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On a functional equation of Abel

Abstract. We are concerned with the general solution and the stability problem for the functional equation

$$f(x+y) = g(xy) + h(x-y)$$

in the case $f, g, h : [0, \infty) \to S$, where S is an Abelian semigroup with the cancelation law. The set-valued case is considered as an application.

1. General solution

Let $(S, +)$ be an Abelian semigroup with zero satisfying the cancelation law, i.e.,

$$t + s = t' + s \implies t = t'. \tag{1}$$

Consider the Abel functional equation

$$0 \leqslant y \leqslant x \implies f(x+y) = g(xy) + h(x-y), \tag{2}$$

where $f, g, h : [0, \infty) \to S$ are unknown functions. The general solution of the equation

$$f(x+y) = g(xy) + h(x-y)$$

for functions $f, g, h : \mathbb{R} \to \mathbb{R}$ was given by Aczél [1] and by Lajkó (see [5] and [6]). The same equation was also solved in the case $f, g, h : K \to G$, where K is a field and G is an Abelian group (see [4]).

A function $g : [0, \infty) \to S$ satisfying the functional equation

$$2g\left(\frac{x+y}{2}\right) = g(x) + g(y) \tag{3}$$

for all $x, y \in [0, \infty)$, is said to be Jensen.

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THEOREM 1

Let $(S,+)$ be an Abelian semigroup with zero satisfying the cancellation law and let $f,g,h : [0,\infty) \to S$ satisfy equation (2). Then g is a Jensen function and

$$f(x) = g\left(\frac{x^2}{4}\right) + h(0), \tag{4}$$

$$h(x) + g(0) = g\left(\frac{x^2}{4}\right) + h(0) \tag{5}$$

for all $x \in [0,\infty)$. Conversely, if $g : [0,\infty) \to S$ is a Jensen function and

$$f(x) = g\left(\frac{x^2}{4}\right) + \alpha, \tag{6}$$

$$h(x) + g(0) = g\left(\frac{x^2}{4}\right) + \alpha \tag{7}$$

for $x \in [0,\infty)$, where α is an element of S, then functions f, g, h satisfy equation (2).

Proof. Putting $y = 0$ in (2) we get

$$f(x) = g(0) + h(x) \quad \text{for all } x \in [0,\infty). \tag{8}$$

Next, let us fix $u \in [0,\infty)$. The substitution $x = y = \frac{u}{2}$ yields

$$f(u) = h(0) + g\left(\frac{u^2}{4}\right). \tag{9}$$

Take arbitrary $u, v \in [0,\infty)$. We may find x, y such that $0 \leqslant y \leqslant x$ and

$$u = xy, \quad v = \left(\frac{x-y}{2}\right)^2. \tag{10}$$

Indeed, it sufficies to take

$$x = \sqrt{u+v} + \sqrt{v}, \quad y = \sqrt{u+v} - \sqrt{v}.$$

Then we have also $u + v = \frac{1}{4}(x+y)^2$. Relations (2) and (9) imply that

$$\begin{aligned} g(u+v) + h(0) &= g\left(\frac{1}{4}(x+y)^2\right) + h(0) \\ &= f(x+y) = g(xy) + h(x-y) \\ &= g(u) + h(2\sqrt{v}), \end{aligned}$$

whence by (8) and (9)

$$\begin{aligned} g(u+v)+h(0)+g(0) &= g(u)+h(2\sqrt{v})+g(0) \\ &= g(u)+f(2\sqrt{v}) \\ &= g(u)+g(v)+h(0). \end{aligned}$$

From (1) we get

$$g(u+v)+g(0)=g(u)+g(v). \tag{11}$$

Setting $v=u$ we obtain hence

$$2g(u)=g(2u)+g(0).$$

Setting next $u=\frac{x+y}{2}$ in the above equality and applying once more (11) we get

$$2g\left(\frac{x+y}{2}\right)=g(x+y)+g(0)=g(x)+g(y),$$

i.e., g is a Jensen function. Formula (4) coincides with (9) while (5) follows from (4) and (8).

Assume that $g:[0,\infty)\to S$ is a Jensen function and $\alpha\in S$. We have

$$\begin{aligned} g\left(\frac{(x+y)^2}{4}\right)+g(0) &= g\left(\frac{(x-y)^2}{4}+xy\right)+g(0) \\ &= 2g\left(\frac{\frac{(x-y)^2}{4}+xy}{2}\right) \\ &= g\left(\frac{(x-y)^2}{4}\right)+g(xy) \end{aligned}$$

for all $x\geqslant y\geqslant 0$. Suppose that functions f and h are given by (6) and (7). Then

$$\begin{aligned} f(x+y)+g(0) &= g\left(\frac{(x+y)^2}{4}\right)+\alpha+g(0) \\ &= g\left(\frac{(x-y)^2}{4}\right)+g(xy)+\alpha \\ &= g(xy)+h(x-y)+g(0) \end{aligned}$$

for all $x\geqslant y\geqslant 0$ and condition (2) holds.

Let $(G,+)$ be an Abelian group and let $g:[0,\infty)\to G$ be a Jensen function. It is well known that there exists an additive function $a:\mathbb{R}\to G$ such that $g(x)=a(x)+g(0)$ for $x\in[0,\infty)$. Taking $S=G$ we obtain from Theorem 1

COROLLARY 1

Let $(G,+)$ be an Abelian group. Functions $f,g,h:[0,\infty)\to G$ satisfy (2) if and only if there exist an additive function $a:\mathbb{R}\to G$ and constants $\alpha,\beta\in G$ such that

$$g(x)=a(x)+\beta,\quad f(x)=\frac{1}{4}a(x^2)+\beta+\alpha,\quad h(x)=\frac{1}{4}a(x^2)+\alpha$$

for all $x\in[0,\infty)$.

An Abelian semigroup S with zero is said to be an abstract convex cone if a map $(\lambda,s)\to\lambda s$ defined on $[0,\infty)\times S$ into S such that

$$1\cdot s=s,\quad \lambda(\mu s)=(\lambda\mu)s,\quad \lambda(s+t)=\lambda s+\lambda t,\quad (\lambda+\mu)s=\lambda s+\mu s$$

for all $s,t\in S$ and $\lambda,\mu\in[0,\infty)$, is given. We will assume that an abstract convex cone is endowed with a complete metric ρ such that

$$\rho(s+t,s+t')=\rho(t,t')\quad\text{for all } s,t,t'\in S \tag{12}$$

and

$$\rho(\lambda s,\lambda t)=\lambda\rho(s,t)\quad\text{for all } \lambda\in[0,\infty),\ s,t\in S. \tag{13}$$

The following theorem will be usefull (see [9]).

THEOREM W

Assume that S is an abstract convex cone satisfying the cancellation law and that a complete metric ρ is given in S such that (12) and (13) hold. Then, $f:[0,\infty)\to S$ is a Jensen function if and only if there exist an additive function $a:\mathbb{R}\to S$ and a constant $b\in S$ such that

$$f(x)=a(x)+b\quad\textit{for all } x\in[0,\infty).$$

Let us note direct consequences of Theorems 1 and W.

COROLLARY 2

Assume that S is as in Theorem W. Then functions $f,g,h:[0,\infty)\to S$ satisfy (2) if and only if there exist an additive function $a:\mathbb{R}\to S$ and constants $\alpha,\beta\in S$ such that

$$g(x)=a(x)+\beta,\quad f(x)=\frac{1}{4}a(x^2)+\beta+\alpha,\quad h(x)=\frac{1}{4}a(x^2)+\alpha$$

for all $x\in[0,\infty)$.

2. Stability

In what follows, we will need the following theorem due to A. Smajdor (see [8])

THEOREM A

Let $(M,+)$ be an Abelian semigroup with zero and let $(S,+)$ be an abstract convex cone satisfying the cancellation law. Assume that a complete metric ρ such that (12) and (13) hold is given in S. If $f,g,h: M \to S$ satisfy the conditon

$$\rho(f(x+y), g(x)+h(y)) \leqslant \varepsilon$$

for some $\varepsilon \geqslant 0$ and $x,y \in M$, then there exists exactly one additive function $a: M \to S$ such that

$$\rho(a(x)+f(0), f(x)) \leqslant 4\varepsilon, \quad \rho(a(x)+g(0), g(x)) \leqslant 4\varepsilon$$

and

$$\rho(a(x)+h(0), h(x)) \leqslant 4\varepsilon$$

for all $x \in M$.

The theorem in [8] was formulated in the case when f, g, h are set-valued functions. But the closer analysis of the proof shows that this assumption is superfluous and actually the theorem in the above form was proved.

THEOREM 2

Assume that $(S,+)$ is as in Theorem A. If functions $f,g,h: [0,\infty) \to S$ satisfy the condition

$$0 \leqslant y \leqslant x \quad \Longrightarrow \quad \rho(f(x+y), g(xy)+h(x-y)) \leqslant \varepsilon \tag{14}$$

with some $\varepsilon \geqslant 0$, then there exists exactly one additive functon $a: \mathbb{R} \to S$ such that

$$\rho\left(f(x), \frac{1}{4}a(x^2)+g(0)+h(0)\right) \leqslant 17\varepsilon, \tag{15}$$

$$\rho(g(x), a(x)+g(0)) \leqslant 16\varepsilon \tag{16}$$

and

$$\rho\left(h(x), \frac{1}{4}a(x^2)+h(0)\right) \leqslant 18\varepsilon \tag{17}$$

for all $x \in [0,\infty)$.

Proof. Setting $y = 0$ in (14) we get

$$\rho(f(x), g(0)+h(x)) \leqslant \varepsilon \tag{18}$$

for all $x \in [0, \infty)$. Take $u \in [0, \infty)$ and put $x = y = \frac{u}{2}$ in (14). Then

$$\rho\left(f(u), g\left(\frac{1}{4}u^2\right) + h(0)\right) \leqslant \varepsilon. \tag{19}$$

Let us fix $u, v \in [0, \infty)$ and take $x = \sqrt{u+v}+\sqrt{v}$, $y = \sqrt{u+v}-\sqrt{v}$. Relations (12), (14), (18) and (19) yield

$$\begin{aligned}
&\rho\left(g(u+v) + g(0), g(u) + g(v)\right) \\
&\quad = \rho\left(g\left(\frac{(x+y)^2}{4}\right) + g(0) + h(0), g(xy) + g\left(\frac{(x-y)^2}{4}\right) + h(0)\right) \\
&\quad \leqslant \rho\left(g\left(\frac{(x+y)^2}{4}\right) + g(0) + h(0), f(x+y) + g(0)\right) \\
&\qquad + \rho\left(f(x+y) + g(0), g(xy) + h(x-y) + g(0)\right) \\
&\qquad + \rho\left(g(xy) + h(x-y) + g(0), g(xy) + g\left(\frac{(x-y)^2}{4}\right) + h(0)\right) \\
&\quad \leqslant \rho\left(g\left(\frac{(x+y)^2}{4}\right) + h(0), f(x+y)\right) \\
&\qquad + \rho\left(f(x+y), g(xy) + h(x-y)\right) \\
&\qquad + \rho\left(h(x-y) + g(0), f(x-y)\right) \\
&\qquad + \rho\left(f(x-y), g\left(\frac{(x-y)^2}{4}\right) + h(0)\right) \\
&\quad \leqslant 4\varepsilon.
\end{aligned}$$

In virtue of Theorem A there exists an additive function $a : \mathbb{R} \to S$ such that

$$\rho\left(a(x) + g(0), g(x)\right) \leqslant 16\varepsilon$$

for $x \in [0, \infty)$. Hence, by (12), (18) and (19)

$$\begin{aligned}
&\rho\left(f(x), \frac{1}{4}a\left(x^2\right) + g(0) + h(0)\right) \\
&\quad \leqslant \rho\left(f(x), g\left(\frac{x^2}{4}\right) + h(0)\right) + \rho\left(g\left(\frac{x^2}{4}\right) + h(0), \frac{1}{4}a\left(x^2\right) + g(0) + h(0)\right) \\
&\quad \leqslant \varepsilon + 16\varepsilon \\
&\quad = 17\varepsilon
\end{aligned}$$

and

$$\rho\left(h(x), \frac{1}{4}a\left(x^2\right) + h(0)\right) = \rho\left(h(x) + g(0), \frac{1}{4}a\left(x^2\right) + h(0) + g(0)\right)$$

$$\leqslant \rho\left(h(x)+g(0), f(x)\right) + \rho\left(f(x), \frac{1}{4}a\left(x^2\right)+g(0)+h(0)\right) \leqslant 18\varepsilon$$

for $x \in [0, \infty)$.

To prove uniqueness of a suppose that (16) holds with an additive function $a : \mathbb{R} \to S$. Since

$$a(nx) = na(x)$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$\rho\left(g(nx), na(x)+g(0)\right) \leqslant 16\varepsilon$$

for the same n and $x \in [0, \infty)$. Dividing by n and passing to the limit as $n \to \infty$, we get

$$a(x) = \lim_{n\to\infty} \frac{1}{n} g(nx), \quad \text{for all } x \in [0, \infty).$$

This shows the uniqueness of a and completes the proof.

3. Applications

Now let X be a real Banach space and let $\mathcal{K}(X)$ denote the set of all non-empty, convex, bounded and closed subsets of X. Introduce a binary operation $\overset{*}{+}$ in $\mathcal{K}(X)$ by the formula

$$A \overset{*}{+} B = \operatorname{cl}(A+B)$$

($\operatorname{cl} A$ denotes the closure of the set A). It is easy to see that $(\mathcal{K}(X), \overset{*}{+})$ is an Abelian semigroup with zero. The cancellation law for this semigroup follows from a generalization of the Rådström lemma (cf. [7], [10]). Moreover, the semigroup $\mathcal{K}(X)$ is an abstract convex cone. In particular, the multiplication $[0, \infty) \times \mathcal{K}(X) \ni (\lambda, A) \longmapsto \lambda A \in \mathcal{K}(X)$ has the following properties

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B, \quad (\lambda+\mu)A = \lambda A \overset{*}{+} \mu A$$

for all $\lambda, \mu \in [0, \infty)$ and $A, B \in \mathcal{K}(X)$.

The Hausdorff distance of A and B is defined by

$$\delta(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup\{\inf\{\|a-b\| : b \in B\} : a \in A\}$. δ is a metric on $\mathcal{K}(X)$. It is well known that $\mathcal{K}(X)$ is a complete metric space with the distance δ (cf. [3]). Furthermore

$$\delta(A \dot{+} B, C \dot{+} B) = \delta(A, C), \quad \delta(\lambda A, \lambda B) = \lambda\delta(A, B)$$

for all $\lambda \in [0, \infty)$, $A, B, C \in \mathcal{K}(X)$ (see [2] for the proof of the first equality, the second one is clear).

The following theorem follows from Corollary 2.

THEOREM 3

If X is a Banach space, then set-valued functions $F, G, H : [0, \infty) \to \mathcal{K}(X)$ satisfy the equation

$$0 \leqslant y \leqslant x \implies F(x+y) = G(xy) \dot{+} H(x-y)$$

if and only if there exists a set-valued function $A : \mathbb{R} \to \mathcal{K}(X)$ and sets $B, C \in \mathcal{K}(X)$ such that

$$F(x) = \frac{1}{4}A\left(x^2\right) \dot{+} B \dot{+} C, \quad G(x) = A(x) \dot{+} B, \quad H(x) = \frac{1}{4}A\left(x^2\right) \dot{+} C$$

for all $x \in [0, \infty)$ and

$$A(x+y) = A(x) \dot{+} A(y) \quad \textit{for all } x, y \in \mathbb{R}. \tag{20}$$

Concerning a stability problem for set-valued functions satisfying the Abel functional equation we have the following as a consequence of Theorem 2.

THEOREM 4

Let X be a Banach space. If set-valued functions $F, G, H : [0, \infty) \to \mathcal{K}(X)$ satisfy the inequality

$$0 \leqslant y \leqslant x \implies \delta(\, F(x+y), G(xy) \dot{+} H(x-y)\,) \leqslant \varepsilon,$$

where ε is some non-negative number, then there exists a set-valued function $A : \mathbb{R} \to \mathcal{K}(X)$ such that (20) *and*

$$\delta\left(F(x), \frac{1}{4}A\left(x^2\right) \dot{+} G(0) \dot{+} H(0)\right) \leqslant 17\varepsilon,$$

$$\delta\left(G(x), A(x) \dot{+} G(0)\right) \leqslant \varepsilon,$$

$$\delta\left(H(x), \frac{1}{4}A\left(x^2\right) \dot{+} H(0)\right) \leqslant 18\varepsilon$$

for all $x \in [0, \infty)$, hold.

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References

[1] J. Aczél, *The state of the second of Hilbert's fifth problem*, Bull. Amer. Math. Soc. **20** (1989), 153-163.

[2] F. S. De Blasi, *On differentiability of multifunctions*, Pacific J. Math. **66** (1976), 67-81.

[3] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*. Lecture Notes in Math. **580**, Springer-Verlag, Berlin – New York, 1977.

[4] J. K. Chung, B. R. Ebanks, C. T. Ng, P. K. Sahoo, W. B. Zeng, *On a functional equation of Abel*, Results Math. **26** (1994), 241-252.

[5] K. Lajkó, *Remarks on the Hosszú functional equation*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **12** (1987), 192-193.

[6] K. Lajkó, *The general solution of Abel-type functional equations*, Results Math. **26** (1994), 336-341.

[7] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165-169.

[8] A. Smajdor, *The stability of the Pexider equation for set-valued functions*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **13** (1993), 277-286.

[9] W. Smajdor, *Note on Jensen and Pexider functional equations*, to appear in Demonstratio Mathematica.

[10] R. Urbański, *A generalization of Minkowski-Rådström-Hörmander theorem*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **24** (1976), 709-715.

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