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Walter Benz

DeSitter distances in Hilbert spaces

Dedicated to Zenon Moszner
on the occasion of his 70th birthday, in friendship

Abstract. All 2-point invariants of an arbitrary DeSitter manifold are determined without any regularity assumption, especially those which are additive on geodesics.

1. Introduction

Two functional equations play a role in this note: the functional equation of 2-*point invariants* of a DeSitter manifold will be solved, and, moreover, the functional equation of *additivity along lines* of such a manifold will be studied within the set of solutions of the first equation. This then leads us to the notion of *distance* in a DeSitter World over an arbitrary pre-Hilbert space X of dimension at least 3. In the case $\dim X < \infty$ the problems above are treated in chapter 4 of the book [1]. However, further methods and ideas are needed in the present situation in comparison with the finite-dimensional case.

For the classical theory of DeSitter's World see [3], [4], for modern developments see [5], [6], [7].

2. Points, motions, lines

Let X be a real pre-Hilbert space, i.e. a real vector space furnished with an inner product

$$\delta : X \times X \to \mathbb{R}, \quad \delta(x, y) =: xy,$$

satisfying $x^2 > 0$ for all $x \neq 0$ in X. We assume that the dimension of X is at least 3. Let t be a fixed element of X such that $t^2 = 1$. Define

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$$H := t^{\perp} := \{x \in X \mid tx = 0\}.$$

Then $H \oplus \mathbb{R}t = X$ holds true (see section 2 of [2]). Since $x - (tx)\,t$ is in H for all $x \in X$,

$$x = \overline{x} + x_0 t \quad \text{with} \quad \overline{x} \in H \quad \text{and} \quad x_0 \in \mathbb{R}$$

implies $\overline{x} = x - (tx)\,t$ and $x_0 = tx$. Define

$$S\,(X) := \{x \in X \mid \overline{x}^2 - x_0^2 = 1\}$$

as the set of *points* of *DeSitter's Manifold* over X. The Lorentz-Minkowski distance $l\,(x,y)$ of $x, y \in X$ is defined (see [2]) by the expression

$$l\,(x,y) = (\overline{x} - \overline{y})^2 - (x_0 - y_0)^2.$$

A Lorentz transformation of X is a mapping λ from X into itself such that

$$l\,(x,y) = l\,(\lambda\,(x), \lambda\,(x))$$

holds true for all $x, y \in X$. In [2] all Lorentz transformations of X are determined.

The restriction of a surjective Lorentz transformation $\lambda : X \to X$ on $S\,(X)$ will be called a *motion* of $S\,(X)$ provided that $\lambda\,(0) = 0$. Motions of $S\,(X)$ must be bijective. We hence get the group $\Delta\,(X)$ of motions of $S\,(X)$.

The points a, b of $S\,(X)$ are called *separated* if, and only if, $b \notin \{a, -a\}$. Such a pair must be linearly independent. Otherwise an equation $\alpha a = \beta b$ holds true with real α, β which are not both 0. But this implies

$$\alpha^2 = l\,(\alpha a, 0) = l\,(\beta b, 0) = \beta^2,$$

i.e. $\beta = \alpha$ or $\beta = -\alpha$.

Suppose that a, b are points of $S\,(X)$ which are separated. Then every ellipse, every euclidean line, every branch of a hyperbola in

$$\{\xi a + \eta b \mid \xi, \quad \eta \in \mathbb{R}\} \cap S\,(X) \tag{1}$$

is called a *line* of $S\,(X)$. All $\xi a + \eta b$ in (1) are characterized by the equation

$$(\xi + \gamma\eta)^2 + (1 - \gamma^2)\,\eta^2 = 1, \tag{2}$$

where $\gamma := \gamma\,(a,b) := \overline{a}\,\overline{b} - a_0 b_0$ designates the *pseudo-euclidean scalar product* of a, b. In the cases $\gamma^2 < 1$, $\gamma^2 = 1$, $\gamma^2 > 1$, respectively, we get an ellipse (a *closed line*), two euclidean lines (*null-lines*), two branches of a hyperbola (*open lines*), respectively, of $S\,(X)$. The lines of $S\,(X)$ are also called its *geodesics*.

3. The functional equation of 2-point invariants

Suppose that $W \neq \emptyset$ is a set and that

$$d : S(X) \times S(X) \to W$$

satisfies

$$d(x,y) = d(f(x), f(y)) \tag{3}$$

for all motions f of $S(X)$ and all $x, y \in S(X)$. Then d is called a *2-point invariant* of $S(X)$.

THEOREM 1

$\Delta(X)$ acts transitively on $S(X)$. If a, b and c, e are pairs of separated points, then there exists $\delta \in \Delta(X)$ with $\delta(a) = c$ and $\delta(b) = e$ if, and only if,

$$\gamma(a,b) = \gamma(c,e)$$

holds true.

Proof. a) In step c) of the proof of Theorem 3 in [2] we showed that to $x, y \in X \backslash \{0\}$ there exists a bijective Lorentz transformation λ with $\lambda(0) = 0$ and $\lambda(x) = y$ if, and only if, $l(x,0) = l(y,0)$. Suppose that x, y are points of $S(X)$. Then

$$l(x,0) = 1 = l(y,0)$$

holds true. There hence exists a motion δ with $\delta(x) = y$.

b) If a, b and c, e are pairs of separated points, and if $\delta \in \Delta(X)$ satisfies $\delta(a) = c$ and $\delta(b) = e$, then

$$l(a,b) = l(c,e),$$

since δ is a Lorentz transformation. Hence

$$(\overline{a} - \overline{b})^2 - (a_0 - b_0)^2 = (\overline{c} - \overline{e})^2 - (c_0 - e_0)^2,$$

i.e. $\gamma(a,b) = \gamma(c,e)$, in view of $a, b, c, e \in S(X)$.

c) Let a, b and c, e be pairs of separated points satisfying $\gamma(a,b) = \gamma(c,e)$. Because of Theorem 1 in [2], Lorentz transformations of X fixing 0, must be linear. Separated points x, y must thus be transformed into separated points under motions. In view of step a) we hence may assume $a = c$ without loss of generality, by observing

$$\gamma(x,y) = \gamma(f(x), f(y)) \tag{4}$$

for $x, y \in S(X)$ and motions f. The situation now is that a, b and a, e are pairs of separated points such that $\gamma(a,b) = \gamma(a,e)$. If $h \in H$ satisfies $h^2 = 1$, which especially implies $h \in C(X)$, then we even may assume that $a = h$, in view of step a). Then $\gamma(a,b)=\gamma(a,e)$ reads $h\bar{b}=h\bar{e}$, i.e. $hb=he$ because of $h\perp t$. We now consider the pre-Hilbert space

$$X_0 := \{x-(xh)\,h \mid x\in X\}.$$

Obviously, $t\in X_0$. Again, we would like to apply Theorem 3 of [2], but this time for X_0 and for the points

$$\xi := b-(bh)\,h \quad\text{and}\quad \eta := e-(eh)\,h.$$

Observe

$$\begin{aligned} l(\xi,0) &= \bar{\xi}^2 - b_0^2 = (\bar{b}-(bh)\,h)^2 - b_0^2 \\ &= 1-(bh)^2 = 1-(eh)^2 \\ &= l(\eta,0). \end{aligned}$$

There hence exists a Lorentz transformation λ_0 of X_0 satisfying $\lambda_0(0)=0$ and

$$\lambda_0(b-(bh)\,h) = e-(eh)\,h. \tag{5}$$

The problem now is to extend λ_0 to a Lorentz transformation λ of X by putting

$$\lambda(x) := \lambda_0(x-(xh)\,h) + (xh)\,h$$

for all $x\in X$. That λ is an extension of λ_0 follows from

$$xh=0 \quad\text{for all } x\in X_0.$$

Put $x_h := x-(xh)\,h$ for $x\in X$. Then

$$\overline{\lambda(x)} = \overline{\lambda_0(x_h)} + (xh)\,h =: x_1 + (xh)\,h.$$

Put $[\lambda_0(x_h)]_0 =: x_2$. Then

$$\begin{aligned} l(\lambda(x),\lambda(y)) &= (x_1+(xh)\,h-y_1-(yh)\,h)^2 - (x_2-y_2)^2 \\ &= l(\lambda_0(x_h),\lambda_0(y_h)) + A \end{aligned}$$

with $A=2(x_1-y_1)\,h\,(xh-yh)+(xh-yh)^2=(xh-yh)^2$ since

$$x_1 = \overline{\lambda_0(x_h)} \in X_0$$

implies $x_1h=0$. Similarly, we get

$$l(x,y) = l(x_h,y_h) + A,$$

i.e. $l(x,y)=l(\lambda_0(x_h),\lambda_0(y_h))+A=l(\lambda(x),\lambda(y))$, and λ must hence be a Lorentz transformation of X. We finally would like to show

$$\lambda(h)=h \quad\text{and}\quad \lambda(b)=e.$$

In fact,

$$\lambda(h) = \lambda_0(h - h^2 \cdot h) + h^2 h = \lambda_0(0) + h = h,$$

and

$$\lambda(b) = \lambda_0(b - (bh)\,h) + (bh)\,h = [e - (eh)\,h] + (eh)\,h = e,$$

in view of (5) and $bh = eh$.

We now would like to solve the functional equation (3) of 2-point invariants.

THEOREM 2

Let $g : \mathbb{R} \to W$ be a function and let w_0, w_1 be fixed elements of W. Then

$$d(x,y) = \begin{cases} g(\overline{x}\,\overline{y} - x_0 y_0) & \text{for } x, y \text{ separated} \\ w_0 & \text{for } x = y \\ w_1 & \text{for } x = -y \end{cases} \tag{6}$$

with $x, y \in S(X)$ is a solution of (3). *If, on the other hand,*

$$d : S(X) \times S(X) \to W$$

solves (3), *then there exists a function $g : \mathbb{R} \to W$ and elements $w_0, w_1 \in W$ such that* (6) *holds true.*

Proof. Obviously, (6) solves (3) for all motions f and all $x, y \in S(X)$, in view of (4). Assume now that $d : S(X) \times S(X) \to W$ is a solution of (3). Take elements $i, j \in H$ with $i^2 = 1$, $j^2 = 1$, $ij = 0$. For $k \in \mathbb{R}$ define $g(k)$ by means of

$$g(k) := d(i, ki + j + kt). \tag{7}$$

Observe here $i \in S(X)$, $ki + j + kt \in S(X)$ and

$$\gamma(i, ki + j + kt) = k. \tag{8}$$

Moreover, put $w_0 := d(i,i)$ and $w_1 := d(i,-i)$. If $x \in S(X)$, there exists $f \in \Delta(X)$ with $f(i) = x$ on account of Theorem 1. Hence

$$d(x,x) = d(f(i), f(i)) = d(i,i) = w_0.$$

Since f is linear, we also get

$$d(x,-x) = d(f(i), f(-i)) = d(i,-i) = w_1.$$

Suppose now that $x, y \in S(X)$ are separated. If $\gamma(x,y) =: k$, then, according to (8) and Theorem 1, there exists $f \in \Delta(X)$ satisfying

$$f(i) = x \quad \text{and} \quad f(ki + j + kt) = y.$$

Hence $d(x,y) = d(f(i), f(ki+j+kt)) = d(i, ki+j+kt) = g(k)$, in view of (7). Thus

$$d(x,y) = g(k) = g(\gamma(x,y)) = g(\overline{x}\,\overline{y} - x_0 y_0).$$

4. The additivity equation

Since motions are linear, images of null-lines must be null-lines, and images of closed lines, open lines must be closed lines, open lines, respectively.

Suppose that $s : S(X) \times S(X) \to \mathbb{R}_{\geqslant 0} := \{r \in \mathbb{R} \mid r \geqslant 0\}$ satisfies the following property.

(A) *If x, y are separated points on a line l, and if $z \in l$ is between x, y, then*

$$s(x,y) = s(x,z) + s(z,y) \tag{9}$$

holds true.

Then s is called *additive*.

If l is null or open, the usual betweenness relation is meant. If l is closed, then z is assumed to be an element of the smaller part of the ellipse in question of the two parts defined by the points $x, y \in l$.

THEOREM 3

If $s : S(X) \times S(X) \to \mathbb{R}_{\geqslant 0}$ is an additive 2-point invariant, then there exist non-negative constants r_1, r_2 such that

$$s(x,y) = 0, \quad s(x,y) = r_1 \mathrm{ch}^{-1}(\gamma(x,y)), \quad s(x,y) = r_2 \cos^{-1}(\gamma(x,y)),$$

provided x, y are separated points on a null-line, an open line, a closed line, respectively.

Here $\mathrm{ch}^{-1}\alpha = \beta$ with $\alpha \geqslant 1$ and $\beta \geqslant 0$ is defined by $\mathrm{ch}\,\beta = \alpha$. Similarly, $\cos^{-1}\alpha = \beta$ with $\beta \in [0, \pi]$ and $\alpha \in [-1, 1]$ means $\cos\beta = \alpha$.

Proof. a) Let $a + \mathbb{R}v := \{a + \lambda v \mid \lambda \in \mathbb{R}\}$, $v \neq 0$, be a null-line. Hence

$$\gamma(a,a) = 1, \quad \gamma(a,v) = 0 = \gamma(v,v). \tag{10}$$

Any two distinct points of this line l must be separated. Moreover,

$$\gamma(a + \lambda_1 v, a + \lambda_2 v) = \gamma(a + \lambda_3 v, a + \lambda_4 v) \tag{11}$$

in the case $\lambda_1 \neq \lambda_2$ and $\lambda_3 \neq \lambda_4$, in view of (10).

Suppose that $x \neq y$ are on l and that $z \notin \{x, y\}$ is on l between x and y. Then (11) and Theorem 1 imply the existence of motions δ_1, δ_2 satisfying

$$\delta_1(x) = x, \quad \delta_1(y) = z, \quad \delta_2(x) = z, \quad \delta_2(y) = y.$$

Hence $s(x,y) = s(x,z)$ and $s(x,y) = s(z,y)$. Thus $s(x,y) = 0$, in view of (9).

b) Let (1) be an open line satisfying (2) with $\gamma(a,b) =: k$. Up to a motion we may assume $a = i$ and $b = ki + j + kt$ with $i, j \in H$, $i^2 = 1$, $j^2 = 1$, $ij = 0$. The two branches of (2) are given by

$$\operatorname{ch}\lambda \cdot i + \operatorname{sh}\lambda \cdot v \quad \text{and} \quad \operatorname{ch}\lambda \cdot (-i) + \operatorname{sh}\lambda \cdot v$$

with $\lambda \in \mathbb{R}$ and

$$v := \frac{j + kt}{\sqrt{k^2 - 1}}.$$

Without loss of generality we may work with the first version, since in the other case $-i$ can be replaced by i. We hence get

$$p(\lambda) := \operatorname{ch}\lambda \cdot i + \operatorname{sh}\lambda \cdot v, \quad \lambda \in \mathbb{R}, \tag{12}$$

with $\gamma(i,i) = 1$, $\gamma(i,v) = 0$, $\gamma(v,v) = -1$. Two points $p(\lambda_1), p(\lambda_2)$ with $\lambda_1 \neq \lambda_2$ are separated. Moreover, in this case

$$\gamma(p(\lambda_1), p(\lambda_2)) = \operatorname{ch}(\lambda_1 - \lambda_2) > 1 \tag{13}$$

holds true. ($k > 1$ need not to be true: if a, b in (1) are on different branches of the underlying hyperbola, then $k < -1$.)

Let x, y be different points on (12). Because of $\gamma(x,y) = \gamma(y,x)$ and Theorem 1 there exists a motion that interchanges x and y. Hence $s(x,y) = s(y,x)$. We hence may assume $x = p(\lambda_1)$ and $y = p(\lambda_2)$ with $\lambda_1 < \lambda_2$. Theorem 2 implies

$$s(x,y) = g(\operatorname{ch}(\lambda_1 - \lambda_2)) \geqslant 0,$$

in view of (13). Put $\varphi(\lambda) = g(ch\lambda)$ for $\lambda \geqslant 0$. On account of (9) we get for $\lambda_2 \leqslant \lambda_3 \leqslant \lambda_1$,

$$\varphi(\lambda_1 - \lambda_2) = \varphi(\lambda_1 - \lambda_3) + \varphi(\lambda_3 - \lambda_2).$$

A standard procedure now leads to $\varphi(\lambda) = r_1\lambda$ with a non-negative constant r_1. Hence

$$s(x,y) = g(\operatorname{ch}(\lambda_1 - \lambda_2)) = r_1(\lambda_1 - \lambda_2) = r_1 \operatorname{ch}^{-1}(\gamma(x,y)),$$

on account of (13).

c) Let (1) be a closed line satisfying (2) with $\gamma(a,b) =: k$. Mutatis mutandis to the case b), we may work with the line l,

$$p(\lambda) = \cos\lambda \cdot i + \sin\lambda \cdot v, \quad \lambda \in \mathbb{R}, \tag{14}$$

with

$$v := \frac{j + kt}{\sqrt{1 - k^2}}.$$

Hence $\gamma(i,i) = 1$, $\gamma(i,v) = 0$, $\gamma(v,v) = 1$. Take separated points x, y on the line l with

$$x = p(\lambda_1) \quad \text{and} \quad y = p(\lambda_2),$$

$\lambda_2 < \lambda_1$, $\lambda_1 - \lambda_2 < \pi$. Then

$$\gamma(p(\lambda_1), p(\lambda_2)) = \cos(\lambda_1 - \lambda_2) \in]-1, 1[.$$

Theorem 2 implies $s(x,y) = g(\cos(\lambda_1 - \lambda_2)) \geqslant 0$. Put $\varphi(\lambda) = g(\cos\lambda)$ for $\lambda \in [0,\pi]$. Again (9) together with a standard procedure leads to $\varphi(\lambda) = r_2\lambda$ with a non-negative constant r_2. Hence

$$s(x,y) = g(\cos(\lambda_1 - \lambda_2)) = r_2(\lambda_1 - \lambda_2) = r_2 \cos^{-1}(\gamma(x,y)).$$

In the case $r_1 = r_2 = 1$ we call $s(x,y)$ *DeSitter's distance function.*

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Mathematisches Seminar
der Universität Hamburg
Bundesstraße 55
D-20146 Hamburg
Germany
E-mail: benz@math.uni-hamburg.de

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