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On some inequalities related to Cauchy–Schwarz inequality

Dedicated to Professor Zenon Moszner on the occasion of his 70th anniversary

Abstract. Using some results of functional equations we solve some inequalities in real normed spaces of dimension 2 and 3 which generalize the classical Cauchy–Schwarz inequality and which characterize some special norms.

1. Introduction

The classical Cauchy–Schwarz inequality in $\mathbb{R}^n$ says that

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leqslant \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}, \tag{1}$$

for all $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$. First we observe that since the absolute value in $\mathbb{R}$ is a subadditive and multiplicative function, (1) is in fact *equivalent* to

$$\sum_{i=1}^{n} |x_i||y_i| \leqslant \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2}, \tag{2}$$

i.e., the inequality is mainly concerned with vectors with non-negative components, i.e., in $(\mathbb{R}^+)^n$.

Second, we observe that after showing (2) in $(\mathbb{R}^+)^2$, i.e.,

$$|x_1||y_1| + |x_2||y_2| \leqslant \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2} \tag{3}$$

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the inequality (2) for $n = 3$ derives from (3) by the following argument which uses (3) twice:

$$\begin{aligned}\sum_{i=1}^{3} |x_i||y_i| &\leqslant \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2} + |x_3||y_3| \\ &\leqslant \sqrt{\left(\sqrt{x_1^2 + x_2^2}\right)^2 + (|x_3|)^2} \cdot \sqrt{\left(\sqrt{y_1^2 + y_2^2}\right)^2 + (|y_3|)^2} \qquad (4) \\ &= \sqrt{\sum_{i=1}^{3} x_i^2} \cdot \sqrt{\sum_{i=1}^{3} y_i^2}.\end{aligned}$$

Thus from (3) we have (2) for all $n \geqslant 1$. In (4) note that the first and last terms correspond to the inner product of $\mathbb{R}^3$ and the norms of $\mathbb{R}^3$, respectively, while the middle term in the double inequality is precisely the inner product in $\mathbb{R}^2$ of the vectors $(\sqrt{x_1^2 + x_2^2}, |x_3|)$ and $(\sqrt{y_1^2 + y_2^2}, |y_3|)$.

All this yields to the following problem. Let $(\mathbb{R}^3, \|\ \|_3)$ and $(\mathbb{R}^2, \|\ \|_2)$ be real normed spaces and associated to these norms consider the functions $\rho_i : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}^i$, $i = 2, 3$, defined by

$$\rho_i(\vec{x}, \vec{y}) = \lim_{\lambda \to 0^+} \frac{\|\vec{x} + \lambda \vec{y}\|_i^2 - \|\vec{x}\|_i^2}{2\lambda}, \quad i = 2, 3. \qquad (5)$$

These functionals (5) generalize inner products and satisfy the generalizations of (1):

$$|\rho_i(\vec{x}, \vec{y})| \leqslant \|\vec{x}\|_i \cdot \|\vec{y}\|_i, \quad i = 2, 3. \qquad (6)$$

Indeed there exist many interesting characterizations of inner product spaces based upon special properties of these functionals (see, e.g., (Amir, 1986)). We want to study the following simultaneous inequalities for non-negative coordinates:

$$\begin{aligned}\rho_3((x_1, x_2, x_3), (y_1, y_2, y_3)) &\leqslant \rho_2((\|(x_1, x_2)\|_2, x_3), (\|(y_1, y_2)\|_2, y_3)) \\ &\leqslant \|(x_1, x_2, x_3)\|_3 \, \|(y_1, y_2, y_3)\|_3 \,. \end{aligned} \qquad (7)$$

In the usual euclidean structure (7) gives the Cauchy–Schwarz inequality (1). On the other hand (6), which is implied for $i = 3$ by (7), holds in any normed structure, so, to what extent (7) determine the structures of the spaces involved? We will answer this question under some special conditions on the norms by solving completely (7).

2. On the inequalities (7)

From now on we will assume that $(\mathbb{R}^3, \| \ \|_3)$ and $(\mathbb{R}^2, \| \ \|_2)$ are normed spaces and that $\| \ \|_3$ is *symmetric*, i.e.,

$$\|(x_1, x_2, x_3)\|_3 = \|(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})\|_3 \,, \tag{8}$$

for any permutation σ in $\{1,2,3\}$, and $\| \ \|_3$ is strictly increasing in $(\mathbb{R}^+)^3$ in the sense that for any $x_1, x_2, x_3 \geqslant 0$, $t > 0$ we have

$$\|(x_1 + t, x_2, x_3)\|_3 > \|(x_1, x_2, x_3)\|_3 \,. \tag{9}$$

We can show now the following

THEOREM

Let $(\mathbb{R}^3, \| \ \|_3)$ and $(\mathbb{R}^2, \| \ \|_2)$ be real normed spaces such that $\| \ \|_3$ is symmetric and strictly increasing in $(\mathbb{R}^+)^3$. Then (7) holds for all vectors in $(\mathbb{R}^+)^3$ if and only if there exists a real number $\alpha \geqslant 1$ such that

$$\|(x, y)\|_2 = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}} \,, \quad \|(x_1, x_2, x_3)\|_3 = (x_1^\alpha + x_2^\alpha + x_3^\alpha)^{\frac{1}{\alpha}} \,, \tag{10}$$

$$\rho_2\left((a,b),(c,d)\right) = (a^\alpha + b^\alpha)^{\frac{2-\alpha}{\alpha}} (a^{\alpha-1}c + b^{\alpha-1}d), \tag{11}$$

and

$$\begin{aligned} &\rho_3\left((x_1, x_2, x_3), (y_1, y_2, y_3)\right) \\ &\quad = (x_1^\alpha + x_2^\alpha + x_3^\alpha)^{\frac{2-\alpha}{\alpha}} (x_1^{\alpha-1} y_1 + x_2^{\alpha-1} y_2 + x_3^{\alpha-1} y_3), \end{aligned} \tag{12}$$

for all, x, y, x_1, x_2, x_3, a, b, c, d, y_1, y_2, y_3 in $\mathbb{R}^+$.

Note that when $\alpha > 1$, (7) is just Hölder's inequality.

Proof. If (7) holds then for $(x_1, x_2, x_3) = (y_1, y_2, y_3)$ in $(\mathbb{R}^+)^3$ we obtain, by using the fact that $\rho_i(\vec{a}, \vec{a}) = \|\vec{a}\|_i^2$, $i = 2, 3$,

$$\|(x_1, x_2, x_3)\|_3 = \|(\|(x_1, x_2)\|_2, x_3)\|_2 \,. \tag{13}$$

In particular $\|(1,0,0)\|_3 = \|(\|(1,0)\|_2, 0)\|_2 = \|(1,0)\|_2^2$. Let $A = \|(1,0)\|_2$. Then

$$\begin{aligned} \|(A,1,0)\|_3 &= \|(1,0,A)\|_3 = \|(\|(1,0)\|_2, A)\|_2 = \|(A,A)\|_2 \\ &= A\|(1,1)\|_2 = \|(\|(1,1)\|_2, 0)\|_2 \\ &= \|(1,1,0)\|_3 \end{aligned}$$

and since $\| \ \|_3$ is strictly increasing we deduce $A = 1$, i.e., then

$$\|(1,0)\|_2 = 1 \quad \text{and} \quad \|(1,0,0)\|_3 = 1. \tag{14}$$

Next from (8), (13) and (14) we can derive the symmetry of $\| \ \|_2$:

$$\begin{aligned} \|(x,y)\|_2 &= \|(x,y)\|_2 \cdot \|(1,0)\|_2 = \|(\|(x,y)\|_2, 0)\|_2 \\ &= \|(x,y,0)\|_3 = \|(y,x,0)\|_3 = \|(\|(y,x)\|_2, 0)\|_2 \\ &= \|(y,x)\|_2. \end{aligned} \tag{15}$$

Let us define $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $F(x,y) = \|(x,y)\|_2$. Then by (15) and (13) we have for all x, y and z in $\mathbb{R}^+$

$$\begin{aligned} F(x, F(y,z)) &= \|(x, \|(y,z)\|_2)\|_2 = \|(\|(y,z)\|_2, x)\|_2 \\ &= \|(y,z,x)\|_3 = \|(x,y,z)\|_3 \\ &= \|(\|(x,y)\|_2, z)\|_2 = F(F(x,y),z). \end{aligned} \tag{16}$$

Thus F is associative as a binary operation on $\mathbb{R}^+$. Moreover we have by (14)

$$F(x,0) = \|(x,0)\|_2 = x\|(1,0)\|_2 = x,$$

and by (15) F is commutative and since $\| \ \|_3$ is strictly increasing we have

$$\begin{aligned} F(x+t, y) &= \|(x+t, y)\|_2 = \|(\|(x+t,0)\|_2, y)\|_2 = \|(x+t, 0, y)\|_3 \\ &> \|(x,0,y)\|_3 = \|(\|(x,0)\|_2, y)\|_2 = \|(x,y)\|_2 \\ &= F(x,y). \end{aligned}$$

Therefore F is a binary operation on $\mathbb{R}^+$ which is associative, commutative, strictly increasing, 0 is a unit element and $F(x,y) = \|(x,y)\|_2$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$. By the celebrated theorem of Aczél on the associativity equation (Aczél, 1966) there exists a strictly increasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$ and F can be represented in the form

$$F(x,y) = f^{-1}(f(x) + f(y)). \tag{17}$$

But since F is homogeneous of degree one ($F(\lambda x, \lambda y) = \lambda F(x,y)$, for all $x, y, \lambda > 0$) it is well-known (Aczél, 1966) that the only homogeneous functions representable in the form (17) are given by $f(x) = Bx^\alpha$, $B, \alpha > 0$. Thus for all $x, y \geqslant 0$

$$\|(x,y)\|_2 = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}, \tag{18}$$

and since we have that $\| \ \|_2$ is a norm the unit ball must be convex and therefore we need to have (18) with $\alpha \geqslant 1$.

Moreover by (18) and (13) we obtain for all $x_1, x_2, x_3 \geqslant 0$

$$\|(x_1, x_2, x_3)\|_3 = (x_1^\alpha + x_2^\alpha + x_3^\alpha)^{\frac{1}{\alpha}}, \tag{19}$$

with $\alpha \geqslant 1$ and (10) follows, ρ_2 and ρ_3 are given in this case by (11) and (12). respectively (after (18), (19) and (15) this is a calculus exercise of computing two directional derivatives).

Conversely, if we have (10), (11) and (12) then to see (7) we observe that in this very peculiar case we have $\|(x_1, x_2, x_3)\|_3 = \|(\|(x_1, x_2)\|_2, x_3)\|_2$ and the second inequality in (7) follows from the general property (6) for ρ_2. To see the first inequality we need to check only the relation

$$\sum_{i=1}^{2} x_i^{\alpha-1} y_i \leqslant (x_1^\alpha + x_2^\alpha)^{\frac{\alpha-1}{\alpha}} (y_1^\alpha + y_2^\alpha)^{\frac{1}{\alpha}}. \tag{20}$$

If $\alpha = 1$ then (20) is a trivial inequality. If $\alpha > 1$ then (20) is just a special case of the classical Hölder inequality (with $p = \frac{\alpha}{\alpha-1} > 1$; $q = \alpha$ and $\frac{1}{p} + \frac{1}{q} = 1$). The theorem is proved.

Thus inequalities (7) for symmetric and strictly increasing norms $\| \ \|_3$ characterize the norms $\| \ \|_2$ and $\| \ \|_3$ as well as the associated functionals ρ_2 and ρ_3 on $(\mathbb{R}^+)^2$ and $(\mathbb{R}^+)^3$. If the norms satisfied additional conditions like $\|(x, -y)\|_2 = \|(x, y)\|_2$, or $\|(x, y, -z)\|_3 = \|(x, y, z)\|_3$ then we would have them determined in all $\mathbb{R}^2$ and $\mathbb{R}^3$.

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