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Concave iteration semigroups of Jensen set-valued functions

Abstract. Let S be a closed convex cone such that $\operatorname{int} S \neq \emptyset$ in a real separable Banach space. A necessary and sufficient condition is given under which a family $\{F^t : t \geqslant 0\}$ of set-valued functions $F^t : S \to cc(S)$ is a concave iteration semigroup of continuous Jensen set-valued functions.

1. A subset S of a real vector space X is called a *cone* iff $tS \subset S$ for all $t \in (0, \infty)$. Let X and Y be two real vector spaces and let $S \subset X$ be a convex cone. $n(Y)$ denotes the family of all nonempty subsets of Y. A set-valued function $F : S \to n(Y)$ is called *Jensen* iff it satisfies the equation

$$\frac{1}{2}[F(x) + F(y)] = F\left(\frac{x+y}{2}\right)$$

for all $x, y \in S$, where

$$F(x) + F(y) = \{u + v : u \in F(x), v \in F(y)\}.$$

A set-valued function $F : S \to n(Y)$ is said to be *concave* iff

$$F(\lambda x + (1-\lambda)y) \subset \lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in S$ and $\lambda \in (0, 1)$.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers. All vector spaces are supposed to be real. If X is a vector normed space, then $c(X)$ denotes the set of all compact members of $n(X)$ and $cc(X)$ stands for the set of all convex sets of $c(X)$. For $A_n, A \in n(X)$, $n \in \mathbb{N}$, the symbol

$$\lim_{n\to\infty} A_n = A$$

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means that

$$\lim_{n\to\infty} d(A_n, A) = 0,$$

where d denotes the Hausdorff metric derived by the norm in X. In what follows the continuity of a set-valued function with compact values denotes the continuity with respect to the Hausdorff metric d.

We need the following lemmas.

LEMMA 1 (cf. [9])

Let A, B and C be subsets of a real topological vector space such that

$$A + B \subset C + B.$$

If C is convex closed and B is non-empty bounded, then

$$A \subset C.$$

LEMMA 2 (see e.g. Lemma 3 in [5])

Let X be a normed space. If (A_n) is a sequence of elements of the set $c(X)$ such that $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

LEMMA 3 (Lemma 3 in [10])

Let S be a closed convex cone such that $\operatorname{int} S \neq \emptyset$ *in Banach space X and let Y be a normed space. If (A_n) is a sequence of continuous additive set-valued functions $A_n : S \to cc(Y)$ such that $A_{n+1}(x) \subset A_n(x)$ for all $x \in S$ and $n \in \mathbb{N}$, then the formula*

$$A_0(x) := \bigcap_{n=1}^{\infty} A_n(x), \quad x \in S$$

defines a continuous additive set-valued function $A_0 : S \to cc(Y)$. Moreover,

$$\lim_{n\to\infty} A_n(x) = A_0(x), \quad x \in S \tag{1}$$

and the convergence in (1) is uniform on every nonempty compact subset of S.

LEMMA 4 (Lemma 4 in [10])

Let D be a nonempty set and Y be a normed space. Suppose that $F_0, F_n : D \to c(Y)$ are set-valued functions. If the sequence (F_n) uniformly converges to F_0 on D, then

$$\lim_{n\to\infty} F_n(D) = F_0(D).$$

The following lemmas are also true.

LEMMA 5

Let X be a real separable Banach space and Y be a normed space. Assume that S is a closed convex cone in X such that $\operatorname{int} S \neq \emptyset$. Moreover, let $A : S \to cc(Y)$ be a continuous additive set-valued function. Then there exists a constant $M \in (0, \infty)$ such that

$$d(A(x), A(y)) \leqslant M||x - y||$$

for $x, y \in S$.

Proof. By Lemma 7 in [8] there exists a constant $M \in (0, \infty)$ such that

$$d(A(x), A(y)) \leqslant M||x - y||$$

for $x, y \in \operatorname{int} S$. Since $S = \operatorname{cl}(\operatorname{int} S)$ (see Theorem 5.1.6 in [3] for $X = R^n$; the same proof works for arbitrary X) the continuity of A implies this lemma.

LEMMA 6 (see e.g. [11])

Let (S, ρ) be a metric space and let F be a set-valued function from S into S. If there is $M > 0$ such that the inequality

$$d(F(x), F(y)) \leqslant M\, \rho(x, y)$$

holds for all $x, y \in S$, then

$$d(F(A), F(B)) \leqslant M\, d(A, B)$$

for every non-empty subsets A, B of S, where d denotes the Hausdorff metric derived by the metric ρ in S.

Let X be a Banach space and let $[a, b] \subset [0, \infty)$ be a closed interval. Suppose that a set-valued function $G : [a, b] \to cc(X)$ is continuous. Then there exists the Hukuhara version of the Riemann integral

$$\int_a^b G(t)\, dt$$

(see [2]).

The following four lemmas describe some important properties of this integral.

LEMMA 7 (see [2] p. 211)

If $C \in cc(X)$ and $F(t) = C$ for every $t \in [a, b]$, then

$$\int_a^b F(t)\, dt = (b - a)C.$$

LEMMA 8 (see [2] p. 211)

For every continuous set-valued functions $F, G : [a,b] \to cc(X)$ we have

$$d\left(\int_a^b F(t)\,dt, \int_a^b G(t)\,dt\right) \leqslant \int_a^b d(F(t), G(t))\,dt.$$

LEMMA 9 (see [2] p. 212)

If $F : [a,b] \to cc(X)$ is continuous and $a < c < b$, then

$$\int_a^b F(u)\,du = \int_a^c F(u)\,du + \int_c^b F(u)\,du.$$

LEMMA 10

If $F : [a,b] \to cc(X)$ is continuous, $u := \phi(u) = \alpha v + \beta$ for $t \leqslant v \leqslant s$, where $a = \phi(t)$ and $b = \phi(s)$, then

$$\int_a^b F(u)\,du = \alpha \int_t^s F(\alpha v + \beta)\,dv.$$

The easy proof is left to the reader.

Let X, Y, Z be nonempty sets. We recall that the *superposition* $G \circ F$ of set-valued functions F from X into Y and G from Y into Z is defined by the formula

$$(G \circ F)(x) = \bigcup \{G(y) : \ y \in F(x)\}, \quad x \in X.$$

Let us note the following obvious lemma

LEMMA 11

Let X and Y be two Banach spaces and let S be an open convex cone in X. If $F : [a,b] \to cc(S)$ is a continuous set-valued function and $A : S \to cc(Y)$ is a continuous additive set-valued function, then

$$\int_a^b (A \circ F)(t)\,dt = A\left(\int_a^b F(t)\,dt\right).$$

2. Let S be a nonempty set. A family $\{F^t : \ t \geqslant 0\}$ of set-valued functions $F^t : S \to n(S)$ is said to be an *iteration semigroup* iff

$$(F^t \circ F^s)(x) = F^{t+s}(x) \tag{2}$$

for all $x \in S$ and $t, s \geqslant 0$.

An iteration semigroup $\{F^t : t \geqslant 0\}$ is *concave* iff the set-valued function $t \mapsto F^t(x)$ is concave for every $x \in S$.

Let Y be a metric space. An iteration semigroup $\{F^t : t \geqslant 0\}$ of set-valued functions $F^t : Y \to cc(Y)$ is said to be *continuous* iff the function $t \mapsto F^t(y)$ is continuous for every $y \in Y$.

LEMMA 12 (see [7])

Let X be a real separable Banach space and let S be a closed convex cone such that $\operatorname{int} S \neq \emptyset$. *Suppose that $\{A^t : t \geqslant 0\}$ is a concave iteration semigroup of continuous additive set-valued functions $A^t : S \to cc(S)$ with $A^0(x) = \{x\}$. Then there exists a set-valued function $G : S \to cc(S)$ such that*

$$G(x) = \lim_{t \to 0+} \frac{A^t(x) - x}{t}$$

for all $x \in S$ and the convergence is uniform on every nonempty compact subset of S. Moreover, G is additive and continuous and

$$G(x) = \bigcap_{t>0} \frac{A^t(x) - x}{t}$$

for every $x \in S$.

Now let X be a real Banach seperable space and let S be a closed convex cone in X such that $\operatorname{int} S \neq \emptyset$. Consider a concave iteration semigroup $\{F^t : t \geqslant 0\}$ of Jensen continuous set-valued functions $F^t : S \to cc(S)$. Since S is a convex cone with zero, each F^t is of the form

$$F^t(x) = A^t(x) + \phi(t) \tag{3}$$

for all $x \in S$, $t \geqslant 0$, where $A^t : S \to cc(X)$ is an additive set-valued function and $\phi(t) \in cc(X)$ for $t \geqslant 0$ (see Theorem 5.6 in [6]). Similarly as in [10] we can show that actually $\phi(t) \in cc(S)$ and $A^t(x) \in cc(S)$ for all $x \in S$ and $t \geqslant 0$. Since set-valued function F^t is continuous, set-valued functions A^t is also continuous for all $t \geqslant 0$. The concavity of the iteration semigroup implies that

$$\begin{aligned}
\phi(\lambda t + (1-\lambda)s) &= F^{\lambda t + (1-\lambda)s}(0) \\
&\subset \lambda F^t(0) + (1-\lambda)F^s(0) \\
&= \lambda\phi(t) + (1-\lambda)\phi(s)
\end{aligned}$$

for all $t, s \geqslant 0$ and $\lambda \in (0,1)$. Now, fix arbitrarily $x \in S$, $t, s \geqslant 0$ and $\lambda \in (0,1)$. Then

$$\begin{aligned} A^{\lambda t+(1-\lambda)s}(x) &= \lim_{n\to\infty}\left[A^{\lambda t+(1-\lambda)s}(x)+\frac{1}{n}\phi(\lambda t+(1-\lambda)s)\right] \\ &= \lim_{n\to\infty}\frac{1}{n}F^{\lambda t+(1-\lambda)s}(nx) \\ &\subset \lim_{n\to\infty}\left[\lambda\frac{1}{n}F^{t}(nx)+(1-\lambda)\frac{1}{n}F^{s}(nx)\right] \\ &= \lambda A^{t}(x)+(1-\lambda)A^{s}(x). \end{aligned}$$

Thus the set-valued functions ϕ and $t\mapsto A^t(x)$, $x\in S$ are concave. A consequence of equations (2) and (3) is the system

$$A^s[A^t(x)] = A^{s+t}(x)$$

and

$$\phi(s+t) = A^s[\phi(t)]+\phi(s) \tag{4}$$

for all $x\in S$ and $t,s\geqslant 0$.

By their concavity the set valued functions $t\mapsto F^t(x)$, $t\mapsto A^t(x)$, $x\in S$ and ϕ are continuous on $(0,\infty)$ (see Theorem 4.4 in [6]).

Now suppose that

$$F^0(x)=\{x\} \tag{5}$$

for $x\in S$. By (3)

$$\{x\} = A^0(x)+\phi(0), \tag{6}$$

and

$$A^0(x) = \lim_{n\to\infty}\left[A^0(x)+\frac{1}{n}\phi(0)\right] = \lim_{n\to\infty}\frac{1}{n}F^0(nx) = \{x\},$$

whence by (6)

$$\phi(0)=\{0\}. \tag{7}$$

Now we shall prove the following

THEOREM 1

Let X be a real separable Banach space and let S be a closed convex cone in X such that $\operatorname{int} S\neq\emptyset$. Assume that $\{A^t:\ t\geqslant 0\}$ is a concave iteration semigroup of continuous additive set-valued functions $A^t: S\to cc(S)$ with $A^0(x)=\{x\}$ for $x\in S$. A set-valued function $\phi:[0,\infty)\to cc(S)$ is a concave continuous at zero solution of (4) *satisfying condition* (7) *if and only if there exists a set $D\in cc(S)$ such that*

$$\phi(s)=\int_0^s A^u(D)\,du, \tag{8}$$

and

$$D \subset A^s(D) \tag{9}$$

for all $s \geqslant 0$.

Proof. At first assume that $\phi : [0, \infty) \to cc(S)$ is a concave continuous at zero solution of equation (4) such that $\phi(0) = \{0\}$. Let $0 < t < s$. Then we have

$$\begin{aligned} \phi(t) &= \phi\left(\frac{t}{s}s\right) = \phi\left(\frac{t}{s}s + \left(1 - \frac{t}{s}\right)0\right) \\ &\subset \frac{t}{s}\phi(s) + \left(1 - \frac{t}{s}\right)\phi(0) \\ &= \frac{t}{s}\phi(s), \end{aligned}$$

whence

$$\frac{\phi(t)}{t} \subset \frac{\phi(s)}{s},$$

i.e., the function

$$t \mapsto \frac{\phi(t)}{t}$$

is increasing in $(0, \infty)$. It is easy to see that the set $D := \bigcap_{t>0} \frac{\phi(t)}{t}$ belongs to $cc(S)$. By Lemma 2

$$D = \lim_{t \to 0+} \frac{\phi(t)}{t}. \tag{10}$$

Now we have

$$\begin{aligned} d\left(A^t(x), \{x\}\right) &= t\, d\left(\frac{A^t(x) - x}{t}, \{0\}\right) \\ &\leqslant t\, d\left(\frac{A^t(x) - x}{t}, G(x)\right) + t\, d\left(G(x), \{0\}\right) \end{aligned}$$

for all $x \in D$ and positive t, where G is given in Lemma 12. Lemmas 12 and 4 yield the equality

$$\lim_{t \to 0+} A^t(D) = D. \tag{11}$$

Let s be a nonnegative number. There exists the smallest number $L(s) \geqslant 0$ such that

$$d(A^s(x), A^s(y)) \leqslant L(s)\|x - y\|$$

for all $x, y \in S$ (see Lemma 5). The function L is measurable since

$$L(s) = \sup_{n \neq m} \frac{d(A^s(x_n), A^s(x_m))}{\|x_n - x_m\|},$$

where $\{x_n : n \in \mathbb{N}\}$ is a dense subset of S. Moreover, this function is sub-multiplicative. Now, we can define the function $m(s) := \log(L(s) + 1)$. It is a finite measurable subadditive function. By Theorem 7.4.1 in [1] the function m is bounded above on any compact subset of $(0, \infty)$. Consequently for every positive s there exists a positive number K such that

$$L(u) \leqslant K$$

whenever $u \in \left[\frac{s}{2}, s\right]$. Let s be a positive number and $t \in (0, \frac{s}{2})$. By Lemma 6

$$\begin{aligned} d(A^{s+t}(D), A^s(D)) &= d(A^s(A^t(D)), A^s(D)) \\ &\leqslant L(s)d(A^t(D), D) \end{aligned}$$

and

$$\begin{aligned} d(A^s(D), A^{s-t}(D)) &= d(A^{s-t}(A^t(D)), A^{s-t}(D)) \\ &\leqslant L(s-t)d(A^t(D), D) \\ &\leqslant Kd(A^t(D), D). \end{aligned}$$

These inequalities and (11) imply that the function $t \mapsto A^t(D)$ is continuous in $[0, \infty)$.

For every positive t and s we have

$$\begin{aligned} \phi(s) + \frac{t}{s}\phi(s) &= (s+t)\frac{\phi(s)}{s} \\ &\subset (s+t)\frac{\phi(s+t)}{s+t} \\ &= \phi(s+t) \\ &= A^s(\phi(t)) + \phi(s) \end{aligned}$$

and by Lemma 1

$$\frac{\phi(s)}{s} \subset A^s\left(\frac{\phi(t)}{t}\right),$$

whence

$$D \subset A^s\left(\frac{\phi(t)}{t}\right). \tag{12}$$

On the other hand

$$d\left(A^s\left(\frac{\phi(t)}{t}\right), A^s(D)\right) \leqslant L(s)d\left(\frac{\phi(t)}{t}, D\right)$$

for $t > 0, s \geqslant 0$, hence

$$\lim_{t\to 0+} A^s\left(\frac{\phi(t)}{t}\right) = A^s(D).$$

This and (12) yield (9) for all $s \geqslant 0$.

Write

$$\psi(t) = \int_0^t A^u(D)\,du$$

and

$$h(t) = d(\phi(t), \psi(t))$$

for nonnegative t. The functions ψ and h are continuous. By Lemmas 9 and 10

$$\begin{aligned}\psi(t+s) &= \int_0^{t+s} A^u(D)\,du = \psi(s) + \int_0^t A^s(A^u(D))\,du \\ &= \psi(s) + A^s(\psi(t)),\end{aligned}$$

i.e., the set-valued function ψ is a solution of functional equation (4).

To end the proof of the necessity it suffices to show that $\psi = \phi$. Lemmas 7 and 8 imply that

$$\begin{aligned}d\left(\frac{\psi(t)}{t}, D\right) &= d\left(\frac{1}{t}\int_0^t A^u(D)\,du, D\right) \\ &\leqslant \frac{1}{t}\int_0^t d\left(A^u(D), D\right)\,du,\end{aligned}$$

whence, with respect to (11),

$$\lim_{t\to 0+} \frac{\psi(t)}{t} = D. \tag{13}$$

Moreover,

$$\begin{aligned}h^+(s) &:= \limsup_{t\to 0+} \frac{h(s+t) - h(s)}{t} \\ &= \limsup_{t\to 0+} \frac{d(\psi(s+t), \phi(s+t)) - d(\psi(s), \phi(s))}{t} \\ &\leqslant \limsup_{t\to 0+} \frac{d(A^s(\psi(t)), A^s(\phi(t)))}{t} \\ &\leqslant \lim_{t\to 0+} L(s) d\left(\frac{\psi(t)}{t}, \frac{\phi(t)}{t}\right).\end{aligned}$$

Thus by (10) and (13) we have $h^+(x) \leqslant 0$. Since h is a continuous function with

$$h_+(s) := \liminf_{t\to 0+} \frac{h(s+t)-h(s)}{t} \leqslant 0,$$

according to Zygmund's lemma (see [4]) the function h is non-increasing. Consequently $h(s) \leqslant h(0) = 0$ for $s \geqslant 0$, i.e., $\psi = \phi$. Now we assume that for some $D \in cc(S)$ inclusion (9) holds for $s \geqslant 0$. As in the first part of the proof the function $t \to A^t(D)$ is continuous. Let ϕ be given by formula (8). Of course, $\phi(0) = \{0\}$ and ϕ satisfies equation (4). By (9)

$$\int_t^s A^t(D)\,du \subset \int_t^s A^u(D)\,du$$

whenever $0 \leqslant t \leqslant s$. Fix s and t such that $0 \leqslant t \leqslant s$ and $\lambda \in (0,1)$. By Lemma 10 we have

$$\int_t^{\lambda t+(1-\lambda)s} A^u(D)\,du = (1-\lambda)\int_t^s A^{\lambda t+(1-\lambda)v}(D)\,dv$$

and the concavity of the function $u \mapsto A^u(D)$ yields

$$\begin{aligned}\int_t^{\lambda t+(1-\lambda)s} A^u(D)\,du &\subset \lambda(1-\lambda)\int_t^s A^t(D)\,dv + (1-\lambda)^2\int_t^s A^v(D)\,dv\\ &= (1-\lambda)\int_t^s A^v(D)\,dv,\end{aligned}$$

whence

$$\begin{aligned}\int_0^{\lambda t+(1-\lambda)s} A^u(D)\,du &= \lambda\int_0^t A^u(D)\,du + (1-\lambda)\int_0^t A^u(D)\,du\\ &\quad + \int_t^{\lambda t+(1-\lambda)s} A^u(D)\,du\\ &\subset \lambda\int_0^t A^u(D)\,du + (1-\lambda)\int_0^t A^u(D)\,du\\ &\quad + (1-\lambda)\int_t^s A^u(D)\,du\\ &= \lambda\int_0^t A^u(D)\,du + (1-\lambda)\int_0^s A^u(D)\,du,\end{aligned}$$

i.e., ϕ is concave. The proof of the continuity of ϕ defined by formula (8) is analogous to the proof of the similar assertion for Riemann integral. This completes the proof.

3. In this part we prove the main result of the paper.

THEOREM 2

Let X be a separable Banach space and let S be a closed convex cone in X such that $\operatorname{int} S \neq \emptyset$. *One-parameter family $\{F^t : t \geqslant 0\}$ of set-valued functions $F^t : S \to cc(S)$ is a concave iteration semigroup of continuous Jensen set-valued functions $F^t : S \to cc(S)$ such that $F^0(x) = \{x\}$ if and only if there exist a concave iteration semigroup $\{A^t : t \geqslant 0\}$ of continuous additive set-valued functions $A^t : S \to cc(S)$ with $A^0(x) = \{x\}$ and a set $D \in cc(S)$ for which conditions*

$$F^t(x) = A^t(x) + \int_0^t A^u(D)\, du, \tag{14}$$

and (9) *hold.*

Proof. Necessity. Suppose that a family $\{F^t : t \geqslant 0\}$ is a concave iteration semigroup of continuous Jensen set-valued functions $F^t : S \to cc(S)$ such that $F^0(x) = \{x\}$ for $x \in S$. This iteration semigroup is continuous on $(0, \infty)$. Since F^t are Lipschitz functions this semigroup is continuous (see [11]). As it was showed in section 2 of the paper there exist a concave iteration semigroup $\{A^t : t \geqslant 0\}$ of continuous additive functions $A^t : S \to cc(S)$ and a concave solution $\phi : [0, \infty) \to cc(S)$ of equation (4). Theorem 1 and (3) imply that

$$F^s(x) = A^s(x) + \phi(s) = A^s(x) + \int_0^s A^u(D)\, du$$

for $x \in S$ and $s \geqslant 0$, where $D \in cc(S)$ is such that $D \subset A^s(D)$ for all non-negative s.

Sufficiency. Suppose that there exist a concave iteration semigroup $\{A^t : t \geqslant 0\}$ of continuous additive set-valued functions $A^t : S \to cc(S)$ with $A^0(x) = \{x\}$ and a set $D \in cc(S)$ for which condition (9) holds. It is easy to check that formula (14) defines a concave iteration semigroup of continuous Jensen set-valued function such that $F^0(x) = \{x\}$ for $x \in S$.

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