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Note on some functional equations of Gołąb–Schinzel type

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. In the present paper, we consider the following functional equation of Gołąb–Schinzel type:

$$f(f(x)^n y + f(y)^k x) = \phi(f(x), f(y)) \tag{1}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a given function, $f : E \to \mathbb{R}$ is the unknown function, E is a real vector space, k and n are given positive integers. We prove that, under suitable hypotheses on ϕ, if k and n are distinct, the only solutions of (1) having some regularity properties are the constant functions.

1. Introduction

Let E be a vector space over a commutative field K. The functional equation:

$$f(f(x)y + x) = f(x)f(y) \quad (x, y \in E) \tag{GS}$$

where f is a mapping from E into $\mathbb{R}$, is called *the functional equation of Gołąb-Schinzel*. It has been first considered by J. Aczél in 1957 (cf. [1]), and then by S. Gołąb and A. Schinzel in 1959 (cf. [8]) who gave its continuous solutions $f : \mathbb{R} \to \mathbb{R}$. The general solution $f : E \to K$ has been characterized (cf. [9], [11]), and all the continuous solutions $f : E \to \mathbb{R}$ have been explicitely obtained (cf. [10]) when E is a real topological vector space.

Generalizing (GS), the following functional equation has been then considered:

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$$f(f(x)^n y + f(y)^k x) = \lambda f(x) f(y) \quad (x, y \in E) \tag{2}$$

where the unknown function f maps a real topological vector space E into $\mathbb{R}$.

When $n = k = 1$ and $\lambda > 0$, all the continuous solutions $f : E \to \mathbb{R}$ have been obtained in [2].

Following A.M. Bruckner and J.G. Ceder ([6]), we denote by $\mathcal{DB}_1$ the set of all functions from $\mathbb{R}$ into $\mathbb{R}$ which are in class I of Baire and have the Darboux property.

In the case where n and k are arbitrary nonnegative integers and λ is a nonnegative real number, all the solutions of (2) in $\mathcal{DB}_1$ and all the continuous solutions $f : E \to \mathbb{R}$ of (2) have been given in [3].

The arguments used in [3] do not permit to obtain the solutions of (2) when λ is an arbitrary real number.

In what follows, when x is an element of a real vector space E, we shall consider, for a function $f : E \to \mathbb{R}$, the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by:

$$f_x(t) = f(tx) \quad (t \in \mathbb{R}) \tag{P}$$

In [7], J. Brzdęk proved that, when E is a real vector space, k and n are distinct positive integers and λ is a nonzero real number, the only solutions $f : E \to \mathbb{R}$ of (2) such that the functions f_x defined by (P) are continuous for all x in E, are the constant functions.

In the case where E is a real topological vector space, k and n are arbitrary nonnegative integers and λ is an arbitrary real number, the author obtained in [4] (see also [5]) all the solutions of (2) in $\mathcal{DB}_1$ and all the continuous solutions $f : E \to \mathbb{R}$ of (2).

More generally, according to the definition given in [3], a functional equation is of *Gołąb–Schinzel type* if it is of the following form:

$$f(f(x)^n y + f(y)^k x) = \phi(x, y, f(x), f(y), f(xy)) \quad (x, y \in E)$$

where E is a real algebra, $f : E \to \mathbb{R}$ is the unknown function, k and n are given nonnegative integers, and $\phi : E^2 \times \mathbb{R}^3 \to \mathbb{R}$ is a given function.

In the present paper, we shall study the following functional equation of Gołąb–Schinzel type:

$$f(f(x)^n y + f(y)^k x) = \phi(f(x), f(y)) \quad (x, y \in E) \tag{1}$$

where E is a real vector space, $f : E \to \mathbb{R}$ is the unknown function, k and n are given positive integers, and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a given symmetric function satisfying some further hypotheses.

Let us denote by $\mathcal{LDB}_1$ the set of all functions $f : E \to \mathbb{R}$ which have the property that the functions f_x defined by (P) belong to $\mathcal{DB}_1$ for all x in E.

Under some hypotheses on ϕ, k and n, we obtain all the solutions of the functional equation (1) in the class of functions $\mathcal{LDB}_1$. Our results generalize those obtained by J. Brzdęk in [7] in the case $\phi(x,y) = \lambda xy$ where λ is a nonzero real number. Our arguments are similar to those used by J. Brzdęk, and thus, we prove thereby that the method of [7] is in fact very general and does not depend on the special form of ϕ considered in [7]. Some different examples of functions ϕ are also given.

2. Hypotheses and preliminary results

We suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is symmetric and satisfies the following set (H) of hypotheses:

(i) $\phi(x,y) = 0 \Longleftrightarrow xy = 0$,

(ii) $\forall\, y \in \mathbb{R} \setminus \{0\}$, the function $\phi(\cdot, y)$ is one-to-one,

(iii) $\forall\, x \in \mathbb{R}$, $\left|\dfrac{\phi(x,y)}{y}\right|$ is bounded when $|y|$ goes to ∞.

Assumption (iii) means that, for every real number x, there exist $M > 0$ and $R > 0$ such that:

$$|y| > R \Rightarrow \left|\frac{\phi(x,y)}{y}\right| \leqslant M.$$

We shall also consider the case where the hypothesis (ii) is replaced by the following:

(ii bis) $\forall\, y \in \mathbb{R} \setminus \{0\}$, $\phi(x_1, y) = \phi(x_2, y) \Rightarrow |x_1| = |x_2|$.

The set (H) of hypotheses with (ii) replaced by (ii bis) will be denoted by (H').

Lemma 1

Let us suppose that either ϕ satisfies the set (H) *of hypotheses and k, n are arbitrary positive integers, or ϕ satisfies the set* (H') *of hypotheses and k, n are even positive integers.*

Let us define: $\varphi(x,y) = f(x)^n y + f(y)^k x$. If f is a nonidentically zero solution of (1) *in $\mathcal{DB}_1$, the functions $\varphi(\cdot, x_o)$ and $\varphi(x_o, \cdot)$ are one-to-one and continuous when x_o is any real number satisfying $f(x_o) \neq 0$.*

Proof. Since f is in $\mathcal{DB}_1$, its graph is connected (cf. [6]). The continuity of the function: $(t,s) \in \mathbb{R}^2 \to (t, f(x)^n t + s^k x)$ implies that the graph of the function $\varphi(x, \cdot)$ is connected. Therefore, $\varphi(x, \cdot)$ has the Darboux property. Let us suppose that there exist x and y in $\mathbb{R}$ such that

$$\varphi(x_o, x) = \varphi(x_o, y) \tag{3}$$

where x_o is any real number satisfying $f(x_o) \neq 0$.

We deduce by (1): $\phi(f(x_o), f(x)) = \phi(f(x_o), f(y))$. Under (H), this implies: $f(x) = f(y)$. Under (H'), we get: $|f(x)| = |f(y)|$ and therefore $f(x)^k = f(y)^k$.

In both cases, we obtain with (3): $f(x_o)^n x = f(x_o)^n y$ and therefore $x = y$. We deduce that $\varphi(x_o, \cdot)$ is one-to-one. The same thing holds for $\varphi(\cdot, x_o)$.

So, the functions $\varphi(\cdot, x_o)$ and $\varphi(x_o, \cdot)$ are one-to-one and have the Darboux property. Therefore, they are continuous (cf. [6]).

Corollary 2

Under the hypotheses of Lemma 1, if f is a solution of (1) *in* $\mathcal{DB}_1$, *the functions* $f(\cdot)^n$ *and* $f(\cdot)^k$ *are continuous.*

3. Solutions of (1) in $\mathcal{DB}_1$ and in $\mathcal{LDB}_1$ in the case $k \neq n$.

In this section, the following hypotheses will be considered.

Hypothesis 3

We suppose that either ϕ satisfies the set (H) *of hypotheses and k, n are distinct positive integers, or ϕ satisfies the set* (H') *of hypotheses and k, n are distinct even positive integers.*

We start with the following important result.

Proposition 4

Let us suppose the Hypothesis 3 and let f be a nonidentically zero solution of (1) *in* $\mathcal{DB}_1$. *We define:* $Z = \{x \in \mathbb{R};\ f(x) \neq 0\}$. *Then f is not one-to-one on Z.*

Proof. For an indirect proof, we suppose that f is one-to-one on Z.

We remark first that, by (1) and the hypotheses on ϕ, if x and y belong to Z, $\varphi(x, y)$ and $\varphi(y, x)$ belong also to Z. The fact that f is one-to-one on Z implies by (1): $\varphi(x, y) = \varphi(y, x)$ for all x, y in Z. We deduce:

$$(f(x)^n - f(x)^k)y = (f(y)^n - f(y)^k)x$$

for all x, y in Z. Since f has the Darboux property and is not constant, there exists y_o in $Z \setminus \{0\}$ such that $p = \frac{f(y_o)^n - f(y_o)^k}{y_o} \neq 0$. Therefore, we obtain:

$$f(x)^n - f(x)^k = px \quad \text{for every } x \text{ in } Z \tag{4}$$

Let us define: $g(x) = \frac{x^n - x^k}{p}$ $(x \in \mathbb{R})$. We have by (4):

$$g(f(Z)) = Z \tag{5}$$

Let us suppose $Z = \mathbb{R}$. Since f has the Darboux property, $f(Z)$ is an interval of $\mathbb{R}$ which does not contain 0. So, $f(Z)$ is contained either in $(-\infty, 0)$ or in $(0, +\infty)$. It is not difficult to see that neither $g((-\infty, 0))$ nor $g((0, +\infty))$ is equal to $\mathbb{R}$. This contradicts (5).

Therefore, we have $Z \neq \mathbb{R}$. So, there exists x_o in $\mathbb{R}$ such that $f(x_o) = 0$. By taking $x = y = x_o$ in (1), we get:

$$f(0) = 0 \tag{6}$$

So, Z does not contain 0. Then, by (5), $f(Z)$ does not contain 1. Since f has the Darboux property, $f(\mathbb{R}) = f(Z) \cup \{0\}$ is an interval of $\mathbb{R}$ which does not contain 1. (6) implies that we have:

$$f(\mathbb{R}) \subset (-\infty, 1). \tag{7}$$

Let us prove that Z is necessarily unbounded. The continuity of the function $f(\cdot)^n$ (Corollary 2) implies that Z is a nonempty open subset of $\mathbb{R}$. Let (α, β) be the greatest open interval included in Z. If α and β are finite real numbers, the definition of (α, β) implies: $f(\alpha) = f(\beta) = 0$. Since f is one-to-one on Z, f is one-to-one on (α, β) and has the Darboux property . Therefore, f is continuous on (α, β) and strictly monotonic (cf. [6]). This contradicts: $f(\alpha) = f(\beta) = 0$. We deduce that the interval (α, β) is unbounded. Therefore, Z is an unbounded subset of $\mathbb{R}$.

By (5), we deduce that $f(Z)$ is unbounded. By (6) and (7), we get:

$$(-\infty, 0] \subset f(\mathbb{R}) \tag{8}$$

If k and n are both odd or both even, (5) implies that $f(Z)$ does not contain -1, which contradicts (8). Therefore, either k is odd and n is even, or k is even and n is odd. By (5) and (8), we have either $(-\infty, 0) \subset Z$ or $(0, +\infty) \subset Z$. We deduce:

$$\mathbb{R} \setminus Z \subset [0, +\infty) \quad \text{or} \quad \mathbb{R} \setminus Z \subset (-\infty, 0]. \tag{9}$$

Let us suppose for example that k is odd and n is even. We define:

$$A_k = \{f(x)^k;\ x \in Z\}.$$

(8) implies:

$$(-\infty, 0) \subset A_k \tag{10}$$

By (1) and the hypotheses on ϕ, we have: $f(f(y)^k x) = 0$ $(x \in \mathbb{R} \setminus Z,\ y \in Z)$. We deduce: $A_k \cdot (\mathbb{R} \setminus Z) \subset \mathbb{R} \setminus Z$ where $A_k \cdot (\mathbb{R} \setminus Z) = \{xy;\ x \in A_k,\ y \in \mathbb{R} \setminus Z\}$. By (9) and (10), this implies: $\mathbb{R} \setminus Z = \{0\}$ or $Z = \mathbb{R} \setminus \{0\}$, which is impossible by (5) and (7), which completes the proof of Proposition 4.

LEMMA 5

Under the Hypothesis 3, if f is a nonidentically zero solution of (1) *in* $\mathcal{DB}_1$*, there exists a nontrivial interval* $[a,b]$ *such that f is constant and nonzero on* $[a,b]$.

Proof. By Proposition 4, f is not one-to-one on Z. Therefore, there exist a,b in $\mathbb{R}$, $a<b$, such that $f(a)=f(b)\neq 0$.

Let us suppose that f is not constant on $[a,b]$. Since f has the Darboux property, the function $f(\cdot)^k$ is not constant on $[a,b]$, and there exists c in (a,b) such that $f(c)^k \neq f(a)^k$. Without loss of generality, we may suppose $n<k$. We consider the following function:

$$V(x,y,z)=\frac{f(x)^n}{x(y-z)+f(y)^k-f(z)^k} \quad (x\in\mathbb{R}\setminus\{0\};\ y,z\in\mathbb{R})$$

We shall prove first that:

$$\text{for every } r>0,\ \text{there exists } x_r \text{ in } Z\setminus\{0\} \text{ such that } \left|\frac{f(x_r)^n}{x_r}\right|<r \tag{11}$$

For an indirect proof of (11), let us suppose that there exists a positive real number r such that:

$$\left|\frac{f(x)^n}{x}\right|\geqslant r \quad \text{for every } x \text{ in } Z\setminus\{0\} \tag{12}$$

Let us suppose that Z is bounded. Since the function $f(\cdot)^n$ is continuous and not identically zero by Corollary 2, Z is a nonempty open subset of $\mathbb{R}$. So, $\sup Z$ and $\inf Z$ are not both equal to 0. Let a be an element of $\{\sup Z, \inf Z\}\setminus\{0\}$. There exists a sequence $\{x_p\}_{p\in\mathbb{N}}$ in $Z\setminus\{0\}$ converging to a. The definition of Z and the continuity of the function $f(\cdot)^n$ imply that the sequence $\left\{\frac{f(x_p)^n}{x_p}\right\}_{p\in\mathbb{N}}$ converges to 0. This contradicts (12).

Therefore, Z is unbounded and there exists a sequence $\{x_p\}_{p\in\mathbb{N}}$ in $Z\setminus\{0\}$ such that $\{|x_p|\}_{p\in\mathbb{N}}$ tends to ∞. By (1), we have for all x in $Z\setminus\{0\}$ and for every p in $\mathbb{N}$:

$$\left|\frac{f(\varphi(x,x_p))^n}{\varphi(x,x_p)}\right| = \left|\frac{\phi(f(x).f(x_p))}{f(x_p)^n}\right| \frac{1}{|f(x_p)^{k-n}x+f(x)^n f(x_p)^{-n}x_p|}$$

By (12) and since $n<k$, the sequences $\{|f(x_p)|\}_{p\in\mathbb{N}}$ and $\{|f(x_p)|^{k-n}\}_{p\in\mathbb{N}}$ tend to ∞. We have also by (12): $|x_p f(x_p)^{-n}|\leqslant \frac{1}{r}$ for every p in $\mathbb{N}$. We deduce by the hypothesis (iii) on ϕ that the sequence $\left\{\left|\frac{f(\varphi(x,x_p))^n}{\varphi(x,x_p)}\right|\right\}_{p\in\mathbb{N}}$ converges to 0. This contradicts (12).

Therefore, we have proved (11).

By (11), there exists x_o in $Z \setminus \{0\}$ such that:

$$\left|\frac{f(x_o)^n}{x_o}(c-a)\right| < |f(c)^k - f(a)^k| \quad \text{and} \quad \left|\frac{f(x_o)^n}{x_o(b-c)}\right| < |f(b)^k - f(c)^k|$$

We deduce:

$$\operatorname{sgn} V(x_o, c, a) = \operatorname{sgn}(f(c)^k - f(a)^k) \quad \text{and} \quad \operatorname{sgn} V(x_o, b, c) = \operatorname{sgn}(f(b)^k - f(c)^k)$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Since $\operatorname{sgn}(f(c)^k - f(a)^k) \neq \operatorname{sgn}(f(b)^k - f(c)^k)$, we have:

either

$$\operatorname{sgn} V(x_o, b, a) \neq \operatorname{sgn} V(x_o, c, a) \tag{13}$$

or

$$\operatorname{sgn} V(x_o, b, a) \neq \operatorname{sgn} V(x_o, b, c) \tag{14}$$

Let us suppose that (13) occurs (when (14) does, the argument is similar). By Corollary 2, the function: $y \in \mathbb{R} \to V(x_o, y, a)$ is continuous. Therefore, (13) implies that there exists y_o in (c, b) such that $V(x_o, y_o, a) = 0$. We deduce: $\varphi(x_o, y_o) = \varphi(x_o, a)$. This is impossible since, by Lemma 1, the function $\varphi(x_o, \cdot)$ is one-to-one.

Therefore, f is constant on the interval $[a, b]$.

THEOREM 6

Under the Hypothesis 3, if f is a solution of (1) *in $\mathcal{DB}_1$, f is a constant function.*

Proof. Let f be a nonidentically zero solution of (1) in $\mathcal{DB}_1$.

By Lemma 5, there exists a nontrivial interval $[a, b]$ such that f is constant and nonzero on $[a, b]$. Let us suppose that there exists x_o in $\mathbb{R}$ such that $f(x_o) \neq f(a)$.

By the Darboux property of f, there exists a nontrivial real interval J included in $f(\mathbb{R})$ which contains $f(a)$, but does not contain 0. On the set $f^{-1}(J)$, we consider the following equivalence relation:

$$x \sim y \iff f(x) = f(y).$$

By the axiom of choice, there exists a bijection g from the quotient set $(f^{-1}(J)/\sim)$ onto a subset Y of $f^{-1}(J)$. The function $f^* : (f^{-1}(J)/\sim) \to J$ defined by: $f^*(g^{-1}(y)) = f(y)$ $(y \in Y)$ is a bijection from $(f^{-1}(J)/\sim)$ onto J. Therefore, we have:

$$\operatorname{Card} Y = \operatorname{Card} J > \operatorname{Card} \mathbb{N} \tag{15}$$

$$f(Y) = f^*(f^{-1}(J)/\sim) = J \tag{16}$$

$$f(x) \neq f(y) \quad \text{for } x, y \in Y, x \neq y. \tag{17}$$

By (1), we have:

$$f(\varphi(x,y)) = \phi(f(a), f(y)) \quad (x \in [a,b],\ y \in \mathbb{R}) \tag{18}$$

By (16) and Lemma 1, for each y in Y, $\varphi([a,b],y)$ is a nontrivial real interval since J does not contain 0. Moreover, let y and z be distinct elements of Y and let us suppose that the intervals $\varphi([a,b],y)$ and $\varphi([a,b],z)$ are not disjoint. We have by (18):

$$\phi(f(a), f(y)) = \phi(f(a), f(z))$$

If ϕ satisfies (H), we have by the property (ii) of ϕ: $f(y) = f(z)$ which contradicts (17). If ϕ satisfies (H') we have by the property (ii bis) of ϕ: $|f(y)| = |f(z)|$ but, since J does not contain 0, (16) implies that $f(y)$ and $f(z)$ have the same sign. Therefore, we have again: $f(y) = f(z)$ which contradicts (17). So, if y and z are distinct elements of Y, the intervals $\varphi([a,b],y)$ and $\varphi([a,b],z)$ are disjoint. Therefore, $\mathcal{A} = \{\varphi([a,b],y);\ y \in Y\}$ is a family of disjoint nontrivial real intervals. By (15), we have: $\operatorname{card}\mathcal{A} = \operatorname{card} Y > \operatorname{card} \mathbb{N}$. This is impossible. Therefore, f is a nonzero constant function on $\mathbb{R}$. This completes the proof of Theorem 6.

From Theorem 6, we deduce the following result concerning the solutions of (1) in $\mathcal{LDB}_1$:

THEOREM 7

Under the Hypothesis 3, if E is a real vector space and if f is a solution of (1) *in $\mathcal{LDB}_1$, f is a constant function.*

Proof. Let $f : E \to \mathbb{R}$ be a solution of (1) in $\mathcal{LDB}_1$. It is easy to see that, for every x in E, the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by (P) is a solution of (1) in $\mathcal{DB}_1$. By Theorem 6, f_x is a constant function. Therefore, we have: $f(x) = f_x(1) = f_x(0) = f(0)$ for every x in E. So, f is a constant function.

4. Examples and conclusion

The following functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfy the set (H) of hypotheses:

$\phi(x,y) = \lambda xy \quad (x, y \in \mathbb{R})$ where λ is a given nonzero real number.

$\phi(x,y) = \lambda xy\,(1 + |x| + |y|) \quad (x, y \in \mathbb{R})$ where λ is a given nonzero real number.

$\phi(x,y) = \arctan(xy) \quad (x, y \in \mathbb{R})$

$\phi(x,y) = \operatorname{arcsinh}(xy) \quad (x, y \in \mathbb{R})$.

For the first example, we have already noticed that, for *any given nonnegative integers k and n*, all the solutions of (1) in $\mathcal{DB}_1$ and all the continuous solutions $f : E \to \mathbb{R}$ of (1) have been obtained, when E is a real topological vector space.

But, for the last three examples, it looks very difficult to solve the functional equation (1) in the case $k = n$.

The reason is that, in the first example, we obtain , when $k = n$, for the function $g(x) = f(x)^n$ $(x \in \mathbb{R})$, a functional equation for which we know the solutions in $\mathcal{DB}_1$. For another ϕ, we generally obtain only the result of Theorem 7 for $k \neq n$.

Now, the following functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfy the set (H') of hypotheses:

$\phi(x,y) = \lambda|x||y| \quad (x, y \in \mathbb{R})$ where λ is a given nonzero real number.

$\phi(x,y) = \lambda\,|x|^\alpha|y|^\alpha \quad (x, y \in \mathbb{R})$ where λ is a given nonzero real number and α is given in $(0,1)$.

$\phi(x,y) = \ln(1 + x^2y^2) \quad (x, y \in \mathbb{R})$.

For the first example, we may use Theorems 5 and 6 and the results of [4] concerning the functional equation (2) with f replaced by $|f|$. For this case, we obtain in this way, for any given *even* integers k and n, all the solutions of (1) in $\mathcal{DB}_1$ and all the continuous solutions $f : E \to \mathbb{R}$ of (1) when E is a real topological vector space. It does not seem easy to obtain the solutions of (1) for arbitrary integers k and n since, in particular in this case, the function $\varphi(x,\cdot)$ is not necessarily one-to-one.

We notice now that, as in the case of the hypotheses (H) and for the same reason, it looks difficult, for the two last examples, to solve the functional equation (1) in the case $k = n$. In the present paper we have proved that the functional equation (1) has only constant solutions in $\mathcal{LDB}_1$ when the integers k and n are distinct.

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