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# A couple of functional equations applied to utility theory

*Dedicated to Professor Zenon Moszner*
*in high esteem and friendship*

**Abstract.** We show how characterizations of separable and additive representations of rank-dependent expected utility and of rank-independent expected utility lead to the functional equation

or to its particular case $g$ = identity, respectively. These equations are solved here without any regularity condition other than the nonnegativity of $f$. Nothing needs to be assumed about $q$. In order that $g^{-1}$ be defined, one postulates in the general case that $g$ is strictly monotonic and continuous.

## 1. Introduction

Utility theory is somewhat related to risk theory, although it considers the problems from a more subjective point of view (or, rather, examines in an objective way the subjective reactions). In analogy to the "games of fortune" models of probability theory, it considers **uncertain alternatives** as "**binary gambles**" (in general $n$-ary gambles). The notation for a binary gamble is a quadruple $(x, C; y, \overline{C})$, where $x$ and $y$ are the **consequences** of the chance **events** $C$ or $\overline{C}$ occuring, respectively. The "events" are subsets of a universal set ("universal event") $E$ and "non-$C$" is the event $\overline{C} = E \setminus C$. The consequences are not necessarily monetary or otherwise measurable but

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they form a set **partially ordered by preference**: $x \succ y$, $x \prec y$ or $x \sim y$ according whether the "gambler" prefers $x$ to $y$, $y$ to $x$ or likes them equally (the set is **partially** ordered because the "gambler" may not consider $x$ and $y$ comparable). Similarly, the set of gambles is also partially ordered and the partial order of the consequences is considered to be induced by that of the gambles.

Now the **utility function** $u$ assigns a numerical value (**utility**) to each consequence and $U$, the **expected utility**, to the gambles (often the same symbol is used because the ("first-order") gambles can be considered themselves consequences in "second-order gambles"). For a long time the "subjective expected utility" (SEU) form

$$U[(x,C;y,\overline{C})] = u(x)W(C) + u(y)W(\overline{C}) = u(x)W(C) + u(y)[1 - W(C)] \quad (1)$$

was used for expected utility and initially the values of the **weight function** $W$ have been considered probabilities. Experimental results showed, however, that $W$ is not even finitely additive. At that time the particular case $W(\overline{C}) = 1 - W(C)$ of finite (two-term) additivity was still accepted. Subsequent experiments raised doubts also in this respect, so nowadays the "rank-dependent expected utility" (RDEU)

$$U[(x,C;y,\overline{C})] = \begin{cases} u(x)W(C) + u(y)[1 - W(C)] & \text{if } x \succ y \\ u(x) & \text{if } x \sim y \\ u(x)[1 - W(\overline{C})] + u(y)W(\overline{C}) & \text{if } x \prec y \end{cases} \quad (2)$$

is applied. (In an "experiment" individuals of a homogenous group of people are asked in a laboratory setting how much would they bet on one or another outcome (consequence) of a gamble and how much gain (or loss) would they expect after several repetitions of the gamble. More exactly, they are asked whether they would prefer to take part in a given, potentially gainful gamble or rather accept an amount of money fixed in advance. The amount is then varied depending on the answer. The same applies to losses). An objective of research is to give theoretical underpinnings (axiomatic treatment) to these formulas.

We consider equation (1) and the first line of equation (2) (we look only at that, the third line is similar and the second follows from both). We see that the terms containing $x$ and $y$ are added and each such term is a product of a function of an event and of a function of a consequence. Luce (1998) presented axioms for the existence of a "separable and additive" representation

$$f(U[(x,C;y,\overline{C})]) = f[u(x)W(C)] + f[u(y)W(\overline{C})] \quad (3)$$

(cf. (1)). Here the weight function $W$ and the utility function $u$ maps the set of events resp. consequences, onto the closed interval $[0,1]$ or onto the half-open interval $[0,k[$, respectively. The relation $W(\overline{C}) = 1 - W(C)$ is replaced by a more general one: there exists a function $q : [0,1] \to [0,1]$ such that

$$W(\overline{C}) = q[W(C)] \tag{4}$$

R.D. Luce (personal communication) first asked about the general solutions $f$ and $q$ of equations (3) and (4). For this application it would be perfectly reasonable to suppose that $f$ is strictly increasing, $q$ strictly decreasing, $q$ and $f$ map $[0,1]$ resp. $[0,k[$ onto $[0,1]$ and some $[0,K[$, respectively. We will show that much weaker assumptions suffice.

Later Luce and A.A.J. Marley (Luce and Marley, 1999) recognized that it is more appropriate to study the following generalization of (the first line of) (2) and of equation (3):

$$f_0(U[(x,C;y,\overline{C})]) = f[u(x)W(C)] + f^*[u^*(y)W^*(\overline{C})] \tag{5}$$

with also $u^*$ and $W^*$ mapping onto $[0,k[$ or $[0,1]$, respectively, and with

$$W^*(\overline{C}) = q[W(C)] \tag{6}$$

holding, in place of (4). Just as (6) connects $W$ and $W^*$, they found also the connection

$$u^*(y) = g[u(y)] \tag{7}$$

between $u$ and $u^*$. Notice that also in (5) the terms containing $x$ and $y$ are added and functions of $x$ and $C$ are multiplied and so are those of $y$ and $\overline{C}$, but here not one but three unknown functions $f_0$, $f$ and $f^*$ come in between. Luce this time asked (personal communication) about the general form of $f_0$, $f, f^*, q$ and $g$.

Also for the purpose of this application it would be justified to assume about $f$ and $q$ the conditions enumerated after equations (3) and (4) but we will make do with substantially weaker assumptions here too (as we will see, we will have to know more a priori about $g$).

The present paper answers the two above questions by deriving functional equations from the representations and then outlining their solution (for details of the latter cf. Aczél, Ger and Járai, 1999 and Aczél, Maksa, Ng and Páles, 2000).

## 2. Characterization of subjective expected utility

We assume about the expected utility $U[(x,C;y,\overline{C})]$ in eq. (3) only that, when $y=x$, then $U[(x,C;x,\overline{C})]$ is **independent of** $C$ (since in the particular "gamble" $(x,C;x,\overline{C})$, whatever event happens, the consequence is always $x$). Let $E$ be the "universal", $\emptyset$ the "empty" event (set). It is natural to take $W(E)=1$ and $W(\emptyset)=0$ as their "weight". Here we suppose also $f(0)=0$. These and (3) imply

$$f(U[(x,C;x,\overline{C})])=f(U[(x,E;x,\emptyset])=f[u(x)].$$

From this equation, from (4) and from (3) again

$$f[u(x)]=f(U[(x,C;x,\overline{C})])=f[(u(x)W(C)]+f[u(x)q(W(C))]$$

follow or, with the notation $v=u(x)$, $w=W(C)$, the equation

$$f(v)=f(vw)+f[vq(w)]. \tag{8}$$

Since, as mentioned before, $W$ and $u$ map the set of events resp. consequences **onto** $[0,1]$ or $[0,k[$, respectively, equation (8) is supposed to be satisfied for **all** $w\in[0,1]$, $v\in[0,k[$. Thus our task is to **determine the unknown functions** $f$ **and** $q$ **in the functional equation** (8), **which is supposed to hold for all** $v\in[0,k[$, $w\in[0,1]$. (**The condition** $f(0)=0$, which has been stated before, **follows from** (8), so it does not have to be postulated separately).

As **only** "regularity condition" we will assume that $f$ is **nonnegative** on its domain $[0,k[$. Since (8) is supposed to hold for **all** $v\in[0,k[$, $w\in[0,1]$, **we have to have** $vq(w)\in[0,k[$, thus $q(w)\in[0,1]$ for **all** $w\in[0,1]$. From this **the properties enumerated above,** after (3) and (4), will **follow for all solutions of eq.** (8) **except two trivial ones** and **the functional equation** (8) **will be completely solved.**

In the **first phase** of our deliberations, however, we will suppose much more: that $f$ is **strictly increasing and continuously differentiable on** $[0,k[$ **and** $q$ **differentiable on** $]0,1[$. Differentiating (8) with respect to $v$ or $w$, we then get

$$f'(v)=wf'(vw)+q(w)f'[vq(w)]$$

and

$$0=vf'(vw)+vq'(w)f'[vq(w)], \tag{9}$$

respectively. Eliminating $f'[vq(w)]$ we obtain

$$q'(w)f'(v)=[wq'(w)-q(w)]f'(vw). \tag{10}$$

The expression in square brackets is **nowhere** 0 **on** $]0,1[$ because $w_0q'(w_0) - q(w_0) = 0$ in (10) would imply $q'(w_0) = 0$ (since $f$ is strictly increasing, therefore $f'$ cannot be everywhere 0). But $q'(w_0) = 0$ in (9) would bring along $f'(vw_0) = 0$ for **all** $v \in ]0,k[$ while, again, the derivative of a strictly increasing function cannot be 0 on **any** nondegenerate interval. This contradiction shows that eq. (10) can be divided by $wq'(w) - q(w)$. So we obtain the **Pexider equation**

$$f'(vw) = f'(v)h(w) \quad (v \in ]0,k[,\ w \in ]0,1[).$$

(where $h(w) := q'(w)/[wq'(w) - q(w)]$). Its general continuous solution $f'$ is given by (see e.g. Aczél, 1987, pp. 73-74)

$$f'(v) = av^c \quad (v \in ]0,k[),$$

where $a \neq 0$, since $f'(v) \equiv 0$ has been excluded. Also $c \neq -1$ because $f(v) = a \ln v + \gamma$, following from $c = -1$, is either negative or decreasing in a (right) neighbourhood of 0. Integration gives

$$f(v) = \alpha v^\beta + \gamma \quad (v \in ]0,k[).$$

Strict increasing, continuity at 0, $f(0) = 0$ and nonnegativity give $\alpha > 0$, $\beta > 0$ and $\gamma = 0$. Thus $f(v) = \alpha v^\beta$ $(\alpha > 0,\ \beta > 0)$ for all $v \in [0,k[$. Substituting this into (8) to determine $q$, **we obtain**

$$f(v) = \alpha v^\beta,\ q(w) = (1 - w^\beta)^{1/\beta} \quad (v \in [0,k[,\ w \in [0,1];\ \alpha > 0,\ \beta > 0) \quad (11)$$

**as general solution of** (8) **under the assumptions that** $f$ **is strictly increasing and continuously differentiable on** $[0,k[$ **and** $q$ **differentiable on** $]0,1[$. Notice that $q$ turned out to be continuously differentiable on $]0,1[$ (but not necessarily at 0 or 1) and that $q(0) = 1$, $q(1) = 0$ **followed**.

In the **second phase** we will make do with the substantially weaker assumption that $f$ is **continuous and strictly increasing on** $[0,k[$ (thus **positive** on $]0,k[$ since $f(0) = 0$). Interestingly, this can be reduced to the case already settled by use of the following simple idea:

We introduce the integral mean of $f$ :

$$F(u) = \frac{1}{u}\int_0^u f(v)\,dv \quad \text{if } u \neq 0 \quad \text{and} \quad F(0) = 0. \quad (12)$$

Since $f$ is continuous and positive on $]0,k[$, its integral mean $F$ exists, and is strictly increasing on $[0,k[$, positive on $]0,k[$ and continuously differentiable on $[0,k[$ if we define $F'(0) = f(0) = 0$. Surprisingly, integrating (8) with respect to $v$ and dividing by $u$ we get (with $s = vw$, $t = vq(w)$)

$$
\begin{aligned}
F(u) &= \frac{1}{u}\int_0^u f(v)\,dv \\
&= \frac{1}{u}\int_0^u f(vw)\,dv + \frac{1}{u}\int_0^u f[vq(w)]\,dv \\
&= \frac{1}{uw}\int_0^{uw} f(s)\,ds + \frac{1}{uq(w)}\int_0^{uq(w)} f(t)\,dt \\
&= F(uw) + F[uq(w)],
\end{aligned}
$$

thus, for $F$ in place of $f$, **exactly the equation** (8) (with $u > 0$ but because of $F(0) = 0$ also for $u = 0$). Since $F^{-1}$ is differentiable (because $F'(u) = f(u) \neq 0$ for $u \in\, ]0, k[\,$), therefore

$$
q(w) = \frac{1}{u}F^{-1}[F(u) - F(uw)] \tag{13}
$$

is differentiable on $]0, 1[$. Thus all our phase-one conditions are satisfied and we get, with aid of (12),

$$
F(u) = \alpha u^{\beta}, \quad f(v) = vF'(v) = \alpha\beta v^{\beta} = Av^{\beta} \quad (v \in [0, k[\,; \;\; A > 0, \;\; \beta > 0)
$$

while (13) yields the same $q$ as in (11).

Thus **the general solution of the functional equation** (8) **is given by** (11) **even if only the continuity and strictly increasing monotonicity of $f$ on $[0, k[$ is assumed**.

In what follows we just sketch (for details see Aczél, Ger and Járai, 1999) the **third (and last) phase:** getting rid of the remaining conditions (those assuming $f$ to be continuous and strictly increasing on $[0, k[\,$), leaving the **nonnegativity** of $f$ on $[0, k[$ as only assumption.

First we enumerate the trivial (degenerate) solutions:

$$
f \equiv 0 \quad \text{on } [0, k[ \quad \text{and} \quad q \text{ arbitrary (nonnegative and } \leqslant 1), \tag{14}
$$

$$
\begin{cases}
f(0) = 0, \quad f(v) = c > 0 \quad \text{for } v \in\, ]0, k[, \\
q(w) = 0 \quad (w \in\, ]0, 1]), \quad q(0) \text{ arbitrary (nonnegative, } \leqslant 1).
\end{cases} \tag{15}
$$

We **exclude** these for the time being.

We have $f(0) = 0$ and, since we supposed $f$ to be nonnegative, eq. (8) implies $f(v) \geqslant f(vw)$ for $w \in [0, 1]$, thus $f$ is **increasing (not yet strictly increasing)**. So, if there existed a $v_0 > 0$ for which $f(v_0) = 0$ then we would have $f(v) = 0$ **for all** $v \in [0, v_0]$. Let $v_1$ be the largest number such that $f(v) = 0$ for all $v \in [0, v_1[$, thus $f(v) > 0$ for all $v > v_1$. We prove that $v_1 = k$ so that in this case we get the trivial solution (14): for all other solutions

$f(v) > 0$ on $]0,k[$. Indeed, if $v_1 < k$, fix an arbitray $w_0 \in ]0,1[$ and take a $v \in ]v_1, v_1/w_0[$. Then $v > v_1$ but $vw_0 < v_1$, so $f(v) > 0$ but $f(vw_0) = 0$. Thus (8) implies $f[vq(w_0)] = f(v) > 0$ so that $vq(w_0) > v_1$, $q(w_0) > v_1/v$ and with $v \searrow v_1$ we see that $q(w_0) \geqslant 1$. But no value of $q$ is greater than 1, so $q(w_0) = 1$ and (8) becomes $f(v) = f(vw_0) + f(v)$, that is, $f(vw_0) = 0$ for all $v \in [0,k[$. Since $w_0 \in ]0,1[$ was arbitrary, we have $f = 0$ on $[0,k[$, which is (14). **For all other solutions $f$ is positive on $]0,k[$.**

With some effort one shows that in all solutions, other than the trivial (14) and (15), $q(w) > 0$ for $w \in ]0,1[$ thus, by (8), $f(v) = f(vw) + f(vq(w)] > f(vw)$ ($v \in ]0,k[$, $w \in ]0,1[$). This and $f(0) = 0 < f(v)$ ($v \in ]0,k[$) prove that **$f$ is strictly increasing on** $[0,k[$, which was one phase-two condition.

This result implies that **the function $q$ in** (8) **(strictly) decreases**, so $f$ and $q$ have right-hand limits everywhere (except, of course, $q$ at 1) and also left-hand limits everywhere but at 0. Further manipulation with (8) yields that the left- and right-hand limits of $f$ are equal on $]0,k[$ and that $\lim_{v \searrow 0} f(v) = 0 = f(0)$. Thus **$f$ is continuous on** $[0,k[$ which was the other condition in phase two.

This concludes the proof that (14), (15) **and** (11) **give all solutions of** (8) **in which $f$ in nonnegative**.

Substituting the nontrivial solution (11) into (3) we get, in view of (4), the "**rank-independent expected utility**"

$$U[(x,C;y,\overline{C})]^\beta = u(x)^\beta W(C)^\beta + u(y)^\beta[1 - W(C)^\beta], \tag{16}$$

**a slight generalization** of the representation (1) **of subjective expected utility** (SEU).

## 3. Characterization of rank-dependent expected utility

We will not need much more for solving the more general equation (5) than we did for (3). In addition to (6) and to (7), and to properties of the function $g$ therein, we will identify the expected utility $U[(x,C;x,\overline{C})]$, whose independence of $C$ we stated already, with the utility $u(x)$ of $x$:

$$U[x,C;x,\overline{C})] = u(x).$$

We will, moreover, examine the consequences of (5) more thoroughly than we did those of (3).

Again $W(\emptyset) = 0$, $W(E) = 1$, $f(0) = 0$ and here also $W^*(\emptyset) = 0$, $W^*(E) = 1$, $f^*(0) = 0$. So, when we substitute $y = x$ into

$$f_0(U[(x,C;y,\overline{C})]) = f[u(x)W(C)] + f^*[u^*(y)W^*(\overline{C})], \tag{5}$$

we get

$$f_0[u(x)] = f_0(U[(x,C;x,\overline{C})]) = f_0(U[(x,E;x,\emptyset)]) = f[u(x)] + f^*(0)$$
$$= f[u(x)].$$

Here too, $u$ (and also $u^*$) maps the set of consequences onto $[0,k[$, thus

$$f_0(v) = f(v) \quad \text{for all } v \in [0,k[ \tag{17}$$

follows. From (5) again, but this time using also (7) i.e. $u^*(x) = g[u(x)]$, we get

$$f[u(x)] = f_0(U[(x,C;x,\overline{C})]) = f_0(U[(x,\emptyset;x,E)]) = f(0) + f^*[u^*(x)]$$
$$= f^*(g[u(x)]).$$

So we have also $f(v) = f^*[g(v)]$ $(v \in [0,k[\,)$, in addition to (17). In order to obtain

$$f^*(t) = f[g^{-1}(t)] \quad (t \in [0,k[\,) \tag{18}$$

herefrom, we need that the inverse $g^{-1}$ of $g$ exist. Therefore we suppose that the mapping $g$ of $[0,k[$ **onto** $[0,k[$ is **strictly monotonic.** Thus $g$ is also **continuous**. Taking also (6) into consideration, eq. (5) becomes finally

$$f(U[(x,C;y,\overline{C})]) = f[u(x)W(C)] + f[g^{-1}(g[u(y)]q[W(C)])]. \tag{19}$$

Choosing in particular $y = x$ we get, again with $v = u(x) \in [0,k[, w = W(C) \in [0,1]$,

$$f(v) = f(vw) + f(g^{-1}[g(v)q(w)]) \quad (v \in [0,k[,\ w \in [0,1]). \tag{20}$$

This functional equation reduces, of course, to (8) if $g$ is the identity function. We need, however, new ideas to solve (20).

Since $g$ is strictly monotonic, continuous and maps $[0,k[$ onto $[0,k[$ **we have** $g(0) = 0$ **and** $g$ **is strictly increasing**. So $f(0) = 0$ is a consequence of (20) and needs not to be postulated separately here either. The domain of $f$ being $[0,k[$, we have to have $g^{-1}[g(v)q(w)] \in [0,k[$, thus $g(v)q(w) \in [0,k[$ and $q(w) \in [0,1]$ **for all** $w \in [0,1]$. Since $f$ is nonnegative, it is increasing but again this does not yet mean strict increasing.

The pair of functions $f$, $q$ in (14) and (15), each supplemented by an arbitrary strictly increasing $g$ mapping $[0,k[$ onto $[0,k[$, yield trivial solutions also of the equation (20). For all other solutions the proof that $f$ is **strictly increasing and continuous** is similar to though somewhat more difficult than in case of eq. (8). Since $f$ and $g$ strictly increase and $f(0) = g(0) = 0$, also $q(0) = 1$ and $q(1) = 0$. Ignoring these values for the time being, we sketch now the novel road map leading to all nontrivial solutions of equation (20) (for more details see Aczél. Maksa, Ng and Páles, 2000).

First we "linearize" the functional equation (20) (my coauthors coined the word "additivize") with aid of new functions defined as follows:

$$Q(s) = -\ln q(e^{-s}) \quad (s \in ]0, \infty[ =: \mathbb{R}_+) \tag{21}$$

$$F(t) = f(e^{-t}),\ G(t) = -\ln g(e^{-t}),\ H(t) = f[g^{-1}(e^{-t})] \ (t \in ]-\ln k, \infty[\,). \tag{22}$$

Since the nontrivial solutions $f$ of (20) are strictly increasing and all $g$ strictly increase, $F$ **and** $H$ **will strictly decrease,** $G$ **strictly increase**. With these functions and with $s = -\ln w$. $t = -\ln v$, eq. (20) gets "linearized":

$$F(t) - F(t+s) = H[G(t) + Q(s)] \quad (s \in ]0, \infty[,\ t \in [-\ln k, \infty[\,). \tag{23}$$

From this equation, $t \mapsto F(t) - F(t+s)$ is strictly decreasing since, on the right hand side, $H$ decreases and $G$ increases, both strictly. But then

$$F(t) - F(t+s) > F(t+s) - F[(t+s)+s] \quad (s > 0)$$

or, with $z = t + 2s$.

$$F\left(\frac{t+z}{2}\right) < \frac{F(t) + F(z)}{2} \quad (t > z),$$

that is, $f$ is strictly Jensen-(midpoint-)convex. Being also monotonic, $F$ **is strictly convex** (see e.g. Roberts and Varberg, 1973, p. 219).

This idea (of Zsolt Páles) and result proved to be very useful, since we know a lot about convex functions. For instance they are **continuous**. they **have everywhere** (except maybe at the end of their interval of convexity) **left-side and right-side derivatives.** moreover. they are **differentiable up to at most countably many points** (see Roberts and Varberg 1973. pp. 4-7.)

Of course. $G$ and $H$ being monotonic, they are almost everywhere differentiable (see e.g. Riesz and Szőkefalvi-Nagy, 1990, pp. 5-9). We write eq. (23) as

$$H^{-1}[F(t) - F(t+s)] = G(t) + Q(s) \quad (s \in ]0, \infty[,\ t \in ]-\ln k, \infty[\,) \tag{24}$$

which we can do since $H$ (and thus $H^{-1}$) is strictly monotonic (decreasing) and surjective. So also $H^{-1}$ is almost everywhere differentiable on its domain $\{F(t) - F(t+s) \mid s \in \mathbb{R}_+,\ t \in ]-\ln k. \infty[\,\}$ which, $F$ being continuous. is an interval (non-degenerate because $t \mapsto F(t) - F(t+s)$ is strictly decreasing). One can manipulate the two variables ($s$ and $t$) and the differentiability of $H$

almost everywhere and that of $F$ up to at most countably many places so that the left hand side of (24) turns out to be everywhere differentiable in $s$ and so $Q$ on the right and finally $H^{-1}$ prove to be **differentiable on their entire domain**. We do not have yet $F$ and $G$ everywhere differentiable but **the right-side derivatives $F'_+$ and $G'_+$ exist everywhere** (also the left-side derivatives): $F'_+$ exists because $F$ is convex and $G'_+$ because of (24) and the chain rule.

We differentiate eq. (24) from the right with respect to $s$ and $t$, and eliminate $(H^{-1})'[F(t) - F(t+s)]$ in order to get

$$Q'(s)[F'_+(t+s) - F'_+(t)] = G'_+(t)F'_+(t+s) \quad (s \in ]0, \infty[,\ t \in ]-\ln k, \infty[). \tag{25}$$

Since $F$ is strictly decreasing and convex, **$F'_+$ is negative everywhere** and $F'_+(t+s) - F'_+(t)$ preserves its sign (see e.g. Kuczma, 1985, p. 156 or Roberts and Varberg, 1973, p. 5). Looking at (25) more thoroughly, we see that also **$G'_+$ and $Q'$ preserve their sign**. With the notation

$$L = \frac{1}{F'_+}, \quad M = \frac{G'_+}{F'_+}, \quad N = -\frac{1}{Q'}, \tag{26}$$

we obtain the equation

$$L(t+s) = L(t) + M(t)N(s) \quad (t \in ]0, \infty[,\ s \in ]-\ln k, \infty[). \tag{27}$$

This is a particular case of several known functional equations (see e.g. Aczél and Chung, 1982 and Járai, 1984) and, since by our previous results **$M$ and $N$ preserve their sign** and from (27) itself $L$ is strictly monotonic, we get **the general solutions**

$$L(t) = Ae^{ct} + B, \quad M(t) = Ce^{ct}, \quad N(s) = \frac{A}{C}(e^{cs} - 1) \tag{28}$$

**and**

$$L(t) = At + b, \quad M(t) = Ct, \quad N(s) = \frac{A}{C}s. \tag{29}$$

$(s \in ]0, \infty[,\ t \in ]-\ln k, \infty[)$.

All we have to do now is to "put the tape in reverse" through (26), (22) and (21) and, after passing some hurdles, we arrive at the general solution of the functional equation (20) under the given conditions. (Some "hurdles" are put there just by those conditions and by the unusual domain of validity of equation (27). It is for the latter reason that we applied a theorem by Aczél and Chung (1982) which holds for arbitrary intervals). At the point where we get $F'_+$ and $G'_+$ from (26), (28) and (29), we see that these right-side derivatives are **continuous**, thus (see e.g. Kuczma, 1985, p. 156) **$F$ and $G$**

**are everywhere differentiable** and they can be determined by integration (as can $Q$). We recall also $f(0) = g(0) = q(1) = 0$ and $q(0) = 1$.

The final result is that, if $f$ **is supposed to be nonnegative and** $g$ **is continuous and strictly monotonic then all solutions of the functional equation** (20) **are given by** (14) **and** (15) **(with an arbitrary continuous and strictly increasing** $g$**) and by**

$$f(v) = \alpha v^\beta, \quad g(v) = k^{1-\beta\gamma} v^{\beta\gamma}, \quad q(w) = (1 - w^\beta)^\gamma \tag{30}$$

**and**

$$f(v) = \alpha \ln(1 + \mu v^\beta),\ g(v) = k(\mu + k^{-\beta})^\gamma (\mu + v^{-\beta})^{-\gamma},\ q(w) = (1 - w^\beta)^\gamma \tag{31}$$

($v \in [0, k[$, $w \in [0, 1]$), **where** $\alpha > 0$, $\beta > 0$, $\gamma > 0$ **and** $\mu > -k^c$.

We have herewith completely solved eq. (20) under weak conditions. Notice that the solution (11) of (8) is the particular case $\gamma = 1/\beta$ of (30); eq. (8) has no solution corresponding to (31).

With (30) the representation (19) of expected utility becomes

$$U[(x, C; y, \overline{C})]^\beta = u(x)^\beta W(C)^\beta + u(y)^\beta [1 - W(C)^\beta], \tag{32}$$

that is **the same as** (16) and a slight generalization of the first line of (2) (and as (1)).

The solution (31) yields, however, a suprising **new representation of expected utility**

$$U[(x, C; y, \overline{C})]^\beta = \frac{u(x)^\beta W(C)^\beta + u(y)^\beta [1 - W(C)^\beta] + \mu u(x)^\beta u(y)^\beta W(C)^\beta}{1 + \mu u(y)^\beta W(C)^\beta}$$

Interestingly, the particular case $\mu = 0$ is just (32), while (30) is not a particular case of (31) (although it is, in a sense, its limiting case). After several tries, Luce and Marley (1999) found a rational property which reduces the possible representations to (32). We cannot go into that here.

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