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A functional inequality related to Böttcher's equation

Dedicated to Professor Zenon Moszner
with cordial wishes on his seventieth birthday

Abstract. There is proved a theorem on the form of continuous solutions of the iterative functional inequality (1) in the case where the function f in (1) possesses an asymptotic property at the origin. The formula we present involves continuous solutions to the Böttcher functional equation (2) and those of inequality (3). Comments concerning a related Schröder's equation are added.

We consider the functional inequality

$$b(f(x)) \leqslant [b(x)]^p \tag{1}$$

and the Böttcher equation

$$\beta(f(x)) = [\beta(x)]^p, \tag{2}$$

where $p > 1$ is given, f is a given function, b and β are unknown functions.

We take

$$0 < a \leqslant 1,$$

put

$$I := (0, a); \quad J := [0, a),$$

and assume the following hypotheses.

($\mathbf{H_1}$) The function $f : J \to J$ is continuous on J, $0 < f(x) < x$ in I.

($\mathbf{H_2}$) There is a $c > 0$ such that (with a $p > 1$)

$$f(x) = cx^p + o(x^p), \quad x \to 0+ .$$

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We denote by (f^n) the sequence of iterates of the function f:

$$f^0(x) = x; \quad f^n(x) = f(f^{n-1}(x)), \quad x \in J, \ n \in \mathbb{N}.$$

REMARK 1

If hypothesis $(\mathbf{H_1})$ is satisfied, then

$$\lim_{n\to\infty} f^n(x) = 0, \quad x \in I,$$

whence $f(0) = 0$. Moreover,

$$0 < f^n(x) < x; \quad x \in I, \ n \in \mathbb{N}. \tag{$*$}$$

REMARK 2

If hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are fulfilled, then the function $F : J \to \mathbb{R}$, defined by

$$F(x) = x^{-p} f(x), \quad x \in I; \quad F(0) = c, \tag{$**$}$$

is positive and continuous on J.

We shall need the notion of $\{f\}$-monotonic functions.

DEFINITION

Let hypothesis $(\mathbf{H_1})$ be fulfilled. A function $\eta : J \to \mathbb{R}$ is said to be $\{f\}$ *-increasing* in J if

$$\eta(f(x)) \geqslant \eta(x), \quad x \in J \tag{3}$$

It will be called $\{f\}$*-decreasing* when it satisfies (3) but with the inequality sign reversed.

Before we formulate a representation theorem for inequality (1) we quote here a theorem of this kind concerning the inequality

$$m(f(x)) \geqslant g(x)m(x) \tag{4}$$

which has been proved in [1], cf. also [2], Th. 12.2.4, p. 476.

LEMMA

Let hypothesis $(\mathbf{H_1})$ *be fulfilled and let* $g : J \to \mathbb{R}$ *be a continuous function on* J. *Suppose that there exists the function* $G : J \to (0, \infty)$ *defined by*

$$G(x) := \lim_{n\to\infty} \prod_{i=0}^{n-1} g[f^i(x)], \quad x \in J, \tag{5}$$

and it is continuous in J.

If $m : J \to \mathbb{R}$ *is a continuous solution of inequality* (4), *then there is a unique* $\{f\}$*-increasing continuous function* $\eta : J \to \mathbb{R}$ *such that*

$$m(x) = \frac{\eta(x)}{G(x)}, \quad x \in J. \tag{6}$$

Our theorem reads.

THEOREM

Let hypotheses $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *be fulfilled, and let* $b : I \to (0, \infty)$ *be a continuous solution to* (1).

If there exists a continuous solution $\beta : I \to (0, \infty)$ *of equation* (2) *such that*

$$\lim_{x \to 0+} \frac{b(x)}{\beta(x)} > 0, \tag{7}$$

then

$$b(x) = \beta(x) \exp[\eta(x)(\log x + P(x))], \quad x \in I, \tag{8}$$

where $\eta : J \to \mathbb{R}$ *is a unique* $\{f\}$*-increasing function such that* $\eta(0) = 0$ *and*

$$P(x) := \sum_{n=0}^{\infty} p^{-n-1} \log F[f^n(x)], \quad x \in J. \tag{9}$$

Proof. Take b and β as stated in the Theorem and put

$$m(x) := \frac{\log b(x) - \log \beta(x)}{\log x}, \quad x \in I; \quad m(0) = 0. \tag{10}$$

Since, by (7), $\lim_{x \to 0+} m(x) = 0$, the function m defined by (10) is continuous on J.

We have from (10):

$$b(x) = \beta(x) x^{m(x)}, \quad x \in I. \tag{11}$$

whence, by (1) and (2),

$$\begin{aligned} \beta(x)^p f(x)^{m(f(x))} &= b(f(x)) \\ &\leqslant b(x)^p \\ &= \beta(x)^p x^{pm(x)}, \end{aligned}$$

that is,

$$m(f(x)) \log f(x) \leqslant (p \log x) m(x).$$

whenever $x \in I$.

Denote

$$g(x) := \frac{p \log x}{\log f(x)}, \quad x \in I; \quad g(0) := 1. \tag{12}$$

The continuity of F and f at zero implies that $\log f(x)$ tends to $-\infty$ and $\log F(x)$ approaches $\log c$ when $x \to 0+$. Thus from (12) and (**) we get

$$\begin{aligned}\lim_{x\to 0+} g(x) &= \lim_{x\to 0+} \frac{\log x^p}{\log f(x)}\\ &= \lim_{x\to 0+} \frac{\log f(x) - \log F(x)}{\log f(x)}\\ &= 1,\end{aligned}$$

so that function g defined by (12) is continuous on J. Since $J \subset [0,1)$, it is also positive on J. Consequently, the function m defined by (10) is a continuous solution (on J) of inequality (4), where the function g is given by (12).

Let us put (cf. (5))

$$G_n(x) := \prod_{i=0}^{n-1} g[f^i(x)], \quad x \in J,\ n \in \mathbb{N}.$$

It is readily seen from (12) that

$$G_n(x) = p^n \frac{\log x}{\log f^n(x)}, \quad x \in I,\ n \in \mathbb{N}; \quad G_n(0) = 1. \tag{13}$$

In view of (**) we have

$$f^n(x) = x^{p^n} \prod_{i=0}^{n-1} F(f^i(x))^{p^{n-1-i}}, \quad x \in I,\ n \in \mathbb{N}.$$

Hence

$$\frac{G_n(x)}{\log x} = \left(\log x + \sum_{i=0}^{n-1} p^{-i-1} \log F(f^i(x))\right)^{-1}, \quad x \in I,\ n \in \mathbb{N}. \tag{14}$$

On every interval $[0,t] \subset J$ the (continuous) function $|\log F| : J \to \mathbb{R}$ is bounded, by $M = M(t) > 0$, say , so that

$$|p^{-i-1} \log F(f^i(x))| \leqslant Mp^{-i-1}, \quad x \in [0,t],\ i \in \mathbb{N}.$$

Since $p > 1$, the series in (9) converges uniformly on [0,t], consequently its sum $P : J \to \mathbb{R}$ is continuous on J. In view of (14), the limit $G : J \to \mathbb{R}$ of the sequence (13) is given by

$$G(x) = \frac{\log x}{\log x + P(x)}, \quad x \in I; \quad G(0) = 1. \tag{15}$$

where $P(x)$ is defined by (9). and it represents a continuous function on J (as P does, the continuity of G at the origin resulting by passing to the limit in (15) when $x \to 0+$). The function G does not vanish on J, and by (13) it is positive on J.

We are in position to apply the Lemma which yields

$$m(x) = \frac{\eta(x)}{G(x)} = \frac{\eta(x)}{\log x}(\log x + P(x)), \quad x \in I. \tag{16}$$

where $\eta : J \to \mathbb{R}$ obeys the requirements stated in the Theorem. Equality (11) rewritten in the form

$$b(x) = \beta(x) \exp [m(x) \log x]$$

and (16) imply formula (8).

This completes the proof of the Theorem.

Remark 3

Limit (7) exists, in particular, if the function b has the property

$$b(x) = dx^r + o(x^r), \quad x \to 0+,$$

where r is a real number and $d > 0$, cf. [2], Th. 8.3.3., p. 341.

The following proposition contains a result in a sense converse to that from our Theorem.

Proposition

If $f : J \to J$ is a function satisfying

$$0 < f(x) < x^p, \quad x \in I, \tag{17}$$

with a $p > 1$, $\beta : I \to \mathbb{R}$ is a solution to equation (2), *$\eta : I \to \mathbb{R}$ is an $\{f\}$-increasing function and the series in* (9) *converges in I, then the function $\beta : J \to \mathbb{R}$, defined by* (8) *(with* (9)*) is is a solution of inequality* (1).

Proof. We first check that the function $P : I \to \mathbb{R}$ defined by (9) satisfies the equation

$$P(f(x)) = pP(x) - \log F(x). \tag{18}$$

Indeed, we have:

$$\begin{aligned} P(f(x)) &= \sum_{n=0}^{\infty} p^{-n-1} \log F(f^{n+1}(x)) \\ &= p \sum_{m=1}^{\infty} p^{-m-1} \log F(f^m(x)) \\ &= pP(x) - \log F(x), \end{aligned}$$

as claimed. Note also that, because of (17), the function F defined by (**) maps I into the interval $(0, 1]$.

We now calculate, for $x \in I$,

$$b(f(x))b(x)^{-p} = \exp\{\eta(f(x))[\log f(x) + P(f(x))] - p(\log x + P(x))\}.$$

Since both $\log f$ and $\log F$ are not positive, P satisfies (18) and F is given by $(**)$, we get consecutively

$$\begin{aligned} b(f(x))b(x)^{-p} &\leqslant \exp\{\eta(x)[\log f(x) + P(f(x))] - p(\log x + P(x))\} \\ &= \exp[\eta(x)(\log x^{-p} f(x)) - \log F(x)] \\ &= 1. \end{aligned}$$

Thus inequality (1) is satisfied in I. Since $b(0) = 0$ (cf. (8)), (1) holds in J.

COMMENTS

Since there exists the limit G (cf. (15)) of the sequence (G_n) (cf. (13)), the limit

$$s(x) := -\lim_{n\to\infty} p^{-n} \log f^n(x), \quad x \in I,$$

does as well. Put $x = e^{-t}$, $t \in (0, +\infty)$ and consider the function $h : (0, +\infty] \to (0, +\infty]$ defined as follows:

$$h(t) = -\log[f(e^{-t})], \quad t \in (0, +\infty); \quad h(+\infty) = +\infty.$$

With the conventions: $e^{-\infty} = 0$, $\log 0 = -\infty$, we arrive at the formula

$$h^n(t) = -\log[f^n(e^{-t})], \quad t \in (0, +\infty], \ n \in \mathbb{N} \cup \{0\}.$$

Therefore the *Koenigs algorithm* (cf. [2], p. 367)

$$H(t) := \lim_{n\to\infty} p^{-n} h^n(t) = \lim_{n\to\infty} s_n(e^{-t}), \quad t \in (0, +\infty); \quad H(+\infty) := +\infty \tag{19}$$

works, and yields the *principal solution* (cf. [2], p. 366) of the Schröder equation

$$\chi[h(t)] = p\chi(t). \tag{20}$$

The solution χ of (20) on $(0, +\infty]$ are *conjugate* (cf. [2], p. 332) with those β of (2) on J:

$$\chi(t) = -\log\beta(e^{-t}), \quad t \in (0, +\infty].$$

To summarize: our Theorem implies the following.

COROLLARY

Assume that $h : (0, +\infty] \to (0, +\infty]$ *is a continuous function on* $(0, +\infty)$, $h(+\infty) = +\infty$, *satisfying the condition*

$$h(t) > t, \quad t \in (0, +\infty).$$

If the function $f:[0,1)\to[0,1)$ *defined by*

$$f(x)=\exp\left[-h(-\log x)\right],\qquad x\in[0,1) \tag{21}$$

satisfies Hypothesis $(\mathbf{H_2})$, *then the Schröder equation* (20) *possesses the principal solution* $H:(0,+\infty]\to\mathbb{R}$ *which is given by the formula* (19).

REMARK 4

The asymptotic condition (cf. $(\mathbf{H_2})$) for the function f defined by (21) implies the following one:

$$h(t)=pt+o(t),\qquad t\to+\infty. \tag{22}$$

However, the converse implication does not hold as it is seen from the following example:

For

$$h(t)=pt+\sqrt{t},\qquad t\in[0,+\infty)$$

relation (22) holds true, but we have (cf. (21))

$$f(x)=x^p\exp\left(-\sqrt{-\log x}\right),$$

so that

$$\lim_{x\to0+}x^{-p}f(x)=\lim_{t\to+\infty}\exp\left(-\sqrt{t}\right)=0.$$

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We owe the suggestion of adding the Proposition and the example above to a referee whom we are also thankful for other valuable remarks and corrections.

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