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On decent solutions of a functional congruence

Dedicated to Professor Zenon Moszner
on his seventieth anniversary

Abstract. A Cauchy functional congruence and corresponding characters are discussed from the point of view of decency in the sense of Baker.

1. Introduction

The Cauchy functional congruence

$$\varphi(x+y)-\varphi(x)-\varphi(y)\in\mathbb{Z}\quad(x,y\in\mathbb{R})$$

was considered first by J.G. van der Corput [4]. He described all its solutions $\varphi:\mathbb{R}\to\mathbb{R}$, and he found a necessary and sufficient condition for the solutions to have the property

$$\varphi(x)-cx\in\mathbb{Z}\quad(x\in\mathbb{R})$$

with some real constant c. Few years later L. Vietoris [10] described the characters χ of $\mathbb{Q}$ that admit the representation

$$\chi(x)=\exp(2\pi i f(x))\quad(x\in\mathbb{Q})$$

with an additive $f:\mathbb{Q}\to\mathbb{R}$. (A slight incompleteness was corrected by J. Rätz [8], cf. [9; pp. 185-187] for more information). The study of the Cauchy functional congruence started again in the eighties and continued in the nineties, see, e.g., [6; Chapter 12] by D.H. Hyers, G. Isac, and Th.M. Rassias, and the recent papers [2], [3] by J. Brzdęk.

In the present paper we consider both, the Cauchy functional congruence and characters, from the point of view of decency in the sense of J.A. Baker [1].

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2. The Cauchy functional congruence

By V we denote a vector space over the field $\mathbb{Q}$. We consider the functional congruence

$$\varphi(x+y) \equiv \varphi(x) + \varphi(y) \pmod 1 \quad (x, y \in V)$$

for functions $\varphi : V \to \mathbb{R}$, but we simply write

$$\varphi(x+y) \equiv \varphi(x) + \varphi(y) \quad (x, y \in V). \tag{2.1}$$

If

$$\varphi = f + g, \quad f : V \to \mathbb{R} \text{ additive}, \ g : V \to \mathbb{Z}, \tag{2.2}$$

then φ solves (2.1). In analogy to Baker [1], we call such functions φ *decent* solutions of (2.1).

THEOREM 2.1
A solution $\varphi : V \to \mathbb{R}$ of (2.1) is decent if and only if for every $a \in V$ there is a real α such that

$$\varphi(\xi a) \equiv \xi\alpha \quad (\xi \in \mathbb{Q}). \tag{2.3}$$

Proof. From (2.2) we get immediately (2.3) with $\alpha = f(a)$. To prove the converse, we first observe that the real number α in (2.3) uniquely depends upon $a \in V$: If also

$$\varphi(\xi a) \equiv \xi\beta \quad (\xi \in \mathbb{Q}),$$

then $\xi\alpha \equiv \xi\beta$ $(\xi \in \mathbb{Q})$, i.e. $\xi(\alpha - \beta) \in \mathbb{Z}$ $(\xi \in \mathbb{Q})$, whence $\alpha = \beta$.

When setting $f(a) = \alpha$, we now have a well defined function

$$f : V \to \mathbb{R},$$

and (2.3) can be rewritten as

$$\varphi(\xi a) \equiv \xi f(a) \quad (\xi \in \mathbb{Q}, \ a \in V). \tag{2.4}$$

$\xi = 1$ shows $g = \varphi - f$ to be integer valued, and it is sufficient to prove that f is additive: For $x, y \in V$ we have by (2.4)

$$\varphi(\xi x) \equiv \xi f(x), \quad \varphi(\xi y) \equiv \xi f(y), \quad \varphi(\xi(x+y)) \equiv \xi f(x+y) \quad (\xi \in \mathbb{Q}), \tag{2.5}$$

and by (2.1) we have

$$\varphi(\xi(x+y)) \equiv \varphi(\xi x) + \varphi(\xi y) \quad (\xi \in \mathbb{Q}). \tag{2.6}$$

An obvious combination of (2.5), (2.6) gives

$$\xi f(x+y) \equiv \xi f(x) + \xi f(y) \quad (\xi \in \mathbb{Q}),$$

whence $f(x+y) = f(x) + f(y)$.

If $\alpha \in \mathbb{R}$, then $[\alpha]$ denotes its integer part and $\widetilde{\alpha}$ its fractional part, respectively. So we have the decomposition

$$\alpha = [\alpha] + \widetilde{\alpha},$$

where $[\alpha] \in \mathbb{Z}$ and $0 \leqslant \widetilde{\alpha} < 1$. For solutions $\varphi : V \to \mathbb{R}$ of (2.1) we introduce

$$\widetilde{\varphi}(x) = \widetilde{\varphi(x)} = \varphi(x) - [\varphi(x)] \quad (x \in V).$$

Then $\widetilde{\varphi} : V \to [0,1)$ also solves the functional congruence (2.1), and it is easy to see that $\widetilde{\varphi}(0) = 0$ and (for $x \in V$)

$$\widetilde{\varphi}(-x) = \begin{cases} 0, & \text{if } \widetilde{\varphi}(x) = 0, \\ 1 - \widetilde{\varphi}(x), & \text{if } 0 < \widetilde{\varphi}(x) < 1. \end{cases} \tag{2.7}$$

In the next two theorems we characterize decent solutions φ of (2.1) by using $\widetilde{\varphi}$. By $\mathbb{N}$ we mean the set of natural numbers, i.e. the set of integers $\geqslant 1$.

THEOREM 2.2
Consider a sequence of numbers

$$h_1, h_2, h_3, \ldots \in \mathbb{N}, \quad h_n \geqslant 2 \quad \text{for } n \geqslant 2.$$

and put

$$H_n = h_1 \cdots h_n \quad (n \geqslant 1).$$

Then a solution $\varphi : V \to \mathbb{R}$ *of* (2.1) *is decent if and only if for every* $a \in V$ *there is an* $N \in \mathbb{N}$ *such that*

$$\begin{gathered} \widetilde{\varphi}\left(\frac{a}{H_n}\right) = h_{n+1}\widetilde{\varphi}\left(\frac{a}{H_{n+1}}\right) \quad (n \geqslant N) \\ \text{or} \\ \widetilde{\varphi}\left(\frac{-a}{H_n}\right) = h_{n+1}\widetilde{\varphi}\left(\frac{-a}{H_{n+1}}\right) \quad (n \geqslant N). \end{gathered} \tag{2.8}$$

Proof. 1. Let φ be a decent solution of (2.1), i.e., let (2.2) be satisfied. If $a \in V$, then $f(a) \geqslant 0$ or $f(-a) \geqslant 0$, and it will be sufficient to treat the first case: We choose $N \in \mathbb{N}$ such that

$$0 \leqslant f\left(\frac{a}{H_N}\right) = \frac{1}{H_N} f(a) < 1.$$

Then we get for $n \geqslant N$ also $0 \leqslant f(\frac{a}{H_n}) < 1$, hence

$$\widetilde{\varphi}\left(\frac{a}{H_n}\right) = f\left(\frac{a}{H_n}\right) = f\left(\frac{h_{n+1}a}{H_{n+1}}\right) = h_{n+1} f\left(\frac{a}{H_{n+1}}\right) = h_{n+1}\widetilde{\varphi}\left(\frac{a}{H_{n+1}}\right)$$

$(n \geqslant N)$, and this proves (2.8).

2. Now let φ be a solution of (2.1), and suppose (2.8) to be true (with $N \in \mathbb{N}$ depending upon $a \in V$). We use Theorem 2.1 to show the decency of φ : We fix $a \in V$, and we suppose

$$\widetilde{\varphi}\left(\frac{a}{H_n}\right) = h_{n+1}\widetilde{\varphi}\left(\frac{a}{H_{n+1}}\right) \quad (n \geqslant N). \tag{2.9}$$

Our goal is to find a real number α, such that (2.3) holds. In the other case of (2.8) (i.e., (2.9) with $-a$ instead of a) we then get $\beta \in \mathbb{R}$, such that $\varphi(\xi(-a)) \equiv \xi\beta$ ($\xi \in \mathbb{Q}$); from this (2.3) follows with $\alpha = -\beta$.

3. A general observation concerning solutions $\varphi : V \to \mathbb{R}$ of (2.1) is the following one: If $x \in V$ and $k \in \mathbb{N}$ are such that $\widetilde{\varphi}(kx) = k\widetilde{\varphi}(x)$, then also $\widetilde{\varphi}(mx) = m\widetilde{\varphi}(x)$ for $m = 0, 1, \dots, k$. This is an easy consequence of (2.1) and the inequalities $0 \leqslant m\widetilde{\varphi}(x) < 1$.

4. When iterating (2.9), we get

$$\widetilde{\varphi}\left(\frac{a}{H_N}\right) = h_{N+1}\widetilde{\varphi}\left(\frac{a}{H_{N+1}}\right) = \dots = h_{N+1}\cdots h_{N+n}\widetilde{\varphi}\left(\frac{a}{H_{N+n}}\right), \tag{2.10}$$

i.e.,

$$\widetilde{\varphi}\left(h_{N+1}\cdots h_{N+n}\frac{a}{H_{N+n}}\right) = h_{N+1}\cdots h_{N+n}\widetilde{\varphi}\left(\frac{a}{H_{N+n}}\right).$$

By the foregoing remark we have

$$\widetilde{\varphi}\left(m\frac{a}{H_{N+n}}\right) = m\widetilde{\varphi}\left(\frac{a}{H_{N+n}}\right) \quad (m = 0, 1, \dots, h_{N+1}\cdots h_{N+n}), \tag{2.11}$$

and combining with (2.10) yields

$$\widetilde{\varphi}\left(\frac{m}{h_{N+1}\cdots h_{N+n}}\frac{a}{H_N}\right) = \frac{m}{h_{N+1}\cdots h_{N+n}}\widetilde{\varphi}\left(\frac{a}{H_N}\right)$$

for the numbers m occurring in (2.11). This shows

$$\widetilde{\varphi}\left(\eta\frac{a}{H_N}\right) = \eta\widetilde{\varphi}\left(\frac{a}{H_N}\right) \quad \text{for } \eta \in \Omega_N,\ 0 \leqslant \eta \leqslant 1, \tag{2.12}$$

where

$$\Omega_N = \left\{\frac{m}{h_{N+1}\cdots h_{N+n}} : m \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

Now we get

$$\widetilde{\varphi}(\varrho a) \equiv \varrho H_N\widetilde{\varphi}\left(\frac{a}{H_N}\right) \quad (\varrho \in \Omega_N) \tag{2.13}$$

in the following manner: For $\varrho \in \Omega_N$ we select M such that

$$M \in \mathbb{Z} \setminus \{0\}, \quad \frac{\varrho}{M} \in \Omega_N, \quad 0 \leqslant \frac{\varrho}{M} \leqslant 1.$$

(2.12) implies $\widetilde{\varphi}(\frac{\varrho}{M}\frac{a}{H_N}) = \frac{\varrho}{M}\widetilde{\varphi}(\frac{a}{H_N})$. We multiply by MH_N and use (2.1) to get the congruence in (2.13). When setting $\alpha = H_N\widetilde{\varphi}(\frac{a}{H_N})$, (2.13) can be rewritten as

$$\widetilde{\varphi}(\varrho a) \equiv \varrho\alpha \quad (\varrho \in \Omega_N). \tag{2.14}$$

5. For $\xi \in \mathbb{Q}$, consider the element ξa in V. In analogy to (2.14), there is some $\beta \in \mathbb{R}$ and some $M \in \mathbb{N}$, such that

$$\widetilde{\varphi}(\varrho\xi a) \equiv \varrho\beta \quad (\varrho \in \Omega_M). \tag{2.15}$$

For $L = \max\{M, N\}$ we have $\Omega_L = \Omega_M \cap \Omega_N$, and (2.14), (2.15) imply

$$\widetilde{\varphi}(\varrho a) \equiv \varrho\alpha, \quad \widetilde{\varphi}(\varrho\xi a) \equiv \varrho\beta \qquad (\varrho \in \Omega_L). \tag{2.16}$$

We suppose

$$\xi = \frac{p}{q}, \quad \text{where } p \in \mathbb{Z},\ q \in \mathbb{N}.$$

Then (2.1), (2.16) imply for $\varrho \in \Omega_L$

$$p\varrho\alpha \equiv p\widetilde{\varphi}(\varrho a) \equiv \widetilde{\varphi}(p\varrho a) = \widetilde{\varphi}(q\varrho\xi a) \equiv q\widetilde{\varphi}(\varrho\xi a) \equiv q\varrho\beta,$$

hence

$$(p\alpha - q\beta)\varrho \in \mathbb{Z} \quad (\varrho \in \Omega_L).$$

The set Ω_L being dense in $\mathbb{R}$, this yields $p\alpha = q\beta$, hence $\beta = \xi\alpha$. Now by (2.15) (with $\varrho = 1$) we get $\widetilde{\varphi}(\xi a) \equiv \xi\alpha$. The number $\xi \in \mathbb{Q}$ being arbitrary, (2.3) is established.

REMARK 2.1

$\widetilde{\varphi} : V \to [0, 1)$ being a solution of (2.1), for $x \in V$ and $m \in \mathbb{N}$ equality $\widetilde{\varphi}(mx) = m\widetilde{\varphi}(x)$ is equivalent to $\widetilde{\varphi}(mx) \geqslant m\widetilde{\varphi}(x)$. Therefore (2.8) also can be formulated as

$$\widetilde{\varphi}\left(\frac{a}{H_n}\right) \geqslant h_{n+1}\widetilde{\varphi}\left(\frac{a}{H_{n+1}}\right) \quad (n \geqslant N)$$

or

$$\widetilde{\varphi}\left(\frac{-a}{H_n}\right) \geqslant h_{n+1}\widetilde{\varphi}\left(\frac{-a}{H_{n+1}}\right) \quad (n \geqslant N).$$

REMARK 2.2

The case $h_n = n$ (i.e., $H_n = n!$) of Theorem 2.2 goes back to Sablik [9]. Another important case is $h_n = 2$ (i.e., $H_n = 2^n$); it will be used in the proof of the next theorem.

THEOREM 2.3

Let $\varphi : V \to \mathbb{R}$ be a solution of (2.1). The decency of φ is equivalent to each of the following conditions:

(A) *For every $a \in V$ there is an $\varepsilon > 0$ such that*

$$\widetilde{\varphi}(\xi a) < \frac{1}{2} \quad (\xi \in \mathbb{Q},\ 0 \leqslant \xi \leqslant \varepsilon) \quad \text{or} \quad \widetilde{\varphi}(-\xi a) < \frac{1}{2} \quad (\xi \in \mathbb{Q},\ 0 \leqslant \xi \leqslant \varepsilon).$$

(B) *For every $a \in V$ there is an $N \in \mathbb{N}$ such that*

$$\widetilde{\varphi}\left(\frac{a}{2^n}\right) < \frac{1}{2} \quad (n \in \mathbb{N},\ n \geqslant N) \quad \text{or} \quad \widetilde{\varphi}\left(\frac{-a}{2^n}\right) < \frac{1}{2}\ (n \in \mathbb{N},\ n \geqslant N).$$

(C) *For every $a \in V$ there is an $N \in \mathbb{N}$ such that*

$$\min\left\{\widetilde{\varphi}\left(\frac{a}{2^n}\right), \widetilde{\varphi}\left(\frac{-a}{2^n}\right)\right\} < \frac{1}{3} \quad (n \in \mathbb{N},\ n \geqslant N).$$

(D) *If $a \in V$, then*

$$\widetilde{\varphi}\left(\frac{a}{n}\right) = O\left(\frac{1}{n}\right) \quad \text{or} \quad \widetilde{\varphi}\left(\frac{-a}{n}\right) = O\left(\frac{1}{n}\right).$$

(E) *If $a \in V$, then*

$$\min\left\{\widetilde{\varphi}\left(\frac{a}{n}\right), \widetilde{\varphi}\left(\frac{-a}{n}\right)\right\} = O\left(\frac{1}{n}\right).$$

Proof. 1. It is sufficient to show

$$(2.2) \Longrightarrow (A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (2.2) \Longrightarrow (D) \Longrightarrow (E) \Longrightarrow (C).$$

The steps

$$(A) \Longrightarrow (B), \quad (D) \Longrightarrow (E) \Longrightarrow (C)$$

are trivial, the remaining conclusions

$$(2.2) \Longrightarrow (A), \quad (2.2) \Longrightarrow (D), \quad (B) \Longrightarrow (C), \quad (C) \Longrightarrow (2.2)$$

will be verified now.

2. (2.2) $\Longrightarrow$ (A), (D). Suppose (2.2) to hold, and consider $a \in V$. Then $f(a) \geqslant 0$ or $f(-a) \geqslant 0$, and we only treat the first case. The function f being additive, we have

$$0 \leqslant f(\xi a) = \xi f(a) < \frac{1}{2} \quad \text{for } \xi \in \mathbb{Q},\ 0 \leqslant \xi \leqslant \frac{1}{2f(a)+1},$$

therefore we get condition (A) with $\varepsilon = \frac{1}{2f(a)+1}$. For $n \in \mathbb{N}$ we have $f\left(\frac{a}{n}\right) = \frac{f(a)}{n}$, hence

$$0 \leqslant \widetilde{\varphi}\left(\frac{a}{n}\right) \leqslant \frac{f(a)}{n},$$

and this proves (D).

3. (B) $\Longrightarrow$ (C). Suppose (B) to hold, and consider $a \in V$. Suppose

$$\widetilde{\varphi}\left(\frac{a}{2^n}\right) < \frac{1}{2} \quad (n \in \mathbb{N},\ n \geqslant N) \tag{2.17}$$

(the other case can be treated similarly). Then (2.1), (2.17) imply for $n \geqslant N$

$$\widetilde{\varphi}\left(\frac{a}{2^n}\right) \equiv \widetilde{\varphi}\left(\frac{a}{2^{n+1}}\right) + \widetilde{\varphi}\left(\frac{a}{2^{n+1}}\right) < 1,$$

and therefore the congruence is in fact an equality, which gives

$$\widetilde{\varphi}\left(\frac{a}{2^{n+1}}\right) = \frac{1}{2}\widetilde{\varphi}\left(\frac{a}{2^n}\right) < \frac{1}{4} \quad (n \geqslant N).$$

Replacing $N+1$ by N yields (C).

4. (C) $\Longrightarrow$ (2.2). Let (C) hold. For $a \in V$ we shall show that

$$\widetilde{\varphi}\left(\frac{a}{2^n}\right) = 2\widetilde{\varphi}\left(\frac{a}{2^{n+1}}\right) \quad (n \geqslant N) \tag{2.18}$$

or

$$\widetilde{\varphi}\left(\frac{-a}{2^n}\right) = 2\widetilde{\varphi}\left(\frac{-a}{2^{n+1}}\right) \quad (n \geqslant N). \tag{2.19}$$

Then we can apply Theorem 2.2 with $h_n = 2$ $(n \in \mathbb{N})$ to deduce (2.2). For $a \in V$ we have by (C)

$$\widetilde{\varphi}\left(\frac{a}{2^N}\right) < \frac{1}{3} \tag{2.20}$$

or $\widetilde{\varphi}\left(\frac{-a}{2^N}\right) < \frac{1}{3}$. We shall show that the first case implies (2.18); the second case then implies (2.19). So we start with (2.20), and we like to deduce

$$\widetilde{\varphi}\left(\frac{a}{2^N}\right) = 2\widetilde{\varphi}\left(\frac{a}{2^{N+1}}\right); \tag{2.21}$$

after this, (2.18) follows recursively. From (C) we get

$$\min\left\{\widetilde{\varphi}\left(\frac{a}{2^{N+1}}\right), \widetilde{\varphi}\left(\frac{-a}{2^{N+1}}\right)\right\} < \frac{1}{3}.$$

If $\widetilde{\varphi}\left(\frac{a}{2^{N+1}}\right) < \frac{1}{3}$, then (2.21) follows from $\widetilde{\varphi}\left(\frac{a}{2^N}\right) \equiv 2\widetilde{\varphi}\left(\frac{a}{2^{N+1}}\right)$. If $\widetilde{\varphi}\left(\frac{-a}{2^{N+1}}\right) < \frac{1}{3}$, then we have (cf. (2.20))

$$0 = \widetilde{\varphi}(0) \equiv \widetilde{\varphi}\left(\frac{a}{2^N}\right) + 2\widetilde{\varphi}\left(\frac{-a}{2^{N+1}}\right) < 1,$$

which yields $\widetilde{\varphi}\left(\frac{a}{2^N}\right) = \widetilde{\varphi}\left(\frac{-a}{2^{N+1}}\right) = 0$. But then also $\widetilde{\varphi}\left(\frac{a}{2^{N+1}}\right) = 0$ (cf. (2.7)), and again we have (2.21).

3. Decent characters

J.A. Baker [1] called *decent* a character χ of a group G, if there exists an additive function $f : G \to \mathbb{R}$ such that

$$\chi(x) = \exp(2\pi i f(x)) \quad (x \in G).$$

He also noticed that a function φ satisfies the congruence (2.1) if and only if the mapping $\chi_\varphi : x \to \exp(2\pi i\varphi(x))$ is a character. It is also obvious that a solution φ of (2.1) is decent if and only if the corresponding character χ_φ is decent.

We are going now to express the results of the previous section in the language of characters. Let us first focus on rational characters. Put $\mathbb{Q}_+ := \mathbb{Q} \cap [0, \infty)$, let T stand for the unit circle, and start with the following.

LEMMA 3.1

Let $(S, +)$ be a subsemigroup of $(\mathbb{Q}_+, +)$ such that $0 \in S$ and S is dense in $\mathbb{Q}_+$. Then a character $\chi : S \to T$ is continuous at 0 if and only if it admits the form

$$\chi(x) = \exp(2\pi i c x) \quad (x \in S) \tag{3.1}$$

where c is a real constant.

Proof. If χ has the form (3.1) then it is obviously continuous. To prove the converse, similarly as J.A. Baker did in [1; Lemma 2] (cf. also [5; (23.30)]) we observe that χ is uniformly continuous and extend it to a continuous function $\chi_1 : \mathbb{Q}_+ \to T$. It is clear that χ_1 is a character. Setting $\chi_2(x) = \chi_1(|x|)^{\operatorname{sgn} x}$ for $x \in \mathbb{Q}$ we extend χ_1 to a continuous character of $\mathbb{Q}$. By Lemma 3 in [1] (cf. also [5; (25.26)]) we get our assertion.

To prove the forthcoming Proposition we will use the following result (cf. [9; Proposition 1]).

LEMMA 3.2

Let $(S, +)$ be a topological semigroup with neutral element e. If $\chi : S \to T$ is a character and $\{1\}$ is the only semigroup contained in $\operatorname{cl} \chi(U)$ for a neighbourhood U of e then χ is continuous at e.

We have

PROPOSITION 3.1

Let $(S, +)$ be a subsemigroup of $(\mathbb{Q}_+, +)$ such that S is dense in $\mathbb{Q}_+$ and $0 \in S$. A character $\chi : \mathbb{Q} \to T$ is decent if and only if for every $\xi \in \mathbb{Q}_+$ there exists an $\varepsilon > 0$ such that the only semigroup contained in $\operatorname{cl} \chi(\xi(S \cap [0, \varepsilon)))$ is $\{1\}$.

Proof. The "only if" part easily follows from [1; Corollary in Section 4 on p. 323 and Lemma 3]. To prove sufficiency fix a $\xi \in \mathbb{Q}_+$ and consider the character $\chi_\xi : S \to T$ given by

$$\chi_\xi(x) = \chi(\xi x).$$

From our assumptions and Lemma 3.2 we infer that χ_ξ is continuous at 0. Hence by Lemma 3.1 there exists a constant $c(\xi) \in \mathbb{R}$ such that

$$\chi(\xi x) = \exp(2\pi i c(\xi)x) \quad (\xi \in \mathbb{Q}_+,\ x \in S).$$

Define $f : \mathbb{Q} \to \mathbb{R}$ by $f(\xi) = \operatorname{sgn}(\xi)c(|\xi|)$. It is easy to check that we have indeed

$$\chi(\xi x) = \exp(2\pi i f(\xi)x) \quad (\xi \in \mathbb{Q},\ x \in S). \tag{3.2}$$

Fix some rationals ξ and η. Then by (3.2) we obtain for every $x \in S$

$$\exp(2\pi i(f(\xi+\eta) - f(\xi) - f(\eta))x) = \chi((\xi+\eta)x)\chi(\xi x)^{-1}\chi(\eta x)^{-1} = 1$$

whence

$$(f(\xi+\eta) - f(\xi) - f(\eta))x \in \mathbb{Z} \quad (x \in S).$$

Thus in view of density of S in $\mathbb{Q}_+$ we obtain

$$f(\xi+\eta) = f(\xi) + f(\eta),$$

which means that f is additive. Finally, fix an $x_0 \in S \setminus \{0\}$ and let r be an arbitrary rational. Write $r = \xi x_0$ for some $\xi \in \mathbb{Q}$. Then by (3.2) and additivity of f

$$\chi(r) = \chi(\xi x_0) = \exp(2\pi i f(\xi)x_0) = \exp(2\pi i f(\xi x_0)) = \exp(2\pi i f(r)),$$

or χ is decent in view of additivity of f.

The above Proposition can be easily generalized to the case of linear rational spaces. Indeed, let χ be a character of a rational linear space V and let B be a Hamel base of V. Further, for every $b \in B$ define $\chi_b : \mathbb{Q} \to T$ by

$$\chi_b(\xi) = \chi(\xi b) \quad (\xi \in \mathbb{Q}).$$

Then χ_b is a character for every $b \in B$, and [1; Remark on p. 321] states that χ is decent if and only if χ_b is decent for every $b \in B$. Thus we can extend Proposition 3.1 to the following.

THEOREM 3.1

Let V be a rational linear space and let $\chi : V \to T$ be a character. Further, let B be a Hamel base of V and for every $b \in B$ let S_b be a dense subset of $\mathbb{Q}_+$

such that $0 \in S_b$ *and* $(S_b, +)$ *is a subsemigroup of* $(\mathbb{Q}_+, +)$. *Then* χ *is decent if and only if for every* $b \in B$ *and every* $\xi \in \mathbb{Q}_+$ *there exists an* $\varepsilon > 0$ *such that the only semigroup contained in* $\operatorname{cl} \chi(\xi(S_b \cap [0, \varepsilon))b)$ *is* $\{1\}$.

REMARK 3.1

Theorem 3.1 contains Theorem 2.2. To see this consider a sequence $h_1, h_2, h_3, \ldots$ of positive integers with $h_n \geqslant 2$ for $n \geqslant 2$, put $H_n = h_1 \cdots h_n$ $(n \geqslant 1)$ and observe that the subset S of $\mathbb{Q}_+$ defined by

$$S = \left\{ \frac{k}{H_m} : \ k \in \mathbb{N} \cup \{0\}, \ m \in \mathbb{N} \right\}$$

is dense in $\mathbb{Q}_+$, $0 \in S$, and $(S, +)$ is a subsemigroup of $\mathbb{Q}_+$. Suppose that $\varphi : V \to \mathbb{R}$ is a solution of the congruence (2.1), $\widetilde{\varphi}$ is defined as in Section 2. and $\chi : V \to T$ is a character defined by $\chi(x) = \exp(2\pi i \varphi(x))$. Moreover, assume that for every $a \in V$ there is an $N \in \mathbb{N}$ such that (2.8) is satisfied.

1. Let B be a Hamel base of V and for every $b \in B$ put $S_b = S$. Let $a = \xi b$, where b is an arbitrary chosen element of B and ξ is a fixed number from $\mathbb{Q}_+$.

2. Put $\varepsilon = \frac{1}{H_{N+2}}$. It is now a matter of a simple calculation to check that $s \in S_b \cap [0, \varepsilon)$ if and only if there exist $n \geqslant 3$ and $k \in \{0, \ldots, h_{N+3} \cdots h_{N+n} - 1\}$ such that

$$s = \frac{k}{H_{N+n}}. \tag{3.3}$$

3. Assume first that there is an $N \in \mathbb{N}$ such that (2.9) holds. Fix such an N and let $s \in S_b \cap [0, \varepsilon)$ be given by (3.3) for some $n \geqslant 3$ and $k \in \{0, \ldots, h_{N+3} \cdots h_{N+n} - 1\}$. Using the argument from part 4 of the proof of Theorem 2.2, and since $h_{N+r} \geqslant 2$ $(r \in \mathbb{N})$, we obtain

$$\begin{aligned} 0 \leqslant \widetilde{\varphi}(s\xi b) = \widetilde{\varphi}\left(\frac{ka}{H_{N+n}}\right) &= \frac{k}{h_{N+1} \cdots h_{N+n}} \widetilde{\varphi}\left(\frac{\xi b}{H_N}\right) \\ &< \frac{1}{h_{N+1} h_{N+2} h_{N+3}} \leqslant \frac{1}{8}. \end{aligned}$$

This implies that the only semigroup of T contained in $\operatorname{cl} \chi(\xi(S \cap [0, \varepsilon))b)$ is $\{1\}$. as it is the only subsemigroup of T contained in the halfplane $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$.

In the case where there is an $N \in \mathbb{N}$ such that

$$\widetilde{\varphi}\left(\frac{-a}{H_n}\right) = h_{n+1} \widetilde{\varphi}\left(\frac{-a}{H_{n+1}}\right) \quad (n \geqslant N)$$

we have $0 \leqslant \widetilde{\varphi}(-s\xi b) < \frac{1}{8}$ and, according to (2.7), $\widetilde{\varphi}(s\xi b) \in \{0\} \cup \left(\frac{7}{8}, 1\right)$ for every $s \in S_b \cap [0, \varepsilon)$. This implies that $\{1\}$ is the only semigroup contained in $\operatorname{cl} \chi(\xi(S \cap [0, \varepsilon))b)$.

4. Since b and ξ were chosen arbitrarily, by Theorem 3.1 we get decency of χ, which is equivalent to decency of φ, as we observed at the beginning of this section.

5. Conversely, if φ is decent then so is the corresponding character χ_φ. If $a = 0$ then (2.8) obviously holds. In the case where $a \neq 0$, let B be a Hamel base of V which contains a. Now in view of Theorem 3.1, Lemmas 3.2 and 3.1, we see that there exists a $c \in \mathbb{R}$ such that $\widetilde{\varphi}(\xi a) \equiv \xi c$ ($\xi \in \mathbb{Q}_+$). Hence we have either $\widetilde{\varphi}(\xi a) = \xi c$ or $\widetilde{\varphi}(-\xi a) = -\xi c$ for ξ positive and small enough. This implies (2.8).

Let us state now an analogue of Theorem 2.3 where conditions (A)-(E) are *translated* to the language of characters. To this aim let us introduce the following notation.

$$T_+ = \{z \in T : \ \operatorname{Im} z \geqslant 0\} \setminus \{-1\},$$

$$T_r = \begin{cases} T, & \text{if } r > \frac{1}{2}, \\ \{z \in T : \ |z-1| < |\exp(2\pi i r) - 1|\}, & \text{if } 0 \leqslant r \leqslant \frac{1}{2}. \end{cases}$$

THEOREM 3.2

Let $\chi : V \to T$ be a character. The decency of χ is equivalent to each of the following conditions:

(A′) *For every $a \in V$ there is an $\varepsilon > 0$ such that*

$$\chi(\xi a) \in T_+ \ \ (\xi \in \mathbb{Q},\ 0 \leqslant \xi \leqslant \varepsilon) \quad \textit{or} \quad \chi(-\xi a) \in T_+ \ \ (\xi \in \mathbb{Q},\ 0 \leqslant \xi \leqslant \varepsilon).$$

(B′) *For every $a \in V$ there is an $N \in \mathbb{N}$ such that*

$$\chi\left(\frac{a}{2^n}\right) \in T_+ \ \ (n \in \mathbb{N},\ n \geqslant N) \quad \textit{or} \quad \chi\left(\frac{-a}{2^n}\right) \in T_+ \ \ (n \in \mathbb{N},\ n \geqslant N).$$

(C′) *For every $a \in V$ there is an $N \in \mathbb{N}$ such that*

$$\chi\left(\frac{a}{2^n}\right) \in T_{\frac{1}{3}} \quad (n \in \mathbb{N},\ n \geqslant N).$$

(D′) *If $a \in V$ then there is a $D > 0$ such that*

$$\chi\left(\frac{a}{n}\right) \in T_{\frac{D}{n}} \cap T_+ \ \ (n \in \mathbb{N}) \quad \textit{or} \quad \chi\left(\frac{-a}{n}\right) \in T_{\frac{D}{n}} \cap T_+ \ \ (n \in \mathbb{N}).$$

(E′) *If $a \in V$ then there exists a $D > 0$ such that*

$$\chi\left(\frac{a}{n}\right) \in T_{\frac{D}{n}} \quad (n \in \mathbb{N}).$$

REMARK 3.2

Let us note that the condition (C′) slightly differs from (C). In fact (C) is equivalent to

$$\left\{\chi\left(\frac{a}{2^n}\right), \chi\left(\frac{-a}{2^n}\right)\right\} \cap (T_{\frac{1}{3}} \cap T_+) \neq \emptyset \quad (n \in \mathbb{N},\ n \geqslant N),$$

where $\chi(x) = \exp(2\pi i\bar{\varphi}(x))$ $(x \in V)$. However, since χ is a character, we have $\chi(-x) = \chi(x)^{-1}$, and therefore if one of numbers $\chi\left(\frac{a}{2^n}\right)$, $\chi\left(\frac{-a}{2^n}\right)$ belongs to $T_{\frac{1}{3}}$ then so does the other one. Thus at least one of them must belong to T_+. A similar argument justifies that (E′) is equivalent to (E).

REMARK 3.3

We have used above the fact that a character of a rational linear space is decent if and only if its restrictions to rays passing through elements of a fixed Hamel base B are decent. Therefore in Theorem 3.2 it is enough to assume each of the conditions (A′)-(E′) for $a = \xi b$ where $\xi \in \mathbb{Q}$ and $b \in B$.

In view of [1; Corollary in Section 4 on p. 323] another condition equivalent to decency of a character χ of a rational linear space V is its continuity along (rational) rays, i.e., continuity of functions $\xi \to \chi(\xi a)$, mapping $\mathbb{Q}$ into T (for $a \in V$). Let us conclude with a result characterizing decency as continuity with respect to some topology in V. This topology is called *core-topology*, and it is defined as follows (cf., e.g., [7; Chapter I]). A point $x \in A \subset V$ is said to be *algebraically interior* to A iff for every $a \in V$ there exists an $\varepsilon > 0$ such that

$$\xi a + x \in A$$

for every $\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$. By core A we denote the set of all points which are algebraically interior to A. We say that A is *algebraically open*, if core $A = A$. The family of all algebraically open subsets of V is a topology, usually called the *core-topology*. This is the finest topology in V such that with respect to this topology, and the natural topology in $\mathbb{Q}$, the mappings

$$\xi \to \xi x + y$$

of $\mathbb{Q}$ into V are continuous for every x and y in V. This characterization of the core-topology and the equality

$$\chi(y)\chi(\xi x) = \chi(\xi x + y)$$

yield immediately the announced characterization of decency.

THEOREM 3.3
A character of a rational linear space is decent if and only if it is continuous with respect to the core-topology.

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