ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE
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REPORT OF MEETING

6th International Conference on Functional Equations and Inequalities, Muszyna-Złockie, June 1-7, 1997

The Sixth International Conference on Functional Equations and Inequalities in the series of those organized by the Institute of Mathematics of the Pedagogical University in Kraków was held from June 1 to June 7, 1997, in the hotel "Geofizyk" at Muszyna-Złockie. The preceding ICFEI took place at: Sielpia (1984), Szczawnica (1987), Koninki (1991), Krynica (1993) and Muszyna-Złockie (1995).

The Conference was opened by the address of Prof. Dr. Eugeniusz Wachnicki, the Dean of the Faculty of Mathematics, Physics and Technics of the Pedagogical University in Kraków. The first International Conference on Functional Equations held in Poland in October 1967 at Zakopane, and organized by Professors Stanisław Gołąb and Marek Kuczma was then reminded by Professor B. Choczewski.

There were 63 participants who came from: Austria (1), France (1), Germany (3), Hungary (5), Italy (4), Puerto Rico (1), Slovenia (1), the U.S.A. (1), Venezuela (1), Yugoslavia (1); and from Poland: Bielsko-Biała (2), Częstochowa (2), Gdańsk (2), Gliwice (2), Katowice (11), Kraków (19), Rzeszów (5), Zielona Góra (1). During 19 sessions 52 talks were delivered, mainly on: functional equations, their stability and iteration theory (each of the three topics also for multivalued mappings); iterative functional equations; algebra; general inequalities; applications of functional equations.

Problems-and-remarks parts of sessions attracted attention of 9 contributors. The problems were often commented and/or partially solved during the meeting.

The Organizing Committee consisted of Professors Dobiesław Brydak (chairman) and Bogdan Choczewski (member) and Dr. Zbigniew Leśniak (secretary). In a closing address of the Chairman cordial thanks were expressed to all the participants for their coming and contributing to the success of the Conference and for the friendship and hearthiness they were manifesting, also during social events. The participants highly appreciated a substantial help and assistance offered by the conference office: Miss Ewa Dudek, Miss Janina Wiercioch and Mr. Władysław Wilk. Thanks were extended to the managers of "Geofizyk" for providing very good conditions of work and rest. Professor Brydak's announcement that the 7th ICFEI is expected to be held in 1999 again in "Geofizyk" was therefore accepted with satisfaction.

The abstract of talks in alphabetical order, the problems and remarks in chronological order and the list of participants complement the report. The materials have been prepared by Dr. Jacek Chmieliński and Mr. Władysław Wilk. Some of the papers presented at the Conference which were submitted by the authors to the Journal and accepted for publication after a usual refereeing procedure are then printed as separate items.

Bogdan Choczewski

Abstracts of Talks

Roman Badora *On the separation with n-additive functions*

In the talk we are concerned with the problem of the stability of the Hyers-Ulam type for n-additive mappings defined on a product of amenable semigroups. Next, we are going to study the problem of the separation of a pair of n-subadditive and n-superadditive functions defined on a product of amenable semigroups with values in a complete vector lattice by an n-additive mapping.

Karol Baron *On regularity of orthogonally additive mappings*

Joint work with Anna Kucia.

Let E be a real inner product space of dimension at least 2 and G be an abelian group. It is known that if $f : E \longrightarrow G$ satisfies

$$f(x+y) = f(x) + f(y) \qquad \text{for all } x, y \in E \text{ with } x \perp y \tag{1}$$

then there exist exactly one additive mapping $a : \mathbb{R} \longrightarrow G$ and exactly one additive mapping $b : E \longrightarrow G$ such that

$$f(x) = a\left(\|x\|^2\right) + b(x) \qquad \text{for every } x \in E. \tag{2}$$

Assuming additionally that G is a topological group we look for conditions which imposed on a function f satisfying (1) ensure continuity of the additive mappings a and b representing f by (2).

Lech Bartłomiejczyk *Solutions with big graph of iterative functional equations*

We obtain a general result on the existence of solutions with big graph of functional equations of the form

$$g(x, \varphi(x), \varphi(f_1(x)), \varphi(f_2(x)), \ldots) = 0$$

and we apply it to some particular equations recently considered in the theory of functional equations.

Bogdan Batko *Stability of the alternative Cauchy equation on a restricted domain*

We deal with the stability (in the sense of S. M. Ulam and D. H. Hyers) of the functional equation $|f(x+y)| = |f(x) + f(y)|$. We present the following result:

Let X be a topological group and let U be a subset of $X \times X$. Suppose that $f : X \longrightarrow \mathbb{R}$ satisfies with some $\delta > 0$ the inequality

$$\big| |f(x+y)| - |f(x) + f(y)| \big| < \delta \qquad \text{for } (x,y) \in X \times X \setminus U.$$

We prove that, under some additional assumptions, there exsists a unique additive functional $\gamma : X \longrightarrow \mathbb{R}$ such that

$$|f(x) - \gamma(x)| < 40\,\delta \qquad \text{for } x \in X.$$

Zoltán Boros *Shift invariances in binocular space perception*

Various shift invariances were introduced by Heller in some recent studies in characterization of Luneburg's model of binocular vision. Assuming that the perceived distance (or direction) can be expressed in the form $f(\alpha) \pm g(\beta)$, where α and β denote the bipolar coordinates, these invariance properties imply functional equations. These equations are solved in [ABHN] under some weak regularity assumptions. It turns out that pairs of these equations imply the linearity of f and g. In the present talk pairs of these equations are investigated with $F(\alpha, \beta)$ in place of $f(\alpha) \pm g(\beta)$. For instance, supposing appropriate regularity properties of the unknown functions F, Ψ_1 and Ψ_2, the equations

$$F(\alpha + \tau, \beta) = \Psi_1(F(\alpha, \beta), \tau) \qquad \text{and} \qquad F(\alpha, \beta + \tau) = \Psi_2(F(\alpha, \beta), \tau)$$

(on specified triangle domains) imply $F(\alpha, \beta) = f(a\alpha + b\beta)$.

[ABHN] J. Aczél, Z. Boros, J. Heller, C. T. Ng, *Functional Equations in Binocular Space Perception*, J. Math. Psych. **43** (1999), 71-101.

Nicole Brillouet–Belluot *On regular solutions of some simple iterative functional equations*

In 1995, at the 33rd ISFE in Spain, we considered the following simple linear functional equation:

$$\Phi(\alpha x) - \beta\Phi(x) = F(x) \qquad (x \in K) \tag{1}$$

where $F : E \longrightarrow K$ is given, $\Phi : E \longrightarrow K$ is the unknown function, E is a normed linear space over $K = \mathbb{R}$ or $\mathbb{C}$, α and β are elements of K, $\alpha \neq 1$ is a root of 1. n denotes the smallest positive integer satisfying $\alpha^n = 1$. The interesting case concerned the case $\beta^n = 1$, $n > 2$, $K = \mathbb{C}$. Considering an appropriate partition of the space E, we obtained there the general solution of (1) by using simple direct methods.

In this talk, we first generalize our result to the following more general form of (1):

$$\Phi(\alpha x) = L(\Phi(x)) + F(x) \qquad (x \in E) \tag{2}$$

where α and n are as before, E and G are normed linear spaces over $\mathbb{C}$, $F : E \longrightarrow G$ is given, $\Phi : E \longrightarrow G$ is unknown, and $L : G \longrightarrow G$ is a given linear mapping satisfying $L^n = I$ where I denotes the identical mappings in G. When L and F are continuous, we give then its continuous solutions. When L is continuous and F is differentiable, we also explain why we obtain the differentiable solutions of (2) only in the case $E = \mathbb{C}$.

We may extend our results concerning the general and continuous solutions of (2) to the following equations (3) and (4):

$$\Phi(\alpha x) = g(\Phi(x)) \qquad (x \in E) \tag{3}$$

where $g : G \longrightarrow G$ is some given (nonlinear) mappings satisfying $g^n = I$,

$$\Phi(\alpha x) = L(x)(\Phi(x)) \qquad (x \in E) \tag{4}$$

where, for each x in E, $L(x)$ is a given mapping from G into G satisfying some further condition.

However, the method for obtaining the differentiable solutions $\Phi : \mathbb{C} \longrightarrow G$ of (2) does not apply to (3) and (4).

Jacek Chmieliński *On a conditional Wigner equation*

We prove that the class of solutions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of the *Wigner equation*

$$\left|\langle f(u)|f(v)\rangle\right| = \left|\langle u|v\rangle\right|, \qquad u, v \in \mathbb{R}^n$$

coincides with the class of solutions of the following *conditional Wigner equation*

$$\left|\langle f(u)|f(v)\rangle\right| = \left|\langle u|v\rangle\right|, \qquad u, v \in \mathbb{R}^n,\ u \neq v.$$

This is no longer true if one replaces $\mathbb{R}^n$ by an infinite dimensional inner product space as well as in the case $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $n \neq m$ ($n < m$ in fact).

Bogdan Choczewski *The first International Conference on Functional Equations in Poland: Zakopane 1967*

Thirty years ago Marek Kuczma organized the Conference in the mountain hostel "Kalatówki" (Zakopane, Tatra mountains).

The aim of this talk is to remind participants of that meeting and the topics of papers presented there.

Jacek Chudziak *Stability of the homogeneous equation*

The problem of Hyers-Ulam stability of the homogeneous mapping is considered.

Let $\mathbb{K}$ be a real or complex field, let X and Y be normed spaces over $\mathbb{K}$, and let $p, q \in \mathbb{R}$, $\varepsilon \in (0, \infty)$. We study the following inequality

$$\|f(\alpha x) - \alpha f(x)\| \leqslant \varepsilon(|\alpha|^p + \|x\|^q) \tag{$*$}$$

for all $\alpha \in \mathbb{K}$ and $x \in X$ such that $|\alpha|^p + \|x\|^q$ is defined, where $f : X \longrightarrow Y$ is any function.

The following theorem is true.

THEOREM

Let $f : X \longrightarrow Y$ satisfy $(*)$.

(i) *If $p < 1$ or $q < 0$ then*

$$f(\alpha x) = \alpha f(x) \qquad \text{for } \alpha \in \mathbb{K} \setminus \{0\},\ x \in X \setminus \{0\}.$$

Moreover, if

A) $pq < 0$

or

B) $p \in (0, 1)$

or

C) $p = 0$ *and* $q \geqslant 0$,

then

$$f(\alpha x) = \alpha f(x) \qquad \text{for } \alpha \in \mathbb{K},\ x \in X.$$

(ii) *If Y is a Banach space, then in the case $p \geqslant 1$ and $q = 0$ there exists a unique homogeneous mapping $h : X \longrightarrow Y$ such that*

$$\|f(x) - h(x)\| \leqslant \varepsilon \qquad \text{for } x \in X.$$

Moreover, we show that in the case $p \geqslant 1$ and $q > 0$ the homogeneous equation is unstable.

Krzysztof Ciepliński *On the embeddability of a homeomorphism of the unit circle in a disjoint flow*

Let $\mathcal{S}$ be the unit circle and $F : \mathcal{S} \longrightarrow \mathcal{S}$ be an orientation preserving homeomorphism with an irrational rotation number $\alpha(F)$. For this map F there exists a continuous map $\varphi : \mathcal{S} \longrightarrow \mathcal{S}$ such that $\varphi(F(z)) = e^{2\pi i\alpha(F)}\varphi(z)$ for every $z \in \mathcal{S}$.

Definition

A flow $\{F^t,\ t \in \mathbb{R}\}$ of homeomorphisms $F^t : \mathcal{S} \longrightarrow \mathcal{S}$ such that every $F^t \neq \mathrm{id}$ does not have fixed points is said to be disjoint.

Let $L := \{F^n(z),\ n \in \mathbb{Z}\}^d$.

Theorem

If $L = \mathcal{S}$, then F is embeddable in a disjoint flow.

If $L \neq \mathcal{S}$, then F is embeddable in a disjoint flow if and only if there exists a map $c : \mathbb{R} \to \mathcal{S}$ such that

$$c(t+s) = c(t)c(s), \qquad t, s \in \mathbb{R},$$

$$\operatorname{card} \operatorname{Im} c = \aleph_0,$$

$$e^{2\pi i\alpha(F)} = c(1)$$

and

$$\varphi[\mathcal{S} \setminus L] \bullet \operatorname{Im} c = \varphi[\mathcal{S} \setminus L].$$

Moreover, we give the construction of disjoint flows $\{F^t,\ t \in \mathbb{R}\}$ such that $F^1 = F$.

Ioana Cioranescu *Norm inequalities for generators of integrated semigroups*

We prove that if A is the generator of an α-times integrated semigroup of bounded linear operators on a Banach space, then the following inequality holds for $x \in D(A^2)$

$$\|Ax\| \leqslant 8M^2 \left(\frac{\alpha+1}{\alpha+2}\right) \|A^2x\|.$$

A similar inequality is obtained for generators of integrated cosine functions. These results generalize the well-known inequalities of Kallman-Rota [1] and Kurepa [2]. Application is made to the Δ operator in L^p-spaces for $p \neq 2$.

[1] R. R. Kalman, G. C. Rota, *On the inequality* $\|f'\|^2 \leqslant 4\|f\| \cdot \|f''\|$, in: Inequalities II, Academic Press, New York – London (1970), 187-192.

[2] S. Kurepa, *Remark on the Landau Inequality*, Aequationes Math. **4** (1970), 240-241.

Marek Czerni *Regular solutions of a linear functional inequality*

We consider the linear functional inequality

$$\psi[f(x)] \leqslant g(x)\psi(x) \tag{1}$$

in the case where the continuous solution of the corresponding functional equation

$$\varphi[f(x)] = g(x)\varphi(x) \tag{2}$$

depends on an arbitrary function. We assume that

i) $f : I \longrightarrow I$, $g : I \longrightarrow \mathbb{R}$ are continuous, $I = [0, a)$, $0 < a \leqslant \infty$, $g(x) < 0$, $x \in I \setminus \{0\}$ and $0 < f(x) < x$ in $I \setminus \{0\}$ and f is strictly increasing, $f(I) = I$.

ii) $\lim_{n\to\infty} G_n(x) := \prod_{i=0}^{n-1} g[f^i(x)] = 0$ almost uniformly in $I \setminus \{0\}$.

Definition

Let $x_0 \in I \setminus \{0\}$. A continuous solution $\psi : I \longrightarrow \mathbb{R}$ of (1) is called *regular* iff there exists $m \in \mathbb{N}$ such that the function $\varphi_0 : I \longrightarrow \mathbb{R}$, defined by the formula

$$\varphi_0(x) := \begin{cases} \lim_{n\to\infty} \dfrac{\psi[f^{2mn-j}(x)]}{G_{2mn-j}(x)} & \text{for } x \in [f^{2mk+j+1}(x_0), f^{2mk+j}(x_0)], \\ & \quad j = 0, \dots, 2m-1 \\ 0 & \text{for } x = 0 \end{cases}$$

is a continuous solution of (2) in I satisfying

$$(-1)^k \varphi_0(x) > 0, \qquad x \in (f^{k+1}(x_0), f^k(x_0)), \quad k \in \mathbb{Z}.$$

Let us fix $M \in \mathbb{N}$. In this talk some conditions characterizing regular solutions of the system

$$\begin{cases} \psi[f(x)] \leqslant g(x)\psi(x), \\ \psi[f^{2M}(x)] \geqslant G_{2M}(x)\psi(x), \\ \psi[f^{2M+2}(x)] \geqslant G_{2M+2}(x)\psi(x) \end{cases}$$

will be presented.

Stefan Czerwik, Krzysztof Dłutek *Pexider difference operator in L^p spaces*

Assume that $(G,+)$ is a complete measurable group with an finite measure λ and E is a Banach space. For $f,g,h : G \longrightarrow E$ define Cauchy and Pexider differences, respectively

$$C(f)(x,y) = f(x+y) - f(x) - f(y),$$

$$P(f,g,h)(x,y) = f(x+y) - g(x) - h(y), \qquad x,y \in G.$$

Let $P(f,g,h) \in L^P_{\lambda\times\lambda}(G\times G, E)$. Under some assumptions there exists exactly one additive mapping $A : G \longrightarrow E$ such that

$$f - A,\ g - A,\ h - A \in L^P_\lambda(G,E)$$

and

$$\|f - A\| \leqslant \lambda(G)^{-\frac{1}{P}} \|Cf\|,$$
$$\|g - A\| \leqslant \lambda(G)^{-\frac{1}{P}} \|Cg\|,$$
$$\|h - A\| \leqslant \lambda(G)^{-\frac{1}{P}} \|Ch\|.$$

Moreover, results concerning continuity and the existence and continuity of the inverse operator to the Pexider operator will be presented.

Eleonora Dappa *A characterization of a generalized integral mean preserving convexity*

An operator

$$f \longmapsto F(f),$$

given by the formula:

$$F(f)(x) := \begin{cases} 0 & \text{for } x = 0, \\ \frac{n}{x^n} \int\limits_0^x t^{n-1} f(t)\, dt & \text{for } x \neq 0 \end{cases}$$

transforms the class $K(b)$, $b > 0$, of all real functions, convex on $[0,b]$ and vanishing at zero, into itself. In particular, for $n := 1$ this operator reduces to the usual integral mean. More generally, given a suitable function φ (instead of "$x \longmapsto x^n$ ") one may consider an integral operator:

$$F_\varphi(f)(x) := \begin{cases} 0 & \text{for } x = 0, \\ \frac{1}{\varphi(x)} \int\limits_0^x \varphi'(t) f(t)\, dt & \text{for } x \neq 0. \end{cases}$$

This operator was considered by Gh. Toader in [1]. He has proved that the inclusion $F_\varphi(K(b)) \subset K(b)$ forces φ to be proportional to a power function.

We will prove a result similar to that due to Gh. Toader in [1] for the integral mean given by the formula:

$$F_{\varphi,\psi}(f)(x) := \begin{cases} 0 & \text{for } x = 0, \\ \frac{1}{\varphi(x)} \int\limits_0^x \psi(t) f(t)\, dt & \text{for } x \in (0, b] \end{cases}$$

for $f \in C(b)$.

[1] Gh. Toader, *On the hierarchy of convexity of functions*, Anal. Numer. Th. Approx. **15** (1986), 167-172.

Zoltán Daróczy *On a functional equation connected with an identity of Ramanujan*

Joint work with G. Hajdu.

In Ramanujan's third notebook the following entry can be found. If $ad = bc$ then for $k = 2$ and 4

$$(a+b+c)^k + (b+c+d)^k + (a-d)^k = (a+b+d)^k + (a+c+d)^k + (b-c)^k. \quad (1)$$

It would be interesting to know if Ramanujan found all the identities of type (1), if the values a, b, c, d are in a commutative ring. This means the investigation of the functional equation

$$f(a+b+c) + f(b+c+d) + f(a-d) = f(a+b+d) + f(a+c+d) + f(b-c) \quad (2)$$

for all a, b, c, d with $ad = bc$. Several results are presented on the functional equation (2).

Marta Dobosz-Smela, Marek Cezary Zdun *On involutions satisfying a system of functional equations*

THEOREM 1

Let $f_0, f_1, \ldots, f_{p-1}\colon [0,1] \longrightarrow [0,1]$ *be strictly increasing functions,* $f_0(0) = 0$, $f_{k-1}(1) = f_k(0)$, $k = 1, 2, \ldots, p-1$, $f_{p-1}(1) = 1$ *such that*

$$|f_k(x) - f_k(y)| < |x - y|, \quad x, y \in [0,1],\ x \neq y,\ k = 0, \ldots, p-1.$$

The system of the functional equations

$$\begin{cases} N^2(x) = x \\ N(f_k(x)) = f_{p-1-k}(N(x)) \end{cases} \quad x \in [0,1],\ k = 0, \ldots, p-1 \quad (1)$$

has a unique solution. This solution is strictly decreasing and continuous.

In the infinite open interval we have

Theorem 2

If $h_0, h_1, \ldots, h_{p-1} : [0,\infty) \longrightarrow [0,\infty)$ *are strictly increasing and continuous functions,*

$$\begin{aligned} h_0(0) &= 0, \\ \lim_{x\to\infty} h_{k-1}(x) &= h_k(0), \quad k = 1, 2, \ldots, p-1, \\ \lim_{x\to\infty} h_{p-1}(\infty) &= \infty \end{aligned}$$

and there exists a strictly increasing and continuous function β *of* $[0,\infty)$ *onto* $[0,1)$ *such that for* $k = 0, \ldots, p-1$ *the function* $\beta \circ h_k - \beta$ *is strictly decreasing, then the system (1) in* $(0,\infty)$ *with conditions* $N(h_k(0)) = h_{p-k}(0)$, $k = 1, \ldots, p-1$ *has the unique solution* $N : (0,\infty) \longrightarrow (0,\infty)$. *This solution is strictly decreasing and continuous.*

As an application of Theorem 2 we get

Theorem 3

If $k > 1$ *then for every increasing bijection* $f : [0,\infty) \longrightarrow \left[\frac{1}{k}, k\right)$ *such that*

$$\frac{f(x) - f(y)}{1 + f(x)f(y)} < \frac{x - y}{1 + xy} \quad \textit{for } x > y$$

there exists a unique solution of the system

$$\begin{cases} N^2(x) = x \\ N\left(\dfrac{x}{kx+1}\right) = N(x) + k \end{cases} \qquad x \in (0,\infty) \tag{2}$$

such that $N \circ f = f \circ N$ *and* $N(k) = \frac{1}{k}$. *Moreover, for* $k = 1$ *system (2) has exactly one solution* $N(x) = \frac{1}{x}$, $x \in (0,\infty)$.

For $k = 1$ we get another proof of the Volkmann's theorem.

Joachim Domsta *Extremal properties of some regular iteration groups on intervals*

We are concerned with regular iteration groups on $(0,1)$, i.e. groups of homeomorphisms $(h_t;\ t \in \mathbb{R})$, each without fixed points (excepting $h_0 = \mathrm{id}$). Moreover, for some $d > 1$, the derivative of h_t at 0 (equal to $\lim_{x\to 0+} \frac{h(x)}{x}$) is assumed to be equal to d^t, for $t \in \mathbb{R}$. Motivated by some applications, we have solved the following problem:

Problem

For a given $p > 0$, find the regular iteration group with absolutely continuous and positive velocity

$$v(x) := \frac{d}{dt} h_t|_{t=0}(x), \qquad x \in (0,1),$$

for which the ratio of L_2-norm of its derivative v' and the L_p-norm of v is minimal.

For precise setting, let AC_2 be the space of all absolutely continuous functions f on $I := [0,1]$, for which $f(0) = f(1) = 0$ and $\|f'\|_2 < \infty$. Moreover, assume the notation,

$$A_p := \frac{\sqrt{p(p+2)}B_p}{\sqrt[p]{2(p+2)}}, \qquad f_p(x) := \{Q_p(x)(1 - Q_p(x))\}^{\frac{1}{p}},$$

for $x \in I$, $p \in (0, \infty)$, where $B_p := \frac{\Gamma(\frac{2}{p})}{(\Gamma(\frac{1}{p}))^2}$ and Q_p equals the *inverse* p.d.f. for the beta distribution $B(\frac{1}{p}, \frac{1}{p})$. Additionally, let $f_\infty(x) := \min\{x, 1-x\}$, $A_\infty := \frac{1}{2}$.

THEOREM

(i) *If* $0 < p \leqslant \infty$ *and* $f \in AC_2$, *then* $\|f\|_p \leqslant A_p \cdot \|f'\|_2$.

(ii) *If* f *satisfies the equality in* (i), *then* $f = a \cdot f_p$ *with constant* a.

Thus, every solution of the above problem is given by the condition $v = a \cdot f_p|_{(0,1)}$, with arbitrary constant $a > 0$. Consequently, for given p, the groups of homeomorphism corresponding to different a differ by the multiplicative factor of the t-scale, only. Moreover, explicit formula for the group(s) is obtained in terms of a suitable integral expression of f_p. This is performed through a formula for the solution of the Schröder equation for the obtained group(s) in the standard way.

REMARK

For $p = 2$, the Theorem is known as the Wirtinger Inequality. For $1 \leqslant p \leqslant \infty$, the theorem is closely related to the solutions of basic isoperimetric problems by V. M. Tichomirov (1975). The case of $p = 1$ has been applied by A. Lenic (1994) to statistical analysis.

It should be added, that the functional Law of the Iterated Logatithm (LIL) by H. Finkelstein (1971) and the Theorem imply an extension of the Smirnov-Chung LIL with respect to the L_p-distance over all positive p.

Carlos Finol, Marek Wójtowicz *Multiplicative inequalities of real functions*

We give characterizations of the solutions of the functional inequalities $f(x^\alpha y^\beta) \leqslant f(x)^\alpha f(y)^\beta$ and $f(x^\alpha y^\beta) \geqslant f(x)^\alpha f(y)^\beta$, where f is a real function defined on an appropriate interval $I \subset \mathbb{R}^+$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$. These solutions are distinct classes of supermultiplicative functions for the first equation and submultiplicative ones for the second. Our approach yields: a unified proof of some classical inequalities, new inequalities and generalizations of known inequalities. Since many functions satisfy some kind of super- or submultiplicativity, we introduce and study the concept of local sub- and supermultiplicativity.

Margherita Fochi *Remarks on the solutions of the d'Alembert functional equation on the orthogonal vectors*

We deal with the d'Alembert conditional equation

$$(x, y) = 0 \Rightarrow f(x + y) + f(x - y) = 2f(x)f(y) \tag{1}$$

in the class of the real functionals f defined on a real inner product space X.

We can easily prove that both the functionals satisfying

$$f(x + y) + f(x - y) = 2f(x)f(y) \qquad x, y \in X \tag{2}$$

and those satisfying

$$(x, y) = 0 \Rightarrow f(x + y) = f(x)f(y) \tag{3}$$

are solutions of (1).

We have recently characterized the solutions of (2) and those of (3) in the wider class of the solutions of (1).

We purpose now to follow such studies in order to characterize other solutions of (1).

In this context we consider the solutions of (1) which are the product of a non trivial functional satisfying (1), (or (2)) and a solution of (3).

Gian Luigi Forti — Part I, **Luigi Paganoni** — Part II, *On the construction of a class of continuous and monotonic solutions of the functional equation* $\varphi[G(x, y)] = G[\varphi(x), \varphi(y)]$

We consider the system of functional equations

$$\begin{cases} \varphi[G(x, y)] = G[\varphi(x), \varphi(y)] \\ G(x, x) = x \end{cases} \qquad x, y \in \mathbb{R}, \tag{1}$$

where φ and G are continuous and G is strictly monotonic in both variables.

First we present some necessary conditions for the existence of solutions (φ, G) of (1). In particular we prove that for non trivial solutions the function φ must be strictly monotonic.

Hence we present a general procedure which permits, given φ, to construct all functions G such that (φ, G) solves (1).

Joanna Ger *Law of large numbers and a functional equation*

This is a report on a joint work with Maciej Sablik. We deal with the linear functional equation

$$g(x) = \sum_{i=1}^{r} p_i g(c_i x), \tag{E}$$

where $g : (0, \infty) \to (0, \infty)$ is unknown, $(p_1, \ldots, p_r)$ is a probability distribution, and c_i's are positive numbers. The equation (or some equivalent forms) was considered earlier under different assumptions (cf. [1], [2], [3] and [4]). Using Bernoulli's Law of Large Numbers we prove that g has to be constant provided it has a limit at one side of the domain and is bounded at the other side.

[1] J. A. Baker, *A functional equation from probability theory,* Proc. Amer. Math. Soc. **121** (1994), 767-773.

[2] J. Ger, M. Sablik, *On Jensen equation on a graph,* Zeszyty Naukowe Polit. Śląskiej, ser. Matematyka-Fizyka **68** (1993), 41-52.

[3] W. Jarczyk, *On an equation characterizing some probability distribution* in: Report of Meeting, The Thirty-fourth International Symposium on Functional Equations, June 10 to 19, 1996, Wisła-Jawornik, Poland, Aequationes Math. **53** (1997), 171.

[4] M. Laczkovich, *Non-negative measurable solutions of a difference equation,* J. London Math. Soc. **2(34)** (1986), 139-147.

Roman Ger *Orthogonalities in linear spaces and difference operators*

Quite recently C. Alsina, P. Cruells and M. S. Tomás [1], motivated by F. Suzuki's property of isosceles trapezoids, have proposed the following orthogonality relation in a real normed linear space $(X, (\|\cdot\|))$: two vectors $x, y \in X$ are T-orthogonal whenever

$$\|z - x\|^2 + \|z - y\|^2 = \|z\|^2 + \|z - x - y\|^2$$

for every $z \in X$. A natural question arises whether an analogue of T-orthogonality may be defined in any real linear space (without a norm structure). Our proposal reads as follows. Given a functional φ on a real linear linear space X we say that two vectors $x, y \in X$ are φ-orthogonal (and write $x \perp_\varphi y$) provided that $\Delta_{x,y}\varphi = 0$ ($\Delta_{h_1,\ldots,h_n}$ stands here and in the sequel for the superposition $\Delta_{h_1} \circ \ldots \circ \Delta_{h_n}$ of the usual difference operators).

We are looking for necessary and/or sufficient conditions upon the functional φ to generate a φ-orthogonality such that the pair $(X, \perp_\varphi)$ forms an orthogonality space in the sense of J. Rätz (cf. [2]). For instance, the following relationship yields a simple necessary condition of that kind:

$$\Delta_{u,v,w}\varphi = 0 \quad \text{for all } u, v, w \in X\,,$$

i.e., φ has to be a polynomial function of at most second degree. A new characterization of inner product spaces as well as a generalization of some results obtained in [1] will also be presented.

[1] C. Alsina, P. Cruells, M. S. Tomás, *Isosceles trapezoids, norms and inner products*, Arch. Math. (Brno) **35** (1999), 21-27.

[2] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. **28** (1985), 35-49.

Attila Gilányi *On solving linear functional equations with computer*

In the present talk we consider the functional equation

$$\sum_{i=0}^{n+1} f_i(p_i x + q_i y) = 0 \qquad (x, y \in L),$$

where n is a positive integer, $p_0, p_1, \ldots, p_{n+1}, q_0, q_1, \ldots, q_{n+1}$ are rational numbers, L, M are linear spaces over the rationals and $f_0, f_1, \ldots, f_{n+1} : L \longrightarrow M$ are unknown functions. Our aim is to describe a computer-program which gives the polynomial solutions of the functional equation above.

Roland Girgensohn *On the construction of Hölder and proximal subderivates*

Let f be an extended real-valued lower semicontinuous function defined on an open set $U \subset \mathbb{R}$, and let $s > 0$. Then $\xi \in \mathbb{R}$ is called an *s-Hölder* subgradient of f at $x \in U$ if $f(x)$ is finite, and for some $\sigma > 0$ and $\delta > 0$ one has

$$f(y) \geqslant f(x) + \xi \cdot (y - x) - \sigma |y - x|^{1+s} \quad \text{when } |y - x| < \delta,\ y \in U.$$

We write $\xi \in \partial_{hs} f(x)$. When $s = 1$, such a subdifferential is called a *proximal subdifferential*, denoted by $\partial_p f(x)$. These subdifferentials are investigated in connection with optimization problems.

We construct a class of Lipschitz functions on $\mathbb{R}$ such that for all $s > 0$ they are s-Hölder, and so proximally, subdifferentiable only on dyadic rationals and nowhere else.

This is joint work with J. Borwein and X. Wang.

Wojciech Jabłoński *On the stability of the homogeneous equation*

The problem of the stability of the homogeneous equation is considered.

Let f be a function defined on a subset U ($\mathbb{R}U \subset U$) of the real linear space X with the values in a sequentially complete locally convex linear topological Hausdorff space Y.

We have the following

THEOREM

Let a set $A \subset \mathbb{R} \setminus \{0\}$ contain at least one element α such that $|\alpha| \neq 1$, and let $\delta : A \longrightarrow [0, \infty)$ be a mapping. Assume that the function $k : U \longrightarrow [0, \infty)$ satisfies the condition

$$k(\alpha x) \leqslant |\alpha|^p k(x) \quad \text{for } \alpha \in \mathbb{R} \setminus \{0\},\ x \in U \text{ and a certain } p \in \mathbb{R} \setminus \{1\}.$$

Let $V \subset Y$ be a bounded set and let $f : U \longrightarrow Y$ satisfy the condition

$$\alpha^{-1} f(\alpha x) - f(x) \in \delta(\alpha) k(x) V \quad \text{for all } \alpha \in A,\ x \in U.$$

Then there exists a unique function $F : U \longrightarrow Y$ such that

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \in \langle A \setminus \{-1\} \rangle \cup \{0\},\ x \in U$$

and

$$F(x) - f(x) \in c\, k(x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)), \qquad x \in U,$$

where

$$c := \min \left(\inf_{\alpha \in A_p} \frac{\delta(\alpha)}{1 - |\alpha|^{p-1}}, \inf_{\alpha \in A^p} \frac{\delta(\alpha)}{|\alpha|^{p-1} - 1} \right),$$

$$A_p := \{\alpha \in A : \ |\alpha|^{p-1} < 1\}, \qquad A^p := \{\alpha \in A : \ |\alpha|^{p-1} > 1\},$$

(by $\inf \emptyset$ we mean $+\infty$).

Applying this Theorem we obtain some stability results for the homogeneous function.

Witold Jarczyk *Reversibility in the dimension one*

A self-mapping of a set is called reversible iff it is a composition of two involutions. The notion arose naturally in dynamical systems on the real plane. We discuss some aspects of continuous reversible self-mappings of a real interval, especially those interesting from the functional equation point of view.

Zygfryd Kominek *On sets generating all distances*

The well-known theorem of Steinhaus says that if $A \subset \mathbb{R}^p$ has a positive measure then $A - A = \{a - b;\ a, b \in A\}$ has a non-empty interior. This theorem says nothing on the "size" of the open set cointained in $A - A$. The purpose of this talk is to present sufficient condition for a set $A \subset \mathbb{R}^p$ to have $A - A = \mathbb{R}^p$. Some additional results of this type in real normed spaces will be done.

Aleksandar Krapež *Pexider functional equation*

In a paper *On the Pexider equation*, Aequationes Math. **28** (1985), 170-189, A. Krapež and M. A. Taylor investigated the Pexider equation

$$f(x * y) = g(x) \cdot h(y) \tag{P}$$

under various assumptions for (generalized) groupoid operations $*$ and $\cdot$. In particular, they generalized earlier results of Aczél, Vincze and Taylor, giving the general solution of (P) when:

1. $*$ is a semigroup and $\cdot$ a rectangular group
2. $*$ is a groupoid and $\cdot$ is a rectangular band.

We now consider a case when $\cdot$ is a semilattice operation, possibly satisfying some additional condition.

Typical result:

THEOREM

If $*$ is a quasigroup and $\cdot$ a semilattice satisfying (P), *then f is a constant function.*

Janusz Krzyszkowski *Approximately generalized convex functions*

Let F be a two-parameter family on (a, b). A function $g : (a, b) \longrightarrow \mathbb{R}$ is called ε-convex with respect to the family F $(\varepsilon > 0)$ iff for any points $a < x_1 < x_2 < b$ the unique $\varphi \in F$ determined by

$$\varphi(x_i) = g(x_i), \qquad i = 1, 2$$

satisfies the inequality

$$g(x) \leqslant \varphi(x) + \varepsilon, \qquad x \in [x_1, x_2].$$

A function $g : (a, b) \longrightarrow \mathbb{R}$ is called approximately generalized convex iff it is ε-convex with respect to the family F (for an $\varepsilon > 0$).

THEOREM

Let F be a two-parameter family on (a, b) and let $g : (a, b) \longrightarrow \mathbb{R}$ be ε-convex with respect to the family F $(\varepsilon > 0)$. Then there exists a convex function (with respect to the family F) $f : (a, b) \longrightarrow \mathbb{R}$ such that

$$|f(x) - g(x)| \leqslant \varepsilon, \qquad x \in (a, b).$$

Marian Kwapisz *On some difference-delay equations arising in mathematics of finance*

Many problems in mathematics of finance can be described by simple difference equations. In the paper we discuss a problem of capital deposits under a specific condition when the interest calculated after each basic time unit is capitalized only after the conversion period which is equal to some multiple of the basic time unit.

Under this conditions the balance of the capital after n basic time units satisfies some difference-delay equation and a specified initial condition.

We show the explicit formulas for the balance mentioned. The formulas express the balance by the given quantities: initial deposit, interest rate, and the time for which the deposit is made.

The formulas allow us to solve inverse problems consisting in finding the initial deposit, the interest rate or the lenght of the period after which a capital reaches a fixed level.

Our formulas make solving such problems very easy.

A straightforward application of backward recurrence formulas for solving the inverse problems although possible is quite troublesome.

Károly Lajkó *Functional equations in the theory of conditionally specified distributions*

Let us denote by $f_{X,Y}(x,y)$, $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ the joint, marginal and conditional densities of X and Y.

If we write the joint density as a product of a marginal and conditional density in both ways, we find the functional equation

$$f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x). \tag{1}$$

Narumi (1923) was the first to study all possible bivariate distributions with given regression functions $E(X|Y=y) = a(y)$, and $E(Y|X=x) = b(x)$ with conditionals belonging to unspecified locations families. The conditional densities were required to be of the from

$$f_{X|Y}(x|y) = g_1(x-a(y)), \qquad f_{Y|X}(y|x) = g_2(y-b(x)). \tag{2}$$

In this case we have from (1) the functional equation

$$f_Y(y)g_1(x-a(y)) = f_X(x)g_2(y-b(x)) \qquad (x,y\in\mathbb{R}).$$

Suppose instead of (2) we require that the conditional densities satisfy

$$f_{X|Y}(x|y) = g_1\left(\frac{x}{c(y)}\right), \qquad f_{Y|X}(y|x) = g_2\left(\frac{y}{d(x)}\right) \tag{3}$$

or

$$f_{X|Y}(x|y) = g_1\left(\frac{x-a(y)}{c(y)}\right), \qquad f_{Y|X}(y|x) = g_2\left(\frac{y-b(x)}{d(x)}\right) \tag{4}$$

for given functions $a(y)$, $b(x)$ and given positive functions $c(y)$ and $d(x)$. Then we obtain from (1) further functional equations. Solving this functional equations for certain choices of the functions $a(y)$, $b(x)$, (or $a(y)$, $b(x)$, $c(y)$ and $d(x)$) it is possible to determine the nature of the joint distributions associated with (2) or (3) or (4), respectively.

Zbigniew Leśniak *On embedding of free mappings in flows*

We give a construction of continuous iteration groups of a given free mapping f for which there exists a locally parallelizable family of invariant curves filling the plane. To get such groups we use the fact that the plane can be decomposed into a family of simply connected subsets such that in each of them the family of invariant curves is half-parallelizable.

Gyula Maksa *The stability of a sum form functional equation arising in information theory*

This work is joint with I. Kocsis.

In this talk the stability of a sum form equation due to M. Behara and P. Nath will be discussed. The main tools are stability results on Cauchy type functional equations on restricted domains. An unsolved problem will also be presented.

Stanisław Midura *Sur les solutions d'un système d'équations fonctionnelles*

Soit $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$ où $\mathbb{R}$ est l'ensemble des nombres réels. Dans [1] on trouve un système d'équations fonctionnelles

$$\begin{aligned} F\,[y_1x_1, y_1x_2 + y_2x_1^4 &+ 4x_1F(y_1,y_2)g(x_1,x_2) \\ &+ 6x_1^2g(y_1,y_2)F(x_1,x_2) + 3F(y_1,y_2)F^2(x_1,x_2)] \\ &= y_1F(x_1,x_2) + x_1^2F(y_1,y_2) \end{aligned} \tag{1}$$

$$\begin{aligned} g\,[y_1x_1, y_1x_2 + y_2x_1^4 &+ 4x_1F(y_1,y_2)g(x_1,x_2) \\ &+ 6x_1^2g(y_1,y_2)F(x_1,x_2) + 3F(y_1,y_2)F^2(x_1,x_2)] \\ &= y_1g(x_1,x_2) + 3x_1F(y_1,y_2)F(x_1,x_2) + x_1^3g(y_1,y_2) \end{aligned} \tag{2}$$

dans l'ensemble des fonctions $F : \mathbb{R}_0 \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

Dans [1] sont indiquees seulement deux solutions (1) et (2)

$$F(x_1,x_2) = g(x_1,x_2) = 0 \quad \text{pour } x_1 \in \mathbb{R}_0 \text{ et } x_2 \in \mathbb{R}$$

et

$$F(x_1, x_2) = g(x_1, x_2) = \frac{1}{3}(x_1 - x_1^2).$$

THEOREM

La solution générale du systeme d'équations (1) *et* (2) *dans la classe des fonctions telles, que la fonction* F *ne depénd pas de deuxiemme variable, est une famille des couples des fonctions*

$$F(x_1, x_2) = c\,(x_1^2 - x_1)$$
$$g(x_1, x_2) = (3c^2 - b)x_1^3 - 3c^2 x_1^2 + bx_1,$$

où b, c *sont des nombres réels quelconques.*

[1] S. Midura, *Sur les solutions des equations fonctionnelles qui determinent certains sous-demi-groupes à deux paramètres du groupe* L_4^1, Demonstratio Math. **23** (1990), 69-82.

Janusz Morawiec *On singular solutions of a linear functional equation of infinite order*

Given a sequence $(p_n : n \in \mathbb{N})$ of non-negative reals summing up to 1 and a sequence $(f_n : n \in \mathbb{N})$ of self-mappings of $\mathbb{R}$ we consider the functional equation

$$\varphi(x) = \sum_{n=1}^{\infty} p_n \varphi\left[f_n(x)\right]$$

and its solutions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ in the class of bounded functions, vanishing on the interval $(-\infty, 0]$ and taking value 1 on the interval $[1, +\infty)$. Some conditions under which there exists exactly one such a solution were obtained in [Bulletin of the Polish Academy of Sciences. Mathematics **43** (1995), 131-142]. Now we look for additional ones which imply singularity of this unique solution.

Zenon Moszner *L'équation de translation et l'équation de Sincov généralisée*

On donne une liaison entre l'équation de translation

$$F(F(\alpha, x), y) = F(\alpha, x \cdot y),$$

où $F : \Gamma \times S \longrightarrow \Gamma$, Γ étant un ensemble arbitraire et $(S, \cdot)$ une structure algébrique, et l'équation de Sincov généralisée

$$G(\alpha, \beta) \cdot G(\beta, \gamma) = G(\alpha, \gamma),$$

où $G : \Gamma \times \Gamma \longrightarrow S$. Cette liaison nous permet donner une forme générale de la solution de l'équation de translation dans le cas si $(S,\cdot)$ forme un groupe abélien.

Kazimierz Nikodem *Continuity of the superposition of set-valued functions*

This is a joint work with N. Merentes and S. Rivas.

Some results on continuity of the superposition $F(\cdot,\varphi(\cdot))$ are presented. In particular the following theorem is given:

THEOREM

Let Y be a topological vector space and $F:(a,b)\times\mathbb{R}\longrightarrow c(Y)$ be a midconvex set-valued function. Assume that for some continuous function $\varphi:(a,b)\longrightarrow\mathbb{R}$ which is not affine on any interval $I\subset(a,b)$, the superposition $F(\cdot,\varphi(\cdot)):(a,b)\longrightarrow c(Y)$ is bounded on an interval $(\alpha,\beta)\subset(a,b)$. Then the superposition $F(\cdot,G(\cdot))$ is l.s.c. for every l.s.c. set-valued function $G:(a,b)\longrightarrow n(\mathbb{R})$, and it is u.s.c. for every u.s.c. $G:(a,b)\longrightarrow c(\mathbb{R})$.

Jolanta Olko *Selections of an iteration semigroup of linear set-valued functions*

Let $\{F^t:\ t\geqslant 0\}$ be an iteration semigroup of linear set-valued functions defined on a cone with a finite cone-basis. Under additional assumptions there exists the unique iteration semigroup $\{f^t:\ t\geqslant 0\}$ of continuous linear selections f^t of F^t for every $t\geqslant 0$.

Zbigniew Powązka *Über die Funktionalgleichung die mit den trigonometrischen oder additiven Funktionen verbunden ist*

Betrachten wir die Funktionalgleichung

$$af(x)+bf(y)=f(ax+by)g(by-ax),\qquad x,y\in\mathbb{R},\tag{1}$$

wobei a,b die positive Zahlen sind und $f:\mathbb{R}\longrightarrow\mathbb{R}$, $g:\mathbb{R}\longrightarrow\mathbb{R}$ die suchende Funktionen bezeichnen.

Im Vortrag studieren wir die Lösungen von (1) mit

$$a=b$$

und

$$a=1\qquad\text{oder}\qquad b=1.$$

in der Klasse integrierbaren Funktionen.

Thomas Riedel *On a functional equation related to a generalization of Flett's mean value theorem*

This is joint work with M. Sablik (Katowice). Recently Davitt, Powers, Sahoo and Riedel gave the following generalization of Flett's mean value theorem:

THEOREM

Let f be a real valued function differentiable on an open interval containing $[a, b]$, then there is a point $c \in [a, b]$ such that

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a)^2. \tag{1}$$

It is easy to see that if f is a quadratic polynomial then any point c satisfies equation (1), and if f is a cubic polynomial then $c = \dfrac{a + 3b}{4}$ satisfies it. Here we change equation (1) to the Pexider type functional equation

$$f(c) - f(a) = (c - a)h(c) - \frac{1}{2}\frac{h(b) - h(a)}{b - a}(c - a)^2, \tag{2}$$

and upon setting $c = \dfrac{a + 3b}{4}$ we obtain

$$f\left(\frac{a + 3b}{4}\right) - f(a) = \frac{3}{4}(b - a)h\left(\frac{a + 3b}{4}\right) - \frac{9}{32}(b - a)(h(b) - h(a)). \tag{3}$$

We present the general solution of equation (3) as well as some generalizations.

Justyna Sikorska *Stability of a conditional Cauchy equation on balls*

Let $(X, (\cdot|\cdot))$ be a real inner product space with $\dim X = N \geqslant 2$ and let $(Y, \|\cdot\|)$ be a real Banach space. We consider the conditional functional equation

$$(x, y, x + y \in K, \quad x \perp y) \quad \text{implies} \quad \|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon, \tag{$*$}$$

where K is a ball with centre in the origin in X and $\perp$ stands for the usual orthogonality relation defined by inner product in X.

We prove that any solution $f : K \longrightarrow Y$ of $(*)$ has to be uniformly close on K to an orthogonally additive mapping.

Stanisław Siudut *Cauchy difference operator in $\mathcal{L}^p$ spaces*

Let $(G, \cdot, \lambda)$ be a measurable group with a complete, and finite measure λ, and let E be a Banach space. If $1 \leqslant p \leqslant \infty$, $f : G \longrightarrow E$ and the Cauchy difference $\mathcal{C} f(x,y) = f(xy) - f(x) - f(y)$ of f belongs to $\mathcal{L}^p_{\lambda\times\lambda}(G \times G, E)$, then there exists unique additive $A : G \longrightarrow E$ such that $f - A \in \mathcal{L}^p_\lambda(G, E)$ and

$$\|f - A\|_p \leqslant \lambda(G)^{-\frac{1}{p}} \|\mathcal{C} f\|_p.$$

Similar result we also obtain without associativity of $\cdot$ but with $f \in \mathcal{L}^p_\lambda(G, E)$ and with measurability of $\mathcal{C} f$. In this case $A = 0$ and the Cauchy difference $\mathcal{C} : \mathcal{L}^p(G, E) \longrightarrow \mathcal{L}^p(G \times G, E)$ is linear, continuous and continuously invertible on its image.

Examples of applications of these results are given.

Fulvia Skof *On asymptotically isometric operators*

By an *asymptotically isometric* (a.i.) *operator* we mean a function f from a real normed space $X = (X, \|\cdot\|)$ into another such space $E = (E, \|\cdot\|)$ satisfying the limit condition

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \longrightarrow 1 \quad \text{for } \|x - y\| \longrightarrow \infty. \tag{1}$$

We shall assume without loss of generality $f(0) = 0$.

The class of the a.i. operators includes properly the class of the δ-isometries (for some $\delta > 0$) and, obviously, that of the isometries.

In this research we look for a suitable further condition with the aim that every a.i. operator satisfying such condition can be uniformly approached by an isometry on the whole space.

It turn out that the behaviour of the difference

$$f(2x) - 2f(x)$$

plays a central role for this purpose. Some simple examples or counterexamples illustrate different situations which may occur.

Finally, on the ground of the fact that a special structure of E — like strict convexity — may influence significantly the result, we get a characterization of strictly convex spaces in this context.

Andrzej Smajdor *Concave iteration semigroups of Jensen set-valued functions*

Let S be a closed convex cone with the nonempty interior in a real separable Banach space and let $cc(S)$ denote the set of all nonempty compact convex subsets of S. We give a necessary and sufficient condition for a family $\{f^t :\ t \geqslant 0\}$ of set-valued functions $f^t : S \longrightarrow cc(S)$ to be a concave iteration semigroup of Jensen set-valued functions.

Wilhelmina Smajdor *The stability of the Abel functional equation for set-valued functions*

Let Y be a Banach space and let $cc(Y)$ denote the set of all non-empty convex compact subsets of Y. We assume that $\varepsilon \geqslant 0$ and that set-valued functions $F, G, H : [0, \infty) \longrightarrow cc(Y)$ satisfy the inequality

$$x \geqslant q \geqslant 0 \Longrightarrow d(F(x+y), G(xy) + H(x-y)) \leqslant \varepsilon,$$

where d denotes the Hausdorff metric derived by the norm in Y. Then there exists an additive set-valued function $A : [0, \infty) \longrightarrow cc(Y)$ such that

$$d\left(F(x), \frac{1}{4}A(x^2) + H(0) + G(0)\right) \leqslant 17\varepsilon,$$
$$d(G(x), A(x) + G(0)) \leqslant 16\varepsilon,$$
$$d\left(H(x), \frac{1}{4}A(x^2) + H(0)\right) \leqslant 18\varepsilon.$$

In particular, we have the following result

If $f, g, h : [0, \infty) \longrightarrow Y$ satisfy the inequality

$$x \geqslant q \geqslant 0 \Longrightarrow \|f(x+y) - g(xy) - h(x-y)\| \leqslant \varepsilon,$$

then there exists an additive function $a : [0, \infty) \longrightarrow Y$ such that

$$\left\|f(x) - \frac{1}{4}a(x^2) - h(0) - g(0)\right\| \leqslant 17\varepsilon,$$
$$\|g(x) - a(x) - g(0)\| \leqslant 16\varepsilon,$$
$$\left\|h(x) - \frac{1}{4}a(x^2) - h(0)\right\| \leqslant 18\varepsilon.$$

Joanna Szczawińska *On Nemytskii operator*

We will consider the function H generating the Nemytskii operator defined on a familly of functions of the class C^1 with values in a space of set-valued functions. If the Nemytskii operator satisfies the Lipschitz condition, the function H must be Jensen function with respect to the second variable.

Peter Šemrl *Approximate isometries on Euclidean spaces*

We prove a stability result for isometries acting on an Euclidean space without the surjectivity assumption.

Jacek Tabor *Superstability of the isometry equation in integral norm*

Let E be a Hilbert space and let $f : \mathbb{R}^n \longrightarrow E$. We define $\mathcal{I}f$, the isometry difference of f, by the formula

$$\mathcal{I}f(x,y) := |\,\|f(x)-f(y)\| - \|x-y\|\,| \quad \text{for} \quad x,y \in \mathbb{R}^n.$$

We obtain the following results.

THEOREM 1

Let $f : \mathbb{R}^n \longrightarrow E$ be arbitrary and let $1 \leqslant p \leqslant n$. If

$$\mathcal{I}f \in \mathcal{L}^p(\mathbb{R}^{2n}, E)$$

then there exists a unique isometry $I : \mathbb{R}^n \longrightarrow E$ such that

$$f \overset{a.e.}{=} \mathcal{I}.$$

THEOREM 2

For every $p > n$ there exists a Hilbert space E and a function $f : \mathbb{R}^n \longrightarrow E$ such that

$$\mathcal{I}f \in \mathcal{L}^p(\mathbb{R}^{2n}, E)$$

but there is no isometry I such that $f \overset{a.e.}{=} \mathcal{I}$.

Józef Tabor *Stability of the Cauchy equation in the class of differentiable functions*

Let X be a real normed space and Y a real Banach space. By $C_n(X,Y)$ $(n \in \mathbb{N} \cup \{\infty\})$ we denote the space of n-times continuously differentiable functions. We prove that the class C_n has the *double difference property*, that is if $f : X \longrightarrow Y$ is such a function that $Cf(x,y) := f(x+y) - f(x) - f(y) \in C_n(X \times X, Y)$, then there exists a unique additive function $a : X \longrightarrow Y$ such that $f - a \in C_n(X,Y)$ and $D(f-a)(0) = 0$. Similar result is also valid for the Jensen equation. Moreover we consider the stability problem of the Cauchy and Jensen equations in normed subspaces of C_n. These results were obtained jointly with Jacek Tabor.

Eugeniusz Wachnicki *Sur une équation intégro-fonctionnelle*

On considère l'équation

$$\begin{aligned} \frac{u(s)+u(t)}{2} &= u\left(\frac{s+t}{2}\right) F\left(\frac{t-s}{2}\right) \\ &\quad + \int_s^t u(\tau) G\left(\frac{t-s}{2} - \left|\tau - \frac{t+s}{2}\right|\right) d\tau, \end{aligned} \tag{1}$$

$s,t \in I$, $s \leqslant t$, I un intervalle ouvert, G et F sont des fonctions données, définies et continues dans un intervalle $[0,\alpha)$, où α est la demi-longuer de I $(\alpha = +\infty$ si I est non-borné).

Le résultat principial est suivant:

Ou suppose qu'il existe une fonction g définie et continue dans $[0, \alpha)$ telle que

$$g(0) = 1, \quad \lim_{r \to 0^+} \frac{g(r) - 1}{r^2} = \frac{\lambda}{2} \quad F(r) = g(r) - 2\int_0^r g(r)G(r-\rho)d\rho, \quad r \in [0, \alpha).$$

De plus, on suppose que

$$\exists\, r \in [0, \alpha)\ \exists\, M > 0\ \forall\, \rho \in (0, r)\ \ \frac{1}{\rho^2}\int_0^\rho |G(t)|d\tau < M.$$

Alors toute solution u de (1) définie et absolument localement intégrable dans I est de la forme

$$u(t) = \begin{cases} ax + b & \text{si } \lambda = 0, \\ ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}}x & \text{si } \lambda > 0, \\ a\sin(\sqrt{-\lambda}x) + b\cos(\sqrt{-\lambda}x) & \text{si } \lambda < 0. \end{cases}$$

Cela généralise les résultats obtenus dans [2] et [3].

[1] M. Kuczma, *An introduction to the theory of functional equations and inequalities*, PWN, Uniwersytet Śląski, Warszawa - Kraków - Katowice, 1985.

[2] Z. Powązka, *Sur une équation fonctionnelle associée à l'équation de Jensen*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **15** (1998), 119-128.

[3] E. Wachnicki, *Sur un développement de la valeur moyenne*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **14** (1997), 35-47.

Małgorzata Wróbel *The locally defined operators*

A characterization of the locally defined operators of the types

$$K : C^m(I) \longrightarrow C^0(I) \quad \textit{and} \quad K : C^\infty(I) \longrightarrow C^0(I),$$

where $I \subset \mathbb{R}^n$ is an interval, will be presented. Moreover, in the case when $I \subset \mathbb{R}$, under some regularity assumptions, a representation of locally defined operators $K : C^m(I) \longrightarrow C^k(I)$ is given. This is a partial answer to the problem posed in a paper [1].

[1] K. Lichawski, J. Matkowski, J. Miś, *Locally defined operators in the space of differentiable functions*, Bull. Acad. Polon. Sci. Math. **37** (1989), 315-325.

Problems and Remarks

1. Problem. S. Gołąb, *Functional equations in geometry*, Zeszyty Nauk. Uniw. Jagielloń. **14** (1970), 13-19.

Result

$$\dim E < \infty, \qquad \Omega : E \longrightarrow [0, \infty),$$
$$\Omega(x) = 0 \iff x = 0; \qquad \Omega(\lambda x) = \lambda\Omega(x), \quad \lambda > 0.$$

If

$$\varphi(x, y) := \frac{1}{2}\left[\Omega(x)^2 + \Omega(y)^2 - \Omega(y-x)^2\right]$$

satisfies

$$\varphi(x + y, z) = \varphi(x, z) + \varphi(y, z),$$

then Ω is the euclidean norm.

Problems

I. Given a norm $\Omega : E \longrightarrow [0, +\infty)$, and

$$\varphi(x, y) := \Phi(\Omega(x), \Omega(y), \Omega(y + x), \Omega(y - x)) \tag{A}$$

determine Φ such that φ be an inner product. (Distributivity, homogeneity, symmetry lead to functional equations for Φ.)

II. The same question when

$$\varphi(x, y) := \Psi(\Omega(x), \Omega(y), (\nabla\Omega)(x), (\nabla\Omega)(y)). \tag{B}$$

III. Given Φ and Ψ find Ω such that

$$(A) \iff (B).$$

Bogdan Choczewski

2. Problem. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following property:
If $a, b, c \in \mathbb{R}$ with $a + b \neq 2c$, then $f(a) + f(b) \neq 2f(c)$ and

$$f\left(\frac{2ab - ac - bc}{a + b - 2c}\right) = \frac{2f(a)f(b) - f(a)f(c) - f(b)f(c)}{f(a) + f(b) - 2f(c)}.$$

Give an "elementary" proof for that f is a nonconstant linear function.
This problem was published in the *Középiskolai Matematikai Lapok* in Hungary (April, 1997) (this is a mathematical monthly for high school students). The statement is true but I do not know "elementary" proof for it. "Elementary" means the knowledge of facts and methods in mathematics used in high schools ("Mittelschulen", "szkoły średnie").

Zoltán Daróczy

3. Remark. On extensibility of some class of homomorphisms

L. Reich posed the following question (see [1], p. 309): "When does a homomorphism Φ_s of the group $(\mathbb{R},+)$ into L_s^r (of truncated formal power series transformations in r indeterminates) have an extension $\bar{\Phi}_s$ from $(\mathbb{R},+)$ into L_{s+1}^r?"

For $r = 1$ the term "extensibility of homomorphisms" one should understand as follows. Given a homomorphism $\Phi_s = (f_1, \ldots, f_s)$ of the group $(\mathbb{R},+)$ into L_s^1. Does there exist a function f_{s+1} such that $\bar{\Phi}_s = (f_1, \ldots, f_s, f_{s+1})$ is the homomorphism from $(\mathbb{R},+)$ into L_{s+1}^1? If such a function there exists, we call $\bar{\Phi}_s$ the extension of Φ_s, whereas the homomorphism Φ_s-extensible.

Z. Moszner conjectured that $\Phi_s = (1, 0, \ldots, 0, f_{p+2}, \ldots, f_s)$ with $f_{p+2} \neq 0$ can be extended iff a function f_{s-p} is a certain polynomial in f_{p+2}.

We have proved in [2] that the mentioned condition is a necessary one for the extensibility of the homomorphism Φ_s.

[1] *Report of Meeting The Twenty-eight International Symposium on Functional Equaitons, August 23 - September 1, 1990, Graz - Mariatrost, Austria*, Aequationes Math. **41** (1991), 248-310.

[2] W. Jabłoński, *On extensibility of some class of homomorphisms.*

Wojciech Jabłoński

4. Remark. A solution to a problem posed by Stanisław Gołąb

Prof. B. Choczewski has reminded the following question of S. Gołąb (*Functional equations in geometry*, Zeszyty Nauk. Uniw. Jagielloń. **14** (1970), 13-19):

Given a norm $\|\cdot\|$ in a linear space E determine Φ such that the formula

$$\varphi(x,y) := \Phi(\|x\|, \|y\|, \|x+y\|, \|x-y\|), \quad x, y \in E, \qquad (*)$$

defines an inner product in E.

The solution reads as follows.

THEOREM

Let $(E, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\dim E \geqslant 1$, and let $\Phi : [0,\infty)^4 \longrightarrow \mathbb{R}$ be a function. Then the functional $\varphi : E^2 \longrightarrow \mathbb{K}$ given by $(*)$ yields an inner product in E if and only if for every $x, y \in E$ the quadruple $(\|x\|, \|y\|, \|x+y\|, \|x-y\|)$ belongs to P and there exists a posible number α such that

$$\Phi(s,t,u,v) = \begin{cases} \alpha(u^2 - s^2 - t^2) & \text{for } (s,t,u,v) \in P \\ \Psi(s,t,u,v) & \text{for } (s,t,u,v) \in [0,\infty)^4 \setminus P \end{cases} \qquad (**)$$

where

$$P := \{(s,t,u,v) \in [0,\infty)^4 : \ u^2+v^2 = 2s^2+2t^2\}$$

and $\Psi : P \longrightarrow \mathbb{R}$ *is an arbitrary function.*

Proof. Assume that formula $(*)$ defines an inner product in E and put

$$\|x\|_\varphi := \sqrt{\varphi(x,x)}, \quad x \in E.$$

Let $a : [0,\infty) \longrightarrow \mathbb{R}$ be defined by

$$a(s) := \Phi(s,s,2s,0), \quad s \in [0,\infty).$$

Setting $x = y$ in $(*)$ we get

$$\|x\|_\varphi^2 = \Phi(\|x\|,\|x\|,2\|x\|,0) = a(\|x\|), \quad x \in E.$$

Fix arbitrarily an $x_0 \in E$ such that $\|x_0\| = 1$. Then, for any $\lambda \in [0,\infty)$, one has

$$\lambda^2\|x_0\|_\varphi^2 = a(\lambda),$$

i.e.

$$a(\lambda) = c\lambda^2, \quad \lambda \in [0,\infty),$$

where we have put $c := \|x_0\|_\varphi^2$; clearly, $c > 0$. Consequently, we get

$$\|x\|_\varphi^2 = a(\|x\|) = c\|x\|^2, \quad x \in E,$$

whence

$$\begin{aligned}\Phi(\|x\|,\|y\|,\|x+y\|,\|x-y\|) = \varphi(x,y) &= \frac{1}{2}(\|x+y\|_\varphi^2 - \|x\|_\varphi^2 - \|y\|_\varphi^2)\\ &= \alpha(\|x+y\|^2 - \|x\|^2 - \|y\|^2), \quad x,y \in E,\end{aligned}$$

where $\alpha := \frac{1}{2}c > 0$. Since the norms $\|\cdot\|$ and $\|\cdot\|_\varphi$ are proportional and

$$\|x+y\|_\varphi^2 + \|x-y\|_\varphi^2 = 2\|x\|_\varphi^2 + 2\|y\|_\varphi^2, \quad x,y \in E,$$

we infer that the quadruple

$$(\|x\|,\|y\|,\|x+y\|,\|x-y\|) = \frac{1}{\sqrt{c}}(\|x\|_\varphi,\|y\|_\varphi,\|x+y\|_\varphi,\|x-y\|_\varphi)$$

belongs to P. This shows that

$$\Phi(s,t,u,v) = \alpha(u^2 - s^2 - t^2) \qquad \text{for } (s,t,u,v) \in P.$$

Conversely, if Φ is given by $(**)$ with some $\alpha > 0$, and for every $x,y \in E$ the quadruple $(\|x\|,\|y\|,\|x+y\|,\|x-y\|)$ belongs to P, then by the celebrated Jordan-von Neumann theorem, there exists an inner product $(\cdot|\cdot) : E^2 \longrightarrow \mathbb{K}$ such that

$$\|x\|^2 = (x|x), \quad x \in E,$$

whence

$$\varphi(x,y) = \alpha(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = 2\alpha(x|y), \quad x, y \in E,$$

which shows that φ is an inner product, too.

Roman Ger

5. Remark. (to the talk of Ms. Marta Dobosz–Smela).

Among others, the following conditional functional inequality

$$x > y \Longrightarrow \frac{f(x) - f(y)}{1 + f(x)f(y)} < \frac{x - y}{1 + xy}$$

was considered in the class of increasing bijections $f : [0, \infty) \longrightarrow [\frac{1}{k}, k)$, $k \in \mathbb{N} \setminus \{1\}$.

If we look for solutions of the unconditional *weak* inequality

$$\frac{f(x) - f(y)}{1 + f(x)f(y)} \leqslant \frac{x - y}{1 + xy},$$

then, by interchanging x and y, we immediately get the equation

$$\frac{f(x) - f(y)}{1 + f(x)f(y)} = \frac{x - y}{1 + xy}. \tag{$*$}$$

There is no "inner operation" under the f sign, i.e. we deal with a functional equation of order zero.

It is easy to show that (without any regularity assumption) the general real solution f of equation $(*)$ on an interval $I \subset \mathbb{R}$ such that $-1 \notin I \cdot I$ has one of the forms:

$$f(x) = x \qquad \text{for } x \in I,$$

or

$$f(x) = \frac{x + \alpha}{1 - \alpha x} \qquad \text{for } x \in I,$$

where $\alpha \neq 0$ is such that $\frac{1}{\alpha} \notin I$.

Roman Ger

6. Remark. It is well known that for the classical Cantor set C the algebraic sum $C + C = [0, 2]$. We can prove this fact using the representation of C in the form $C = \bigcap_{n \in \mathbb{N}} C_n$, where

$$C_1 = [0, 1] \quad \text{and} \quad C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right), \quad n \in \mathbb{N}.$$

Indeed, for every $n \in \mathbb{N}$ we have $C_n + C_n = [0, 2]$, and hence

$$C + C = \bigcap_{n\in\mathbb{N}} C_n + \bigcap_{n\in\mathbb{N}} C_n = \bigcap_{n\in\mathbb{N}} (C_n + C_n) = [0,2].$$

Now, consider the set $A = \bigcap_{n\in\mathbb{N}} A_n$, where

$$A_1 = [0,1] \quad \text{and} \quad A_{n+1} = \frac{1}{4}A_n \cup \left(\frac{3}{4} + \frac{1}{4}A_n\right), \qquad n \in \mathbb{N}.$$

From many points of view this set is similar to C, but the set $A + A$ is small — it is of the first category and its Lebesgue measure is zero. However, using the some method as above, we can show that $A + A + A = [0,3]$.

More generally, one may show (see R. Ger, *Some remarks on convex functions*, Fund. Math. **66** (1970), 255-262) that given a positive integer n there exists a set $A \subset [0,1]$ such that $B := A + A + \cdots + A$ is of measure zero and nowhere dense but $B + A = [0,1]$.

Kazimierz Nikodem

7. Remark. We show that the solution of the problem posed by B. Batko is negative. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by the formula

$$f(x) := \left(x, \operatorname{sgn} x \cdot \sqrt{|x|}\right).$$

Then one can easily check that

$$|\,\|f(x+y)\| - \|f(x) + f(y)\|\,| \leqslant 4.$$

However, f cannot be approximated by a linear function.

Jacek Tabor

8. Remark. Under the same assumptions as presented in the the talk of B. Batko we were able to prove that if the function $f : G \to \mathbb{R}$ satisfies the inequality

$$|\,|f(x+y)| - |f(x) + f(y)|\,| \leqslant \varepsilon \quad \text{for } (x,y) \in G \times G \setminus U,$$

then there exists an additive function $a : G \to \mathbb{R}$ such that

$$|f(x) - a(x)| \leqslant K\varepsilon$$

for $K = 2$. Moreover, this constant is the best possible.

Bogdan Batko and Jacek Tabor

9. Remark. Let X be a set and $\emptyset \neq D \subset X \times X \times X$. Suppose that for all $(x,y,z) \in D$ an element $[x,y,z] \in X$ corresponds, i.e., a function $[\cdot,\cdot,\cdot] : D \longrightarrow X$ is given. We discussed the problem:

Which functions $f : X \longrightarrow X$ have the following property: If $(x, y, z) \in D$ then $(f(x), f(y), f(z)) \in D$ and $f([x, y, z]) = [f(x), f(y), f(z)]$.

In this generality, at the present moment, we only know that id_X always has this property. In some particular cases however we know the complete answer. For example:

1. The nonconstant affine functions and only these have this property, whenever $X = \mathbb{R}$, $D = \{(x, y, z) \in \mathbb{R}^3 : x + y \neq 2z\}$ and
$$[x, y, z] = \frac{2xy - xz - yz}{x + y - 2z}$$
(see Problem 2 posed by Z. Daróczy).
2. $\mathrm{id}_{\mathbb{R}}$, $-\mathrm{id}_{\mathbb{R}}$ and the constant functions and only these may have this property, whenever $X = \mathbb{R}$, $* : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is any operation with right-sided identity, $D = \{(x, y, z) \in \mathbb{R}^3 : x * y \neq z\}$ and
$$[x, y, z] = \frac{1}{x * y - z}.$$

Zoltán Daróczy and Gyula Maksa

10. Remark. In connection with the talk of Thomas Riedel let us remark the following. The functional equation stemmed by Flett's Mean Value Theorem

$$f(x + 3t) - f(x) = \left[h(x + 3t) - \frac{3}{8}(h(x + 4t) - h(x)\right](3t) \tag{E}$$

may be considered for functions f and h defined in a group G and taking values in a group H or $X = \mathrm{Hom}\,(G, H)$, respectively. We get the following result

THEOREM

Let G and H be abelian groups and assume that G is divisible by 3 and 4 while H is uniquely divisible by 2, 3 and 5. Then $f : G \longrightarrow H$ and $h : G \longrightarrow \mathrm{Hom}\,(G, H)$ solve (E) if and only if

$$h(x) = A_0 + A_1(x) + A_2(x, x), \quad x \in G$$

and

$$f(x) = C + A_0(x) + \frac{1}{2}A_1(x)(x) + \frac{1}{3}A_2(x, x)(x), \quad x \in G,$$

where $C \in H$ is a constant, $A_0 \in \mathrm{Hom}\,(G, H)$, $A_1 : G \longrightarrow \mathrm{Hom}\,(G, H)$ is additive and $A_2 : G \times G \longrightarrow \mathrm{Hom}\,(G, H)$ is symmetric biadditive and

$$A_1(x)(y) = A_1(y)(x), \quad x, y \in G,$$
$$A_2(x,x)(y) = A_2(x,y)(x), \quad x, y \in G.$$

Maciej Sablik

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