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The rotation number of the composition of homeomorphisms

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. In this paper it is shown that if F and G are commuting orientation-preserving homeomorphisms of the unit circle, then

$$\alpha(F \circ G) = \alpha(F) + \alpha(G) \pmod 1,$$

where $\alpha(F)$ denotes the rotation number of F.

Let $\mathbf{S^1}$ be the unit circle with a positive orientation. In the sequel, F^n stands for the n-th iterate of the function F.

Write

$$t(x) := x + 1 \quad \text{for all } x \in \mathbb{R}$$

and observe that the following facts follow immediately by induction:

Remark 1

If $f : \mathbb{R} \to \mathbb{R}$ is a bijection such that $f \circ t = t \circ f$, then

$$f^n \circ t^m = t^m \circ f^n \quad \text{for all } n, m \in \mathbb{Z}.$$

Lemma 1

If $f, g : \mathbb{R} \to \mathbb{R}$ are bijections such that $f \circ t = t \circ f$, $g \circ t = t \circ g$ and $f \circ g = t^k \circ g \circ f$ for some $k \in \mathbb{Z}$, then

(i) $f^n \circ g = t^{kn} \circ g \circ f^n$ *for all* $n \in \mathbb{Z}$,

(ii) $(f \circ g)^n = t^{\frac{n(n+1)}{2}k} \circ g^n \circ f^n$ *for all* $n \in \mathbb{N}$.

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Remark 2

If $f:\mathbb{R}\to\mathbb{R}$ is a bijection such that $f\circ t=t\circ f$ and $f^n(x_0)=t^m(x_0)$ for some $n,m\in\mathbb{Z}$, $x_0\in\mathbb{R}$, then

$$f^{nk}(x_0)=t^{mk}(x_0)\quad \text{for all } k\in\mathbb{Z}.$$

It is known (see for instance [1]) that for every homeomorphism $F:\mathbf{S}^1\to\mathbf{S}^1$ there exists a homeomorphism $f:\mathbb{R}\to\mathbb{R}$ such that $F(e^{2\pi ix})=e^{2\pi if(x)}$ for all $x\in\mathbb{R}$ and

$$f\circ t=t\circ f,\quad \text{if } f \text{ is strictly increasing}$$

and

$$f\circ t=t^{-1}\circ f,\quad \text{if } f \text{ is strictly decreasing.}$$

We will say that the function f *represents* the homeomorphism F. If f is strictly increasing we will say that the homeomorphism F *preserves orientation.*

If $F:\mathbf{S}^1\to\mathbf{S}^1$ is an orientation-preserving homeomorphism represented by a function f then the number $\alpha(F)\in[0,1)$ defined by

$$\alpha(F):=\lim_{n\to\infty}\frac{f^n(x)}{n}\ (\mathrm{mod}\ 1),\quad x\in\mathbb{R}$$

is said to be the rotation number of F. This limit always exists and does not depend on x and f.

Assume that $f:\mathbb{R}\to\mathbb{R}$ represents an orientation-preserving homeomorphism $F:\mathbf{S}^1\to\mathbf{S}^1$ with an irrational rotation number. Since f and t are commuting homeomorphisms without fixed points, we see that for every $x\in\mathbb{R}$ there exists a unique sequence $\{s_k(x)\}_{k\in\mathbb{N}}\subset\mathbb{Z}$ such that

$$t^{s_k(x)+1}(x)\geqslant f^k(x)>t^{s_k(x)}(x)\quad \text{for all } k\in\mathbb{N}. \tag{1}$$

Moreover, there exists a finite non-zero limit

$$\lim_{k\to\infty}\frac{s_k(x)}{k}=:\nu(t,f),$$

which actually does not depend on x (see [3]). Let us mention here that another way of defining $\nu(t,f)$ can be found in [2].

Lemma 2

Let $f:\mathbb{R}\to\mathbb{R}$ represent an orientation-preserving homeomorphism $F:\mathbf{S}^1\to\mathbf{S}^1$ such that $\alpha(F)\notin\mathbb{Q}$. Then

$$\alpha(F)=\nu(t.f)\ (\mathrm{mod}\ 1).$$

Proof. Fix an $x \in \mathbb{R}$ and observe that from (1) we have

$$\frac{x + s_k(x) + 1}{k} \geqslant \frac{f^k(x)}{k} > \frac{x + s_k(x)}{k} \quad \text{for all } k \in \mathbb{N} \setminus \{0\}.$$

Letting $k \longrightarrow \infty$ gives our assertion.

We can now prove our main result.

THEOREM 1

Let $F, G : \mathbf{S}^1 \to \mathbf{S}^1$ be commuting orientation-preserving homeomorphisms. Then

$$\alpha(F \circ G) = \alpha(F) + \alpha(G) \pmod 1. \tag{2}$$

Moreover, the homeomorphisms $f, g : \mathbb{R} \to \mathbb{R}$ which represent F and G, respectively, also commute.

Proof. Since $f \circ g$ represents $F \circ G$ and $g \circ f$ represents $G \circ F$, there exists a $k \in \mathbb{Z}$ for which $f \circ g = t^k \circ g \circ f$.

We distinguish two cases.

1°. Either $\alpha(F)$ or $\alpha(G)$ is rational.

Since F and G commute, we can assume that $\alpha(F) \in \mathbb{Q}$. Therefore $f^q(x_0) = t^r(x_0)$ for some $q \in \mathbb{N} \setminus \{0\}$, $x_0 \in \mathbb{R}$ and $r \in \mathbb{Z}$, which gives $\alpha(F) = \frac{r}{q} \pmod 1$ (see for instance [1]). Moreover, Remark 2 yields

$$f^{qn}(x_0) = t^{rn}(x_0) \quad \text{for all } n \in \mathbb{Z}. \tag{3}$$

Fix an $n \in \mathbb{N} \setminus \{0\}$. According to Lemma 1(ii), (3) and Remark 1, we have

$$\begin{aligned}
\frac{(f \circ g)^{qn}(x_0)}{qn} &= \frac{\left(t^{\frac{qn(qn+1)}{2}k} \circ g^{qn} \circ f^{qn}\right)(x_0)}{qn} \\
&= \frac{qn+1}{2}k + \frac{g^{qn}(t^{rn}(x_0))}{qn} \\
&= \frac{qn+1}{2}k + \frac{t^{rn}(g^{qn}(x_0))}{qn} \\
&= \frac{qn+1}{2}k + \frac{r}{q} + \frac{g^{qn}(x_0)}{qn}.
\end{aligned}$$

Letting n tend to infinity and using the relations

$$\lim_{n \to \infty} \frac{(f \circ g)^{qn}(x_0)}{qn} = \alpha(F \circ G) \pmod 1, \quad \lim_{n \to \infty} \frac{g^{qn}(x_0)}{qn} = \alpha(G) \pmod 1$$

and $\frac{r}{q} = \alpha(F) \pmod 1$, we deduce that $k = 0$ and (2) holds.

2°. Both $\alpha(F)$ and $\alpha(G)$ are irrational.

Fix an $x_0 \in \mathbb{R}$ and let sequences $\{s_k(x_0)\}_{k\in\mathbb{N}}, \{p_k(x_0)\}_{k\in\mathbb{N}} \subset \mathbb{Z}$ be such that

$$t^{s_k(x_0)+1}(x_0) \geqslant f^k(x_0) > t^{s_k(x_0)}(x_0) \quad \text{for all } k \in \mathbb{N},$$
$$t^{p_k(x_0)+1}(x_0) \geqslant g^k(x_0) > t^{p_k(x_0)}(x_0) \quad \text{for all } k \in \mathbb{N}.$$

Lemma 1(ii) and Remark 1 now show that for a fixed $n \in \mathbb{N}\setminus\{0\}$ we have

$$\begin{aligned}(f\circ g)^n(x_0) &= \left(t^{\frac{n(n+1)}{2}k} \circ g^n \circ f^n\right)(x_0)\\ &\leqslant \left(t^{\frac{n(n+1)}{2}k} \circ g^n \circ t^{s_n(x_0)+1}\right)(x_0)\\ &= \left(t^{\frac{n(n+1)}{2}k+s_n(x_0)+1} \circ g^n\right)(x_0)\\ &\leqslant \left(t^{\frac{n(n+1)}{2}k+s_n(x_0)+1} \circ t^{p_n(x_0)+1}\right)(x_0)\\ &= t^{\frac{n(n+1)}{2}k+s_n(x_0)+p_n(x_0)+2}(x_0).\end{aligned}$$

In the same manner we can see that

$$(f\circ g)^n(x_0) > t^{\frac{n(n+1)}{2}k+s_n(x_0)+p_n(x_0)}(x_0).$$

These inequalities yield

$$\begin{aligned}\frac{n+1}{2}k + \frac{s_n(x_0)}{n} + \frac{p_n(x_0)}{n} + \frac{x_0}{n} &< \frac{(f\circ g)^n(x_0)}{n}\\ &\leqslant \frac{n+1}{2}k + \frac{s_n(x_0)}{n} + \frac{p_n(x_0)}{n} + \frac{2}{n} + \frac{x_0}{n}.\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{(f\circ g)^n(x_0)}{n} = \alpha(F\circ G) \pmod 1, \quad \lim_{n\to\infty}\frac{s_n(x_0)}{n} = \nu(t,f)$$

and

$$\lim_{n\to\infty}\frac{p_n(x_0)}{n} = \nu(t,g),$$

the last relation shows that $k=0$ and

$$\nu(t,f) + \nu(t,g) = \alpha(F\circ G) \pmod 1.$$

Lemma 2 now leads to (2).

As an immediate consequence of Theorem 1 we have the following

COROLLARY 1

Let $F : \mathbf{S}^1 \to \mathbf{S}^1$ be an orientation-preserving homeomorphism. Then

$$\alpha(F^{-1}) = \begin{cases} 0 & \text{if } \alpha(F) = 0\\ 1-\alpha(F) & \text{if } \alpha(F) \neq 0\end{cases}.$$

References

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