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Matkowski-Sutô type problem for conjugate arithmetic means

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. Let $I \subset \mathbb{R}$ be an open interval and let $CM(I)$ denote the set of all continuous and strictly monotonic real functions on I. For $\varphi \in CM(I)$ we define $A^*_\varphi(x,y) := \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(\frac{x+y}{2}))$ for all $x, y \in I$, which is called a conjugate arithmetic mean on I. In this paper we solve the functional equation $A^*_\varphi(x,y) + A^*_\psi(x,y) = x + y$ for all $x, y \in I$, where $\varphi, \psi \in CM(I)$ and either φ or ψ is twice continuously differentiable on I.

1. Introduction

Let $I \subset \mathbb{R}$ be an open interval. We say that a function $M : I^2 \to I$ is a *mean* on I if it satisfies the following conditions

(i) If $x \neq y$ and $x, y \in I$ then $\min\{x,y\} < M(x,y) < \max\{x,y\}$;

(ii) $M(x,y) = M(y,x)$ if $x, y \in I$;

(iii) M is continuous on I^2.

The best-known mean is the arithmetic mean

$$A(x,y) := \frac{x+y}{2} \quad (x, y \in I), \tag{1.1}$$

which is defined on any interval I.

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Let $CM(I)$ denote the set of all continuous and strictly monotonic real functions on I. The following definition is well known ([1], [2], [6], [11]).

Definition 1

A function $M : I^2 \to I$ is called a *quasi-arithmetic mean* on I if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) =: A_\varphi(x,y) \quad (x,y \in I). \tag{1.2}$$

Then the function $\varphi \in CM(I)$ is called the *generating function* of the quasi-arithmetic mean (1.2) on I.

Definition 2 ([3])

A function $M : I^2 \to I$ is called a *conjugate arithmetic mean* on I if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right) =: A_\varphi^*(x,y) \quad (x,y \in I). \tag{1.3}$$

Then the function $\varphi \in CM(I)$ is called the *generating function* of the conjugate arithmetic mean (1.3) on I.

Definition 3

Let $\varphi, \psi \in CM(I)$. If there exist real constants $\alpha \neq 0$ and β such that

$$\varphi(x) = \alpha\psi(x) + \beta \quad \text{if } x \in I, \tag{1.4}$$

then we say that φ is *equivalent* to ψ on I; and, in this case, we write: $\varphi \sim \psi$ on I or $\varphi(x) \sim \psi(x)$ if $x \in I$.

The following result is known ([6], [1], [2], [10], [3]).

Theorem 1

If $\varphi, \psi \in CM(I)$ then the equation $A_\varphi \equiv A_\psi$ or $A_\varphi^* \equiv A_\psi^*$ holds on I^2 if, and only if, φ is equivalent to ψ on I.

Matkowski [11] (see earlier Sutô [12]) raised the following problem: For which quasi-arithmetic means M and N does the equality

$$M + N = 2A, \tag{1.5}$$

that is,

$$M(x,y) + N(x,y) = x + y \tag{1.6}$$

hold on I for all $x, y \in I$?

The problem has not been solved in this general form yet. Sutô [12] and Matkowski [11] supposed that the generating functions are several times differentiable.

Definition 4

Let $M : I^2 \to I$ be a quasi-arithmetic (conjugate arithmetic) mean on I. If there exists a generating function $\varphi \in CM(I)$ of M with the property T then we say that the quasi-arithmetic (conjugate arithmetic) mean M has the property T on I.

For example, if there exists a continuously differentiable generating function $\varphi \in CM(I)$ such that $M = A_\varphi$ (or $M = A_\varphi^*$) on I^2 then briefly we say that M is a continuously differentiable quasi-arithmetic (or conjugate arithmetic) mean on I.

The above mentioned problem has partially been answered by the following result (Daróczy-Páles [5], see also [4]).

Theorem 2

If the quasi-arithmetic means M and N satisfy (1.5) *(or* (1.6)*) on I^2 and either of them is continuously differentiable on I then there exists $p \in \mathbb{R}$ for which*

$$M(x,y) = S_p(x,y) \quad \textit{and} \quad N(x,y) = S_{-p}(x,y) \quad (x, y \in I), \tag{1.7}$$

where

$$S_p(x,y) := \begin{cases} \frac{x+y}{2} & \textit{if } p = 0, \\ \frac{1}{p} \log\left(\frac{e^{px}+e^{py}}{2}\right) & \textit{if } p \neq 0 \end{cases} \quad (x, y \in I). \tag{1.8}$$

In this paper we examine the Matkowski-Sutô type problem in the class of conjugate arithmetic means.

2. The functional equation of the problem, lemmas

The subject of this paper is the investigation of the following question. Let M and N be conjugate arithmetic means on I satisfying the equation

$$M + N = 2A \tag{2.1}$$

on I^2. Then there exist generating functions $\varphi, \psi \in CM(I)$ for which $M = A_\varphi^*$ and $N = A_\psi^*$, and they fulfil the functional equation

$$A_\varphi^*(x,y) + A_\psi^*(x,y) = x + y \tag{2.2}$$

for all $x, y \in I$. The problem has not been solved in this general form yet. Clearly, it is enough to solve the functional equation (2.2) up to equivalence of the generating functions φ and ψ.

DEFINITION 5

A pair of functions (φ, ψ) is called a Matkowski-Sutô type pair on I if $\varphi, \psi \in CM(I)$ and the functional equation (2.2) holds for all $x, y \in I$.

Now we prove some lemmas.

LEMMA 1

If (φ, ψ) is a Matkowski-Sutô type pair on I and ψ is continuously differentiable on I, $\psi'(x) \neq 0$ if $x \in I$, then φ is continuously differentiable on I.

Proof. By (2.2) the function $\varphi \in CM(I)$ satisfies

$$\varphi(x) = \varphi\left(\frac{x+y}{2}\right) + \varphi(g_2(x,y)) - \varphi(y) \tag{2.3}$$

for all $x, y \in I$, where

$$g_2(x,y) := x + y - \psi^{-1}\left(\psi(x) + \psi(y) - \psi\left(\frac{x+y}{2}\right)\right) \tag{2.4}$$

if $x, y \in I$. By the assumptions of the lemma, $g_2 : I^2 \to I$ is continuously differentiable and

$$\frac{\partial g_2(x,y)}{\partial x} = 1 - \frac{\psi'(x) - \psi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(A^*_\psi(x,y))} \quad (x, y \in I) \tag{2.5}$$

and a similar expression is valid for $\frac{\partial g_2(x,y)}{\partial y}$.

Now we intend to show that φ is differentiable at each point of I. For, let $x_0 \in I$ be arbitrarily fixed. Then from (2.3) we have

$$\varphi(x_0) = \varphi\left(\frac{x_0+y}{2}\right) + \varphi(g_2(x_0,y)) - \varphi(y) \quad (y \in I). \tag{2.6}$$

Define the function $h : I \to \mathbb{R}$ by

$$h(y) := g_2(x_0, y).$$

Then h is continuously differentiable and $h'(x_0) = \frac{1}{2}$. Therefore h is invertible in a neighbourhood U of x_0 and it inverse is also continuously differentiable on $h(U) = V$, which is a neighbourhood of $h(x_0) = x_0$. Hence h^{-1} is locally Lipschitz in V, whence it follows that $h^{-1}(H)$ is of measure zero whenever $H \subset V$ is of measure zero.

By Lebesgue's Theorem, the function φ is differentiable almost everywhere, that is, the complement of the set D where φ is differentiable is of measure zero. Hence $h^{-1}(V \setminus D)$ is also of measure zero. Thus the set $[(2D - x_0) \cap U] \setminus h^{-1}(V \setminus D)$ cannot be empty. Let y_0 be an arbitrary element of this set. Then $y_0 \in U$, $\frac{x_0+y_0}{2} \in D$, and $y_0 \notin h^{-1}(V \setminus D)$, i.e., $h(y_0) \in D$. Thus φ is differentiable at the points $\frac{x_0+y_0}{2}$ and $g_2(x_0, y_0)$. The functions $x \mapsto \frac{x+y_0}{2}$ and $x \mapsto g_2(x, y_0)$ are differentiable at x_0 and, by (2.3),

$$\varphi(x) = \varphi\left(\frac{x+y_0}{2}\right) + \varphi(g_2(x, y_0)) - \varphi(y_0) \quad (x \in I),$$

from which, by the rule of differentiability of the composite function, we have that φ is differentiable at x_0. Thus φ is differentiable on I everywhere.

Differentiating the equation (2.3) with respect to x, and using (2.5), we get

$$\varphi'(x) = \varphi'\left(\frac{x+y}{2}\right)\frac{1}{2} + \varphi'(g_2(x,y))\left(1 - \frac{\psi'(x) - \psi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(A_\psi^*(x,y))}\right) \tag{2.7}$$

for all $x, y \in I$. The equation (2.7) can be written as

$$\varphi'(x) = f(x, y; \varphi'(g_1(x,y)), \varphi'(g_2(x,y))),$$

where $g_1(x,y) := \frac{x+y}{2}$, g_2 is the function defined in (2.4), and

$$f(x, y; u, v) = \frac{1}{2}u + v\frac{\partial g_2(x,y)}{\partial x}$$

for all $x, y \in I$ and $u, v \in \mathbb{R}$. Since $\varphi' : I \to \mathbb{R}$ is a *measurable* function and $\frac{\partial g_1}{\partial y}(x,x) = \frac{1}{2} \neq 0$ and $\frac{\partial g_2}{\partial y}(x,x) = \frac{1}{2} \neq 0$, moreover f is continuous, thus by the theorem of Járai [8] (Theorem 2), φ' is continuous on I. (See also the papers Járai [7] and [9].)

Lemma 2

If (φ, ψ) is a Matkowski-Sutô type pair on I, ψ is twice differentiable on I, and $\psi'(x) \neq 0$ for $x \in I$, then φ is twice differentiable on I and there exists a constant $c \neq 0$ such that

$$\varphi'(x)\psi'(x) = c \tag{2.8}$$

for all $x \in I$.

Proof. Let

$$x \circ y := A_\psi^*(x,y) = \psi^{-1}\left(\psi(x) + \psi(y) - \psi\left(\frac{x+y}{2}\right)\right)$$

for all $x, y \in I$. By Lemma 1, φ is continuously differentiable on I, and thus both sides of the equation

$$\varphi(x) + \varphi(y) - \varphi\left(\frac{x+y}{2}\right) = \varphi(x + y - x \circ y) \quad (x, y \in I)$$

are differentiable on I with respect to x, that is,

$$\varphi'(x) - \varphi'\left(\frac{x+y}{2}\right)\frac{1}{2} = \varphi'(x+y-x\circ y)\left(1 - \frac{\psi'(x) - \psi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(x\circ y)}\right) \tag{2.9}$$

for all $x, y \in I$. From this we have

$$\varphi'(x) + \frac{\varphi'(x+y-x\circ y)}{\psi'(x\circ y)}\psi'(x) =: x\Box y = y\Box x = \varphi'(y) + \frac{\varphi'(x+y-x\circ y)}{\psi'(x\circ y)}\psi'(y),$$

which implies

$$\frac{\varphi'(x+y-x\circ y)}{\psi'(x\circ y)}(\psi'(y) - \psi'(x)) + (\varphi'(y) - \varphi'(x)) = 0$$

for all $x, y \in I$. Dividing the equation by $(y-x)$ $(y \neq x)$ and taking the limit $y \to x$, by the continuity of φ' and ψ', we have that φ is *twice differentiable* and

$$\frac{\varphi'(x)}{\psi'(x)}\psi''(x) + \varphi''(x) = 0.$$

that is, $(\varphi'(x)\psi'(x))' = 0$ if $x \in I$. Therefore, there exists a constant $c \neq 0$ (since φ and ψ are strictly monotonic, $c = 0$ cannot occur) for which $\varphi'(x)\psi'(x) = c$ if $x \in I$.

Lemma 3

If (φ, ψ) is a Matkowski-Sutô type pair on I and ψ is twice differentiable on I and $\psi'(x) \neq 0$ if $x \in I$, then the functional equation

$$(\psi'(y) - \psi'(x)) \times \left(\psi'(A_\psi^*(x,y)) - \psi'(x) - \psi'(y) + \psi'\left(\frac{x+y}{2}\right)\frac{1}{2} + \frac{\psi'(x)\psi'(y)}{2\psi'\left(\frac{x+y}{2}\right)}\right) = 0 \tag{2.10}$$

holds for all $x, y \in I$.

Proof. By Lemma 2, (2.8) holds, from which

$$\varphi'(x) = \frac{c}{\psi'(x)} \quad (x \in I)$$

follows, where $c \neq 0$. Then (2.9) holds, too, with the notation

$$x \circ y := A_\psi^*(x, y).$$

which implies, putting $\varphi' = \frac{c}{\psi'}$,

$$\frac{1}{\psi'(x)} - \frac{1}{2\psi'\left(\frac{x+y}{2}\right)} = \varphi'(x+y-x\circ y)\left(1 - \frac{\psi'(x) - \psi'\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(x\circ y)}\right) \tag{2.11}$$

for all $x, y \in I$. Using the symmetry of the equation (2.11), we have

$$\left(2\psi'\left(\frac{x+y}{2}\right)-\psi'(x)\right)\left(\psi'(x\circ y)-\psi'(y)+\psi'\left(\frac{x+y}{2}\right)\frac{1}{2}\right)\frac{1}{\psi'(x)}$$
$$=: x\vartriangle y$$
$$= y\vartriangle x$$

for all $x, y \in I$, which (and some care) yields (2.10).

In the following we examine the *differential functional equation* (2.10) of the problem, assuming that $\psi'(x) \neq 0$ if $x \in I$ and $\psi''(x)$ ($x \in I$) exists and ψ'' is *continuous* on I.

3. On a differential functional equation

The investigation of the differential functional equation (2.10) is a problem, interesting itself, assuming the following conditions: $\psi : I \to \mathbb{R}$ is twice continuously differentiable on I and $\psi'(x) \neq 0$ if $x \in I$. Then clearly $\psi \in CM(I)$ and $x \circ y := A^*_\psi(x, y)$ is defined for all $x, y \in I$.

THEOREM 3
If $\psi : I \to \mathbb{R}$ is twice continuously differentiable on I and $\psi'(x) \neq 0$ if $x \in I$ and the differential functional equation (2.10) *holds for all $x, y \in I$ then there exists $p \in \mathbb{R}$ such that $\psi(x) \sim \chi_p(x)$ if $x \in I$, where*

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0, \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I). \tag{3.1}$$

Proof. Let
$$H_\psi := \{x \mid x \in I,\ \psi''(x) = 0\}.$$
Then H_ψ is closed and thus $I \setminus H_\psi =: L$ is open. If $L = \emptyset$ then $\psi''(x) = 0$ for all x, that is, $\psi(x) = \alpha x + \beta$, where $\alpha \neq 0$, β are constants. Then $\psi(x) \sim \chi_0(x)$ if $x \in I$.

If $L \neq \emptyset$ then let $K =]a, b[\subset L$ be a maximal component, i.e., for at least one of its endpoints, for example for b, we have $b \in I$ and $b \notin L$ hold. Obviously, then
$$\psi''(b) = 0. \tag{3.2}$$
Clearly $\psi' : K \to \mathbb{R}$ is *strictly monotonic* on K, by the condition $\psi''(x) \neq 0$ if $x \in K$, therefore from (2.10), we have

$$\psi'(x\circ y) = \psi'(x) + \psi'(y) - \frac{1}{2}\psi'\left(\frac{x+y}{2}\right) - \frac{\psi'(x)\psi'(y)}{2\psi'\left(\frac{x+y}{2}\right)} \tag{3.3}$$

for all $x \neq y$, $x, y \in K$, where $x \circ y = A^*_\psi(x, y)$.

We shall show that then there exists $p \in \mathbb{R}$, $p \neq 0$, such that $\psi(x) \sim \chi_p(x)$ if $x \in K$. This implies the statement of the theorem.

Indeed, then $\psi(x) = \alpha e^{px} + \beta$ ($\alpha \neq 0$, β are constants) if $x \in K$ and $\psi''(x) = \alpha p^2 e^{px}$, which implies, by the continuity of ψ'', $\psi''(b) = \alpha p^2 e^{pb} \neq 0$, and this contradicts (3.2). Thus $K = I$ and $\psi \sim \chi_p$ on I. Let us therefore return to the examination of (3.3), which holds for all $x, y \in K$. Then

$$\psi(x \circ y) = \psi(x) + \psi(y) - \psi\left(\frac{x+y}{2}\right)$$

if $x, y \in K$, from which we have

$$\frac{\partial(x \circ y)}{\partial x} = \frac{\psi'(x) - \psi\left(\frac{x+y}{2}\right)\frac{1}{2}}{\psi'(x \circ y)},$$

whence

$$\frac{\partial^2(x \circ y)}{\partial x \partial y} = \frac{-\frac{1}{4}\psi''\left(\frac{x+y}{2}\right)\psi'(x \circ y) - \frac{\partial \psi'(x \circ y)}{\partial y}\left(\psi'(x) - \psi\left(\frac{x+y}{2}\right)\frac{1}{2}\right)}{(\psi'(x \circ y))^2},$$

from which

$$\begin{aligned} &\frac{\partial^2(x \circ y)}{\partial x \partial y}(\psi'(x \circ y))^2 + \frac{1}{4}\psi''\left(\frac{x+y}{2}\right)\psi'(x \circ y) \\ &\quad = \frac{\partial \psi'(x \circ y)}{\partial y}\left(\frac{1}{2}\psi'\left(\frac{x+y}{2}\right) - \psi'(x)\right) \end{aligned} \tag{3.4}$$

follows for all $x, y \in K$. The left hand side of (3.4) is symmetric with respect to x and y. therefore, by (3.3),

$$\begin{aligned} &\left(\psi''(y) - \frac{1}{4}\psi''\left(\frac{x+y}{2}\right)\right. \\ &\qquad \left. - \frac{\psi'(x)\psi''(y)2\psi'\left(\frac{x+y}{2}\right) - \psi''\left(\frac{x+y}{2}\right)\psi'(x)\psi'(y)}{4\left(\psi'\left(\frac{x+y}{2}\right)\right)^2}\right) \\ &\qquad \times \left(\frac{1}{2}\psi'\left(\frac{x+y}{2}\right) - \psi'(x)\right) =: x * y = y * x \end{aligned} \tag{3.5}$$

for all $x, y \in K$. We introduce the following notation:

$$F(x_1; x_2, x_3, x_4) := \psi''(x_1)\psi'(x_2)\psi'(x_3)\psi'(x_4), \tag{3.6}$$

where $x_1, x_2, x_3, x_4 \in K$. Apparently F is symmetric with respect to the variables x_2, x_3, x_4. Then (3.5) and some calculation show that, using the notation (3.6), the expression

$$\begin{aligned}A(x,y) := {} & 4F\left(y; \frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right) - F\left(\frac{x+y}{2}; \frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right) \\ & - 2F\left(y; x, \frac{x+y}{2}, \frac{x+y}{2}\right) + F\left(\frac{x+y}{2}; x, y, \frac{x+y}{2}\right) \\ & - 8F\left(y; \frac{x+y}{2}, \frac{x+y}{2}, x\right) + 2F\left(\frac{x+y}{2}; \frac{x+y}{2}, \frac{x+y}{2}, x\right) \\ & + 4F\left(y; x, \frac{x+y}{2}, x\right) - 2F\left(\frac{x+y}{2}; x, y, x\right)\end{aligned}$$

is symmetric with respect to x and y. This together with the symmetry of F with respect to x_2, x_3, x_4 implies that

$$\begin{aligned}B(x,y) := {} & 2F\left(y; \frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right) - 5F\left(y; x, \frac{x+y}{2}, \frac{x+y}{2}\right) \\ & + F\left(\frac{x+y}{2}; x, \frac{x+y}{2}, \frac{x+y}{2}\right) + 2F\left(y; x, x, \frac{x+y}{2}\right) \\ & - F\left(\frac{x+y}{2}; x, x, y\right) \\ = {} & B(y,x)\end{aligned}$$

holds for all $x, y \in K$. By the definition (3.6), we have

$$\begin{aligned}& (\psi''(y) - \psi''(x)) \\ & \quad \times \left(2\psi'^3\left(\frac{x+y}{2}\right) - 5\psi'(x)\psi'^2\left(\frac{x+y}{2}\right) + 2\psi'^2(x)\psi'\left(\frac{x+y}{2}\right)\right) \\ & \quad + (\psi'(y) - \psi'(x))\left(5\psi''(x)\psi'^2\left(\frac{x+y}{2}\right) - \psi''\left(\frac{x+y}{2}\right)\psi'^2\left(\frac{x+y}{2}\right)\right) \\ & \quad + (\psi'(y) - \psi'(x)) \\ & \quad \times \left(-2(\psi'(x) + \psi'(y))\psi''(x)\psi'\left(\frac{x+y}{2}\right) + \psi''\left(\frac{x+y}{2}\right)\psi'(x)\psi'(y)\right) \\ & = 0\end{aligned}$$

for all $x, y \in K$. Dividing this equation by $(y - x)$ $(y \neq x)$ and taking the limit as $y \to x$, we have that $\psi'''(x)$ *exists* $(x \in K)$ and

$$-\psi'''(x)\psi'(x) + \psi''^2(x) = 0 \quad (x \in K). \tag{3.7}$$

It can be easily seen that, under the conditions prescribed for ψ, the solutions of (3.7) are

$$\psi(x) = \alpha e^{px} + \beta \quad (x \in K), \tag{3.8}$$

where $\alpha \neq 0$, $p \neq 0$, and β are real constants. This means $\psi \sim \chi_p$ on K for some real constant $p \neq 0$, and that was to be proved.

4. Corollaries

Returning to our problem, first we give an important corollary of Theorem 3.

THEOREM 4

If (φ, ψ) is a Matkowski-Sutô type pair on I and either φ or ψ is twice continuously differentiable on I then there exists $p \in \mathbb{R}$ for which

$$\varphi(x) \sim \chi_{-p}(x), \quad \psi(x) \sim \chi_p(x) \quad (x \in I). \tag{4.1}$$

Proof. Since the roles of φ and ψ can be exchanged, we suppose that ψ is twice continuously differentiable on I. Let

$$N_\psi := \{x \mid \; x \in I, \; \psi'(x) = 0\}.$$

Then N_ψ is closed (since ψ' is continuous on I). Thus $I \setminus N_\psi$ is open. There are two possible cases: either $N_\psi = \emptyset$ or $N_\psi \neq \emptyset$. We show that the latter is impossible.

Suppose that $N_\psi \neq \emptyset$. Then there exists a maximal component $K \subset I \setminus N_\psi$, that is, an open interval $K =]a, b[$ such that at least one of its endpoints — let it be b — belongs to I, and so $b \in N_\psi$.

Then (φ, ψ) is a Matkowski-Sutô type pair on K, ψ is twice continuously differentiable on K, and $\psi'(x) \neq 0$ for $x \in K$. By Lemma 3, (2.10) holds for all $x, y \in K$ and, by Theorem 3, there exists $p \in \mathbb{R}$ such that $\psi \sim \chi_p$ on K. From this, we have

$$\psi(x) = A\chi_p(x) + B \quad (x \in K)$$

where $A \neq 0$ and B are real constants. This yields

$$\psi'(x) = \begin{cases} A & \text{if } p = 0, \\ Ape^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in K).$$

By the continuity of ψ' on I, we have

$$\lim_{\substack{x \to 0 \\ x \in K}} \psi'(x) = \psi'(b) = 0.$$

However, this is a contradiction, because

$$\psi'(b) = \left\{ \begin{array}{ll} A & \text{if } p = 0, \\ Ape^{pb} & \text{if } p \neq 0 \end{array} \right\} \neq 0.$$

Thus $N_\psi = \emptyset$, that is, $\psi'(x) \neq 0$ for all $x \in I$. The same argument with $K = I$ now results that $\psi \sim \chi_p$ on I for some $p \in \mathbb{R}$.

Then, by (2.2), the function $\varphi \in CM(I)$ satisfies

$$A^*_\varphi(x,y) + A^*_{\chi_p}(x,y) = x + y$$

for all $x, y \in I$. By

$$A^*_\varphi(x,y) = x + y - A^*_{\chi_p}(x,y) = -A^*_{-\chi_p}(x,y)$$

and by Theorem 1, from this we have $\varphi \sim \chi_{-p}$ on I.

Now we can state the main result of our investigations.

THEOREM 5

If the conjugate arithmetic means M and N satisfy the functional equation (1.5) on I, and either of them is twice continuously differentiable on I (see Definition 4) then there exists $p \in \mathbb{R}$ such that

$$M(x,y) = D_p(x,y) \quad \textit{and} \quad N(x,y) = D_{-p}(x,y) \quad (x, y \in I), \tag{5.1}$$

where

$$D_p(x,y) := \begin{cases} \frac{x+y}{2} & \textit{if } p = 0, \\ \frac{1}{p}\log\left(e^{px} + e^{py} - e^{p\frac{x+y}{2}}\right) & \textit{if } p \neq 0 \end{cases} \quad (x, y \in I). \tag{5.2}$$

Proof. There exist generating functions $\psi, \varphi \in CM(I)$ such that $M = A^*_\psi$ and $N = A^*_\varphi$, furthermore, the pair (φ, ψ) is a Matkowski-Sutô type pair. By the symmetry, we can suppose that M is twice continuously differentiable on I, that is, $\psi \in CM(I)$ is twice continuously differentiable on I. By Theorem 4 there exists $p \in \mathbb{R}$ such that $\psi \sim \chi_p$ and $\varphi \sim \chi_{-p}$ on I. Obviously $M = A^*_\psi = A^*_{\chi_p} = D_p$ and $N = A^*_\varphi = A^*_{\chi_{-p}} = D_{-p}$, where D_p is the mean defined in (5.2), that is, (5.1) holds.

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References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York – London, 1966.

[2] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.

[3] Z. Daróczy, *On a class of means of two variables*, Publ. Math. Debrecen **55** (1999), 177-197.

[4] Z. Daróczy, Gy. Maksa, *On a problem of Matkowski*, Colloq. Math. **82** (1999), 117-123.

[5] Z. Daróczy, Zs. Páles, *On means that are both quasi-arithmetic and conjugate arithmetic*, submitted to Acta Math. Hung.

[6] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, University Press, Cambridge, 1934.

[7] A. Járai, *On measurable solutions of functional equations*, Publ. Math. Debrecen **26** (1979), 17-35.

[8] A. Járai, *Regularity properties of functional equations*, Aequationes Math. **25** (1982), 52-66.

[9] A. Járai, *On regular solutions of functional equations*, Aequationes Math. **30** (1986), 21-54.

[10] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Uniw. Śląski – Państwowe Wydawnictwo Naukowe, Warszawa – Kraków – Katowice, 1985.

[11] J. Matkowski, *Invariant and complementary quasi-arithmetic means*, Aequationes Math. **57** (1999), 87-107.

[12] O. Sutô, *Studies on some functional equations I-II*, Tôhoku Math. J. **6** (1914), 1-15 and 82-101.

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