \|x-y\| : \ x, y \in X \setminus A \,\} = [0, \infty).$$

Proof. Assume that there exist $r_i \in (0, \infty)$, $i = 1, 2$ such that

$$r_1 \notin D(A) \quad \text{and} \quad r_2 \notin D(X \setminus A).$$

Then for every $x \in X$ with $\|x\| = r_1$ we have

$$(A + x) \cap A = \emptyset,$$

or, equivalently,

$$A + \{x \in X : \ \|x\| = r_1\} \subset X \setminus A =: B. \tag{11}$$

It follows from (11) that

$$A - A \subset B - B.$$

Similarly one can prove that

$$B - B \subset A - A.$$

According to Theorem 3 from [3]

$$B = A + x \quad \text{iff} \quad x \notin A - A = B - B.$$

So, if our assertion does not hold then

$$B = A + x \quad \text{for every } x \in X \text{ with } \|x\| = r_1 . \tag{12}$$

Take an x, $\|x\| = r_1$, and choose a y with $\|y\| = r_1$ such that $\|x + y\| = r_1$ (it is possible since $\dim X \geqslant 2$; the set $\{y : \|y\| = r_1\}$ is connected and $0 = \|x - x\| < r_1 < 2r = \|x + x\|$). Now by (12)

$$B = A + (x + y) = (A + x) + y = B + y = A,$$

a contradiction. The proof of Theorem 2 is complete.

REMARK 2

In the case $X = \mathbb{R}$ the assertion of Theorem 2 does not hold.

For, take $A = \bigcup_{n\in\mathbb{Z}}[2n, 2n + 1)$. Then $B = A + 1$ and, of course, $1 \notin A - A = B - B$.

THEOREM 3

Let A be an arbitrary non-empty subset of a real normed space X such that $B := X \setminus A \neq \emptyset$ and assume that $\dim X \geqslant 2$. *Then $D(A, B) := \{\|x - y\| : x \in A,\ y \in B\} = (0, \infty)$.*

Proof. Evidently, $0 \notin D(A, B)$. Take an $r > 0$ and assume that $r \notin D(A, B)$. Then for every $x \in X$ with $\|x\| = r$ we have

$$(A + x) \cap B = \emptyset \quad \text{and} \quad (B + x) \cap A = \emptyset. \tag{13}$$

Let us denote $S(r) := \{x \in X : \|x\| = r\}$. It follows from (13) that

$$A + S(r) \subset A \quad \text{and} \quad B + S(r) \subset B. \tag{14}$$

If $0 \in A$ then $S(r) \subset A$. We shall show that

$$S(nr) \subset A \quad \text{for every positive integer } n. \tag{15}$$

On account of (14) we note that condition (15) holds true for $n = 1$. Assume (15) for a positive integer n and take $x \in S((n+1)r)$. Then $y := nr\frac{x}{\|x\|} \in S(nr)$ and $\|x - y\| = \|x\| \left(1 - \frac{nr}{\|x\|}\right) = \|x\| - nr = r$. So, $x \notin B$ because $r \notin D(A, B)$. Consequently, $S((n + 1)r) \subset A$ and the proof of (15) is complete. On account of a result of R. Ger [2] (the proof of Lemma 1) we have

$$\{x \in X : \|x\| < 2\varepsilon\} \subset S(\varepsilon) + S(\varepsilon)$$

for every $\varepsilon > 0$, which together with (15), yields the equality $A = X$, a contradiction. In the case $0 \in B$ the proof runs quite similarly.

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