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A functional equation involving three means

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. The functional equation

$$\frac{1}{n}\sum_{i=1}^{n} t(x_i) = h\left(\frac{1}{n}\sum_{i=1}^{n} x_i\,, f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)\right),$$

where f is given and the functions t and h are unknown, is solved under certain conditions for all fixed n.

In a personal communication Professor Udo Ebert (Universität Oldenburg, Germany) asked Professor János Aczél about the solutions of the functional equation

$$\frac{1}{n}\sum_{i=1}^{n} t(x_i) = h\left(\frac{1}{n}\sum_{i=1}^{n} x_i\,, f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)\right), \tag{1}$$

where f is a given function defining a quasiarithmetic mean and the functions t and h are unknown. Indeed, his consideration about a problem of inequality measurement in economics resulted in a problem, which was formulated by Aczél in the form of the functional equation (1) above. We refer readers to [2] for other related works of Ebert where quasiarithmetic means are often used. We wish to thank Aczél for his coordinations and suggestions on this paper.

In what follows $I, J \subset \mathbb{R}$ are intervals, and f is a one-to-one continuous function mapping I onto J. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and let

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$$a(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$b(\mathbf{x}) := f^{-1}\left(\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\right)$$

denote the arithmetic mean and the quasiarithmetic mean generated by f of the sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Using the notation

$$t(\mathbf{x}) = (t(x_1), t(x_2), \ldots, t(x_n)),$$

equation (1) can be written in the form

$$a(t(\mathbf{x})) = h(a(\mathbf{x}), b(\mathbf{x})), \quad \mathbf{x} \in I^n. \tag{2}$$

Although the mission is to find pairs of real valued functions t and h satisfying (1), our emphasis is on the function t, while treating h as an auxiliary function. Because of this, when we say "t is a solution of (1)", it means "t satisfies (1) for some function h", without elaborating on what h is explicitly.

Let us start with some simple observations.

REMARKS

(i) The solutions (t, h) form a linear space, i.e. if (t_1, h_1) and (t_2, h_2) are solutions (of (1)), so is any linear combination $(c_1t_1 + c_2t_2, c_1h_1 + c_2h_2)$.

(ii) The constant map $t = 1$, the identity map $t(x) = x$, and $t = f$ are particular solutions (with appropriate h).

(iii) An additive function A (i.e. satisfying the Cauchy equation $A(x+y) = A(x) + A(y)$) is a solution, i.e. $t = A$ restricted to the interval I is a solution. Thus discontinuous solution t exists.

(iv) If (1) is satisfied by t for some $n = n_0$, then it is satisfied by t for all lower $n < n_0$. (Here, h may change according to n.)

(v) If $n = 1$, any function t is a solution.

(vi) If f is affine on I, then $a(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in I^n$, and equation (2) is reduced to the well-known Pexider equation when $n \geqslant 2$, (cf. [1]). Hence we shall suppose that the given function f is not affine.

In the proof of our main result the following general uniqueness theorem will be applied:

THEOREM 1 ([3], Theorem 1.0)

Let I and K be intervals of $\mathbb{R}$. Let $T : I \times K \to \mathbb{R}$ be a continuous function. If the functional equation

$$F(x) + G(y) = H(T(x,y)), \quad x \in I,\ y \in K \tag{3}$$

has a particular solution (F_0, G_0, H_0) *with continuous nonconstant* F_0, G_0, *then the general continuous solution of* (3) *is given by*

$$F = \alpha F_0 + \beta_1, \quad G = \alpha G_0 + \beta_2, \quad H = \alpha H_0 + \beta_1 + \beta_2 \tag{4}$$

where α, β_1, β_2 *are constants.*

We are now ready to give the main result:

THEOREM 2

If t *is a continuous solution of* (1) *for some fixed* $n \geqslant 4$, *then it is given by* $t(x) = c_1 f(x) + c_2 x + c_3$ *where* c_1, c_2, c_3 *are constants. Conversely, every such* t *is a continuous solution of* (1) *for all fixed* n.

Proof. In view of observations (i) and (ii), the converse statement is clear. In view of (iv) we may now suppose $n = 4$. Let $x, y \in I$ be fixed temporarily. From (1) we get

$$\begin{aligned}
&[t(x+u) + t(x-u)] + [t(y+v) + t(y-v)] \\
&\quad = 4h\left(\frac{x+y}{2}, f^{-1}\left(\frac{f(x+u) + f(x-u) + f(y+v) + f(y-v)}{4}\right)\right) \\
&\quad =: H(f(x+u) + f(x-u) + f(y+v) + f(y-v)) \\
&\qquad\qquad \text{for all } u,\ v \text{ such that } x+u, x-u, y+v, y-v \in I.
\end{aligned} \tag{5}$$

This equation is of the form (3) where

$$T(u,v) := f(x+u) + f(x-u) + f(y+v) + f(y-v).$$

Clearly the triple $F_0(u) := f(x+u) + f(x-u)$, $G_0(v) := f(y+v) + f(y-v)$ and $H_0 :=$ identity map is a solution of (3) with continuous F_0 and G_0. On the other hand (5) asserts that the triple $F(u) := t(x+u) + t(x-u)$ and $G(v) := t(y+v) + t(y-v)$ and H is another solution.

Assume now that x and y are such that $f(x+u) + f(x-u)$ and $f(y+v) + f(y-v)$ are not constant functions in u and v respectively, and apply the above uniqueness theorem, we conclude that there exist constants $\alpha(x,y)$, $\beta_1(x,y)$ and $\beta_2(x,y)$ such that

$$\begin{aligned}
t(x+u) + t(x-u) &= \alpha(x,y)[f(x+u) + f(x-u)] + \beta_1(x,y) \\
t(y+v) + t(y-v) &= \alpha(x,y)[f(y+v) + f(y-v)] + \beta_2(x,y) \\
&\qquad \text{for all } u,\ v \text{ provided the arguments} \\
&\qquad x+u,\ x-u,\ y+v,\ y-v \text{ are in } I.
\end{aligned} \tag{6}$$

In (6), from the first line we get that α depends only on x and from the second line we get that α depends only on y. Similarly β_1 depends on x only and β_2 depends on y only. Hence α is independent of x and y; i.e. $\alpha(x,y) = \alpha$ say, and $\beta_1(x,y) = \beta_1(x)$, $\beta_2(x,y) = \beta_2(y)$. Further comparison of the two lines leads to $\beta_1 = \beta_2 = \beta$ say. So equation (6) can be summarized as

$$\begin{aligned} t(x+u) + t(x-u) = \alpha[f(x+u) + f(x-u)] + \beta(x) \\ \text{for all } u \text{ with } x+u, x-u \in I \end{aligned} \tag{7}$$

for all x such that $f(x+u) + f(x-u)$ is not constant in u. (Since, by Remark (vi), f is not affine, there exists such x where $f(x+u) + f(x-u)$ is not constant in u and we are not speaking in a vacuum.) On the other hand, when x is such that $f(x+u) + f(x-u)$ is constant in u, it follows from (5) that $t(x+u) + t(x-u)$ is constant in u too. Thus (7) can be extended to cover such x. That is, where $f(x+u) + f(x-u)$ and $t(x+u) + t(x-u)$ are both constant in u, we take (7) as the definition of $\beta(x)$.

Now (7) holds for all x without exception in I. Let $S := t - \alpha f$ and (7) takes the form

$$S(x+u) + S(x-u) = \beta(x).$$

Setting $u = 0$ we obtain $\beta(x) = 2S(x)$ and we obtain the Jensen equation

$$S(x+u) + S(x-u) = 2S(x), \quad x, x+u, x-u \in I. \tag{8}$$

As S is continuous, it is affine

$$S(x) = c_2 x + c_3. \tag{9}$$

Thus $(t - \alpha f)(x) = c_2 x + c_3$. This proves that with $c_1 = \alpha$ we have $t(x) = c_1 f(x) + c_2 x + c_3$ as asserted.

To treat the case $n = 3$ we shall use stronger conditions.

PROPOSITION 1

Suppose that f is continuously differentiable with non-vanishing derivative and not affine on any subinterval of I. Then for a function $t : I \to \mathbb{R}$ there exists a continuously differentiable function $h : I \times I \to \mathbb{R}$ such that the functional equation (2) *is satisfied for some fixed $n \geqslant 3$ if, and only if, there exist constants c_1, c_2, c_3 such that $t(x) = c_1 f(x) + c_2 x + c_3$.*

Proof. The "if" part is obvious. To the "only if" part suppose that (2) holds with continuously differentiable h. We may suppose that $n = 3$. Because f, f^{-1} and h are continuously differentiable, we obtain that t is also continuously differentiable. Derivating both sides of equation (2) we obtain that

$$t'(x) = \partial_1 h\,(a(x,y,z), b(x,y,z)) \\ + \partial_2 h\,(a(x,y,z), b(x,y,z))\,\frac{f'(x)}{f'\,(b(x,y,z))} \tag{10}$$

for all $x, y, z \in I$. First we shall prove that each $x_0 \in I$ has a neighbourhood, such that for each x from this neighbourhood there exist continuously differentiable functions φ and ψ such that $x \mapsto a\,(x, \varphi(x), \psi(x))$ and $x \mapsto b\,(x, \varphi(x), \psi(x))$ are constant. Such neighbourhood can be found by the implicit function theorem. Let us choose y_0 and z_0 such that $f'(y_0) \neq f'(z_0)$ be satisfied. Let $u = a(x_0, y_0, z_0)$, $v = b(x_0, y_0, z_0)$, and let us apply the implicit function theorem for the equation system

$$3u = x + y + z,$$
$$3f(v) = f(x) + f(y) + f(z),$$

to express y and z as a function of x and so to obtain φ and ψ. Now, let us substitute in equation (10) — which is valid for any $x, y, z \in I$ — the values $y = \varphi(x)$ and $z = \psi(x)$. Then we obtain the equation

$$t'(x) = \alpha f'(x) + \beta,$$

where $\beta = \partial_1 h(u, v)$ and $\alpha = \frac{\partial_2 h(u,v)}{f'(v)}$. This equation is valid on a neighbourhood of x_0. Now, this equation implies, that there exists a constant γ such that $t(x) = \alpha f(x) + \beta x + \gamma$ on a neighbourhood of x_0. The constants α, β and γ may depend on x_0, but we shall prove that they do not.

Suppose that $t(x) = \alpha_1 f(x) + \beta_1 x + \gamma_1$ on some subinterval I_1 of I and $t(x) = \alpha_2 f(x) + \beta_2 x + \gamma_2$ on another subinterval I_2 of I. Suppose, that I_1 and I_2 are not disjoint. If $\alpha_1 \neq \alpha_2$, then this means that f is locally affine, a contradiction. Hence $\alpha_1 = \alpha_2$ and so $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$.

Now if I_1 and I_2 are disjoint, then choosing a chain of subintervals connecting them we obtain that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. So we have proved that α, β and γ do not depend on x_0.

In the case $n = 2$ arbitrary functions may appear as solutions.

PROPOSITION 2

Suppose that f is strictly convex or strictly concave on I and $n = 2$. Then to an arbitrary function $t : I \to \mathbb{R}$ there exists a (unique) function $h : I \times I \to \mathbb{R}$ such that the functional equation (2) *is satisfied.*

Proof. We shall prove that the mapping $k : (x, y) \mapsto (a(x,y), b(x,y))$ is one-to-one on the set $\{(x,y) : \ x, y \in I, \ x \leqslant y\}$. Suppose, to the contrary, that this is not the case. Then there are $x_1 < x_2 \leqslant u \leqslant y_2 < y_1$ such that

$a(x_1,y_1) = u = a(x_2,y_2)$ and $b(x_1,y_1) = v = b(x_2,y_2)$. This means that $x_1 + y_1 = 2u = x_2 + y_2$ and $f(x_1) + f(y_1) = 2f(v) = f(x_2) + f(y_2)$. Hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(y_1) - f(y_2)}{y_1 - y_2}$$

for $x_1 < x_2 \leqslant u \leqslant y_2 < y_1$. But this is a contradiction to the assumption that f is strictly convex or strictly concave.

Now for an arbitrary t, let us define $h(u,v)$ equal to $\frac{1}{2}[t(w) + t(z)]$, where $(w,z) = k^{-1}(u,v)$. Then for any $x, y \in I$ we have

$$2h\left(a(x,y), b(x,y)\right) = t(\min\{x,y\}) + t(\max\{x,y\}) = t(x) + t(y).$$

Remark

If f is not strictly convex nor strictly concave then t cannot in general be arbitrary. For example, if $f(x) = x^3$ for $x \in I = \mathbb{R}$, then the mapping $k : (x,y) \mapsto (a(x,y), b(x,y))$ is not one-to-one on the set $\{(x,y) : x,y \in I, x \leqslant y\}$, because all pairs $(x,-x)$ are mapped into $(0,0)$. Hence with $c = 2h(0,0)$, the condition $t(x) + t(-x) \equiv c$ has to be satisfied. Conversely, because k is one-to-one on the set $\{(x,y) : x,y \in I, x \leqslant y,\ x \neq -y\}$, if $t(x) + t(-x) \equiv c$ for all $x \in \mathbb{R}$ is satisfied, then defining $2h(u,v)$ equal to $t(w) + t(z)$, where $(w,z) = k^{-1}(u,v)$, for any $x, y \in I$ we have

$$2h\left(a(x,y), b(x,y)\right) = t(\min\{x,y\}) + t(\max\{x,y\}) = t(x) + t(y).$$

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