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An alternative Cauchy equation almost everywhere

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. Let $(S, +)$ be a commutative semigroup and let $\mathcal{I}$ be a proper left translation invariant σ-ideal in S. Suppose that $f : S \to \mathbb{R}$ satisfies

$$|f(x+y)| = |f(x) + f(y)| \quad \Omega(\mathcal{I})\text{-almost everywhere in } S \times S.$$

We prove that there exists exactly one homomorphism $a : S \to \mathbb{R}$ such that

$$a = f \quad \mathcal{I}\text{-almost everywhere in } S.$$

1. Introduction

Let $(S, +)$ be a commutative semigroup and let $f : S \to \mathbb{R}$. In the paper we are dealing with the following alternative Cauchy equation

$$|f(x+y)| = |f(x) + f(y)|. \tag{1}$$

We postulate its validity almost everywhere in $S \times S$ (the notion "almost everywhere" is explained in the sequel) and observe how much such a near-solution differs from the mapping fulfilling (1) with the full exactness. Let us remind that, in fact, the only exact solution of (1) is an additive function (see [11], Theorem 13.9.1) and that this is the case even if we replace the absolute value by a strictly convex norm (see [7], Theorem 1).

Taking up this question we are inspired with many papers devoted to functional equations postulated almost everywhere, especially with these ones concerning additive functions (see [1], [2], [4]-[6], [8]-[10], [14]). Roughly speaking these results give an answer to the problem raised by P. Erdös (see [3]).

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Let us start with basic notions. As usual, let $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of all real numbers and positive integers respectively. A nonempty family $\mathcal{I}$ of subsets of a semigroup $(S,+)$ is called a *proper σ-ideal* provided the following conditions hold (cf. [4], [5], [11], [13]):

(i) $B \subset A,\ A \in \mathcal{I} \implies B \in \mathcal{I}$,

(ii) $A_i \in \mathcal{I},\ i \in \mathbb{N} \implies \bigcup_{i\in\mathbb{N}} A_i \in I$,

(iii) $S \notin \mathcal{I}$.

If additionally we have

(iv) $x \in S,\ A \in \mathcal{I} \implies -x + U \in \mathcal{I}$,

where $-x+U := \{y \in S \mid x+y \in U\}$, then we say that $\mathcal{I}$ is a *proper left translation invariant σ-ideal* in S (p.l.t.i. σ-ideal). Given a family $\mathcal{I}$ in S we say that a condition is satisfied $\mathcal{I}$-almost everywhere in S ($\mathcal{I}$-(a.e.)) iff there exists a set $A \in \mathcal{I}$ such that the condition in question is satisfied for every $x \in S \setminus A$. Finally, if we have a family $\mathcal{I}$ in S, we define

$$\Omega(\mathcal{I}) = \{M \subset S \times S \mid\ M[x] \in \mathcal{I}\ \mathcal{I}\text{-(a.e.) in } S\},$$

where $M[x] = \{y \in S \mid\ (x,y) \in M\}$ and

$$\Pi(\mathcal{I}) = \{A \subset S \times S \mid\ A \subset (U \times S) \cup (S \times U),\ U \in \mathcal{I}\}.$$

Both, $\Omega(\mathcal{I})$ and $\Pi(\mathcal{I})$, are p.l.t.i. σ-ideals in $S \times S$ whenever $\mathcal{I}$ is a p.l.t.i. σ-ideal in S.

While proving our main result we make use of the following theorem obtained by Jacek Tabor, which generalizes the well-known Theorem of Ger (cf. [5], Theorem 1).

THEOREM T (cf. [12], Theorem 1)

Let $(S,+)$ be a left reversible semigroup, i.e.

$$(a+S) \cap (b+S) \neq \emptyset \quad \textit{for } a,b \in S.$$

Let $(H,+)$ be a group and let $\mathcal{I}$ be a left translation invariant family in S satisfying

$$A_1,\dots,A_4 \in \mathcal{I} \implies A_1 \cup \ldots \cup A_4 \neq S.$$

Suppose that $f : S \to H$ is $\Omega(\mathcal{I})$-almost additive in $S \times S$. Then there exists exactly one additive function $F : S \to H$ such that $F = f$ $\mathcal{I}$-almost everywhere in S.

2. Main results

We have the following theorem

THEOREM 1

Let $(S,+)$ be a commutative semigroup and let $\mathcal{I}$ be a p.l.t.i. σ-ideal in S. Suppose that the function $f: S \to \mathbb{R}$ satisfies (1) *$\Omega(\mathcal{I})$-(a.e.) in $S \times S$. Then there exists an unique additive function $a : S \to \mathbb{R}$ such that $a = f$ $\mathcal{I}$-(a.e.) in S.*

Proof. Let us note that every commutative semigroup is left reversible. Thus, due to the Theorem T it suffices to prove that f is $\Omega(\mathcal{I})$-almost additive. For, let $M \in \Omega(\mathcal{I})$ be such that

$$|f(x+y)| = |f(x) + f(y)| \quad \text{for } (x,y) \in S \times S \setminus M. \tag{2}$$

Taking $U \in \mathcal{I}$ with $M[x] \in \mathcal{I}$ for $x \in S \setminus U$ we define the set $A \in \Omega(\mathcal{I})$ by the formula

$$A := U \times S \cup S \times U \cup \{(x,y) \mid \ x+y \in U\} \cup M.$$

We are going to prove that

$$f(x+y) = f(x) + f(y) \quad \text{for } (x,y) \in S \times S \setminus A.$$

For the indirect proof let us suppose that there exists $(x,y) \in S \times S \setminus A$ such that $f(x+y) \neq f(x) + f(y)$. Since, in particular $(x,y) \in S \times S \setminus M$, then using (2) we have

$$f(x+y) = -f(x) - f(y) \neq 0. \tag{3}$$

By the definition of A we get that $x, y, x+y \in S \setminus U$, thus $M[x], M[y], M[x+y] \in \mathcal{I}$. Define

$$B := \bigcup_{n,m \in \mathbb{N} \cup \{0\}} \left(-ny - mx + (M[x] \cup M[y] \cup M[x+y])\right).$$

According to the properties of $\mathcal{I}$ we have $S \setminus B \neq \emptyset$.

Let us consider two complementary cases.

Case 1. Suppose that

$$z \in S \setminus B \implies (f(z) = 0 \ \vee \ f(z) = f(x) = f(y)). \tag{4}$$

Fix arbitrary $z \in S \setminus B$. By the definition of B we have in particular that $y + z \in S \setminus B$ and $x + y + z \in S \setminus B$, thus making use of (4) we obtain what follows

$$f(z) = 0 \quad \text{or} \quad f(z) = f(x) = f(y), \tag{5}$$

$$f(y+z)=0 \quad \text{or} \quad f(y+z)=f(x)=f(y). \tag{6}$$

and

$$f(x+y+z)=0 \quad \text{or} \quad f(x+y+z)=f(x)=f(y). \tag{7}$$

If $f(z)=0$, then taking into account the pair $(x+y,z) \in S \times S \setminus M$ we have

$$f(x+y+z)=e_1(f(x)+f(y)),$$

where $e_1 \in \{-1,1\}$, which according to (7) means that $f(x)+f(y)=0$ or $f(x)=f(y)=0$. In both cases we have a contradiction with (3). Otherwise, if $f(z)=f(x)=f(y)$, then for the pair $(y,z) \in S \times S \setminus M$ we obtain

$$f(y+z)=e_2(2f(y)),$$

where $e_2 \in \{-1,1\}$. Applying (6) we also get a contradiction.

Case 2. If (4) does not hold then

$$\exists\, z_0 \in S \setminus B \quad f(z_0) \neq 0 \ \wedge\ (f(z_0) \neq f(x) \ \vee\ f(z_0) \neq f(y)). \tag{8}$$

Since $(x+y,z_0) \in S \times S \setminus M$, applying (2) and (3) we have

$$f(x+y+z_0)=e_1(f(x+y)+f(z_0))=e_1(-f(x)-f(y)+f(z_0)), \tag{9}$$

where $e_1 \in \{-1,1\}$. On the other hand observe that $(x, y+z_0) \in S \times S \setminus M$ and $(y,z_0) \in S \times S \setminus M$. Thus, by (2) we have

$$f(x+y+z_0)=e_2(f(x)+f(y+z_0))=e_2(f(x)+e_3(f(y)+f(z_0))), \tag{10}$$

where $e_2, e_3 \in \{-1,1\}$. Conditions (9) and (10) imply

$$-f(x)-f(y)+f(z_0)=e_4(f(x)+e_3(f(y)+f(z_0))), \tag{11}$$

where $e_4, e_3 \in \{-1,1\}$. In the case where $e_4=e_3=1$, (11) means that $f(x)+f(y)=0$ and brings a contradiction with (3). Similarly, if $e_4=-1$, $e_3=1$ then we have $f(z_0)=0$ and a contradiction with (8). If $e_4=1$, $e_3=-1$ then we get

$$f(x)=f(z_0). \tag{12}$$

If $e_4=-1$, $e_3=-1$ then we have

$$f(y)=0. \tag{13}$$

Taking into account pairs $(y, x+z_0) \in S \times S \setminus M$, $(x,z_0) \in S \times S \setminus M$ and proceeding as above one can obtain

$$f(y) = f(z_0), \tag{14}$$

or

$$f(x) = 0. \tag{15}$$

Combining (12) and (13) with (14) and (15) we get $f(x) = f(y) = 0$ or $f(x) = f(y) = f(z_0)$ which also contradicts our assumptions. This ends the proof.

REMARK

Putting $\mathcal{I} = \{\emptyset\}$ in Theorem 1 we derive, as a simple corollary, Theorem 13.9.1 from [11].

Let us draw once more analogy between almost additive functions and near-solutions of (1). S. Hartman (cf. [9]) and N.G. de Bruijn (cf. [1]) noticed that the solution of the generalized Erdös problem can further be strengthened if we restrict the family of "small" sets behind which the additivity condition is postulated. Namely, every function mapping a group G into H being $\Pi(\mathcal{I})$ almost additive ($\mathcal{I}$ is an ideal in G) is a homomorphism.

As the Example 1 shows, the exact equivalent of the above with respect to equation (1) fails to be true.

EXAMPLE 1

Let $f : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$f(x) = \begin{cases} x & \text{for } x \neq 1, \\ -1 & \text{for } x = 1. \end{cases}$$

Then f satisfies (1) for $x, y \in \mathbb{R} \setminus \{1\}$ however it does not satisfy (1) for all $x, y \in \mathbb{R}$.

We are going to prove (proceeding similarly as in the proof of Theorem 17.6.3 from [11]) the following theorem.

THEOREM 2

Let $(G, +)$ be a commutative group and let $\mathcal{I}$ be a proper linearly invariant σ-ideal in G, i.e. satisfying conditions (i)-(iii) *and additionally*

$$x \in G,\ U \in \mathcal{I} \Longrightarrow x - U \in \mathcal{I}.$$

Then $f : G \to \mathbb{R}$ satisfies (1) *$\Pi(\mathcal{I})$-(a.e.) in $G \times G$ iff there exists an unique additive function $a : G \to \mathbb{R}$ with $a(x) = f(x)$ $\mathcal{I}$-(a.e.) in G and*

$$|f(x)| = |a(x)| \quad \textit{for } x \in G. \tag{16}$$

Proof. Suppose that $f : G \to \mathbb{R}$ is a function satisfying (1) $\Pi(\mathcal{I})$-(a.e.) in $G \times G$. Directly from Theorem 1 we have the existence and uniqueness of the additive function $a : G \to \mathbb{R}$ such that

$$f(x) = a(x) \quad \text{for } x \in G \setminus U, \tag{17}$$

where U is an element of $\mathcal{I}$. By the assumptions of f we get that there exists a $V \in \mathcal{I}$ with

$$|f(x+y)| = |f(x) + f(y)| \quad \text{for } x, y \in G \setminus V. \tag{18}$$

Choose arbitrary $x \in G$ and define $A \in \mathcal{I}$ as follows

$$A := -(U \cup V) \cup (-x + (U \cup V)).$$

Thus there exists an $y \in G \setminus A$. This implies that $-y \in G \setminus U$ and $x+y \in G \setminus U$, whence by (17)

$$f(-y) = a(-y) \quad \text{and} \quad f(x+y) = a(x+y). \tag{19}$$

Similarly $-y \in G \setminus V$ and $x + y \in G \setminus V$. Using (18), (19) and the additivity of a we get

$$\begin{aligned} |f(x)| &= |f(x+y-y)| = |f(x+y) + f(-y)| \\ &= |a(x+y) + a(-y)| \\ &= |a(x)|. \end{aligned}$$

Now suppose that $a : G \to \mathbb{R}$ is an additive function such that $a(x) = f(x)$ $\mathcal{I}$-(a.e.) in G and $|f(x)| = |a(x)|$ for $x \in G$. Then there exists $V \in \mathcal{I}$ with

$$a(x) = f(x) \quad \text{for } x \in G \setminus V.$$

Fix $x, y \in G \setminus V$. We have

$$\begin{aligned} |f(x+y)| &= |a(x+y)| \\ &= |a(x) + a(y)| \\ &= |f(x) + f(y)| \end{aligned}$$

which means that f satisfies an alternative Cauchy equation $\Pi(\mathcal{I})$-(a.e.) in $G \times G$.

At the end of this section let us point out that in Theorem 2 it is essential to consider $\Pi(\mathcal{I})$ ideal in $G \times G$.

Example 2

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{for } x \neq 1, \\ 2 & \text{for } x = 1 \end{cases}$$

and let $V = \{(x, y) \in \mathbb{R}^2 | \; x = 1 \;\vee\; y = 1 \;\vee\; x + y = 1\}$. Then f satisfies (1) for $(x, y) \in \mathbb{R}^2 \setminus V$ but there does not exist an additive function $a : \mathbb{R} \to \mathbb{R}$ satisfying (16).

References

[1] N.G. de Bruijn, *On almost additive functions*, Colloq. Math. **15** (1966), 59-63.

[2] J. Brzdęk, *On almost additive functions*, Bull. Austral. Math. Soc. **54** (1996), 281-290.

[3] P. Erdös, *Problem 310*, Colloq. Math. **7** (1960), 311.

[4] R. Ger, *Almost additive functions on semigroups and a functional equation*, Publ. Math. Debrecen **26** (1979), 219-228.

[5] R. Ger, *Note on almost additive functions*, Aequationes Math. **17** (1978), 73-76.

[6] R. Ger, *Christensen measurability and functional equations*, Grazer Math. Ber. **289** (1988), 1-17.

[7] R. Ger, *On a characterization of strictly convex spaces*, Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Natur. **127** (1993), 131-138.

[8] R. Ger, *On some functional equations with a restricted domain I, II*, Fund. Math. **89** (1975), 131-149, **98** (1978), 249-272.

[9] S. Hartman, *A remark about Cauchy's equation*, Colloq. Math. **8** (1961), 77-79.

[10] W.B. Jurkat, *On Cauchy's functional equation*, Proc. Amer. Math. Soc. **16** (1965), 683-686.

[11] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN, Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

[12] Jacek Tabor, *k-Proper families and almost additive functions*, Aequationes Math. (submitted).

[13] Józef Tabor, *A generalized notion of conjugate ideals and its applications*, Opuscula Mathematica **16** (1996), 111-117.

[14] Józef Tabor, *Cauchy and Jensen equations on a restricted domain almost everywhere*, Publ. Math. Debrecen **39** (1991), 219-235.

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