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Remarks on quasi-arithmetic means

Dedicated to Professor Zenon Moszner
on the occasion of his seventieth birthday

Abstract. Conditions under which $M_\varphi = M_\psi$ and $M_{\varphi^{-1}} = M_{\psi^{-1}}$ are considered where $M_\varphi, M_\psi, M_{\varphi^{-1}}$, and $M_{\psi^{-1}}$ denote the quasi-arithmetic means generated by the functions $\varphi, \psi, \varphi^{-1}$, and ψ^{-1}, respectively.

1. Introduction

Let I be a real interval and $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function. The function $M_\varphi : I \times I \to I$, defined by

$$M_\varphi(x,y) := \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right),$$

is called *quasi-arithmetic mean* and the function φ its *generator*.

Let $\varphi, \psi : I \to \mathbb{R}$ be generators of the same mean, i.e. $M_\varphi = M_\psi$. Then, in general, the equation

$$M_{\varphi^{-1}} = M_{\psi^{-1}}$$

(which obviously requires that $\varphi(I) = \psi(I)$) does not hold. To show this consider the following

Example 1

Taking $\varphi = \log$, $\psi = -\log$, we have

$$M_\varphi(x,y) = \sqrt{xy} = M_\psi(x,y) \quad \text{for } x,y \in I = (0,\infty)$$

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and, since

$$M_{\varphi^{-1}}(x,y) = \log\left(\frac{\exp x + \exp y}{2}\right), \quad M_{\psi^{-1}} = -\log\left(\frac{\exp(-x) + \exp(-y)}{2}\right)$$

for all $x, y \in \mathbb{R}$, we have

$$M_{\varphi^{-1}} \neq M_{\psi^{-1}}.$$

Actually, the strict convexity of $-\log$ and the strict concavity of log imply that

$$-\log\left(\frac{\exp(-x) + \exp(-y)}{2}\right) < \frac{x+y}{2} < \log\left(\frac{\exp x + \exp y}{2}\right)$$

for all $x, y \in \mathbb{R}, x \neq y$. In particular, the means $M_{\log}$, $M_{-\log}$, and the arithmetic mean A satisfy the inequality

$$M_{-\log} \leqslant A \leqslant M_{\log}.$$

In this paper we examine conditions under which continuous and strictly monotonic functions φ and ψ satisfy the system

$$M_{\varphi} = M_{\psi} \quad \text{and} \quad M_{\varphi^{-1}} = M_{\psi^{-1}}. \tag{1}$$

Theorems 1-3 provide the form of solutions in some special cases, in particular when the interval I is bounded. In the remaining results, when I is unbounded, we assume that either φ and ψ have a special asymptotic behaviour (Theorem 4), or one of φ, ψ satisfies a condition close to geometric convexity (Theorem 5), or one of them is of class C^{∞} (Theorem 6).

2. Preliminaries

Let I and J be real intervals. We start with the following result which is fundamental for further considerations. Its standard proof repeats, in a sense, an argument given yet by G.H. Hardy, J.E. Littlewood, and G. Pólya (see [3; Theorem 83] for sufficiency of (2) and [3; a conclusion after Theorem 89] for necessity of (2) in the case where φ and ψ are homeomorphisms mapping I onto J.

PROPOSITION

Let φ and ψ be bijections mapping I onto J. Then (φ, ψ) satisfies (1) if and only if there are affine functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(x) = g(\varphi(x)) \quad \textit{and} \quad \varphi(x) = \psi(f(x)) \quad \textit{for } x \in I. \tag{2}$$

Proof. Assume that φ and ψ satisfy (1). Then

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)=\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right), \quad x,y\in I,$$

and

$$\varphi\left(\frac{\varphi^{-1}(u)+\varphi^{-1}(v)}{2}\right)=\psi\left(\frac{\psi^{-1}(u)+\psi^{-1}(v)}{2}\right), \quad u,v\in J,$$

or, equivalently,

$$\psi\circ\varphi^{-1}\left(\frac{u+v}{2}\right)=\frac{\psi\circ\varphi^{-1}(u)+\psi\circ\varphi^{-1}(v)}{2}, \quad u,v\in J,$$

and

$$\psi^{-1}\circ\varphi\left(\frac{x+y}{2}\right)=\frac{\psi^{-1}\circ\varphi(x)+\psi^{-1}\circ\varphi(y)}{2}, \quad x,y\in I.$$

Then $\psi\circ\varphi^{-1}$ and $\psi^{-1}\circ\varphi$ satisfy the Jensen equation and by [5; Theorem XIII.2.1] they are the restrictions of affine functions $g,f:\mathbb{R}\to\mathbb{R}$ to J and I, respectively, i.e. (2) holds true.

Conversely, if (2) is satisfied with affine functions $f,g:\mathbb{R}\to\mathbb{R}$ then also

$$\psi^{-1}(u)=\varphi^{-1}(g^{-1}(u)) \quad\text{and}\quad \varphi^{-1}(u)=f^{-1}(\psi^{-1}(u)) \quad\text{for } u\in J,$$

whence

$$\begin{aligned}M_\psi(x,y)&=\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=\varphi^{-1}\left(g^{-1}\left(\frac{g(\varphi(x))+g(\varphi(y))}{2}\right)\right)\\&=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\\&=M_\varphi(x,y)\end{aligned}$$

for all $x,y\in I$ and

$$\begin{aligned}M_{\varphi^{-1}}(u,v)&=\varphi\left(\frac{\varphi^{-1}(u)+\varphi^{-1}(v)}{2}\right)\\&=\psi\left(f\left(\frac{f^{-1}(\psi^{-1}(u))+f^{-1}(\psi^{-1}(v))}{2}\right)\right)\\&=\psi\left(\frac{\psi^{-1}(u)+\psi^{-1}(v)}{2}\right)\\&=M_{\psi^{-1}}(u,v)\end{aligned}$$

for all $u,v\in J$.

Having in mind the form of continuous additive functions and, consequently, continuous affine functions in the real case, we get the following obvious consequence of the Proposition.

Corollary 1

Let φ and ψ be homeomorphisms mapping I onto J. Then (φ, ψ) satisfies (1) if and only if there are real numbers a, b, c, d such that

$$\psi(x) = c\varphi(x) + d \quad \text{for } x \in I \tag{3}$$

and

$$\varphi(x) = \psi(ax + b) \quad \text{for } x \in I. \tag{4}$$

Remark 1

Let φ and ψ be bijections mapping I onto J.

(i) Clearly, if $f, g : \mathbb{R} \to \mathbb{R}$ are affine functions satisfying (2) then

$$f(I) = I \tag{5}$$

and

$$g(J) = J. \tag{6}$$

Moreover, the functions $f_{|I}$ and $g_{|J}$ are invertible.

(ii) Conditions (2) are equivalent to

$$\varphi^{-1}(u) = \psi^{-1}(g(u)) \quad \text{and} \quad \psi^{-1}(u) = f(\varphi^{-1}(u)) \quad \text{for } u \in J.$$

Remark 2

Let φ and ψ be homeomorphisms mapping I onto J.

(i) The numbers a and c occuring in (3) and (4) have to be non-zero; actually, since φ and ψ are strictly monotonic, we have even more: it follows from (3) and (4) that $ac > 0$.

(ii) Conditions (3) and (4) are equivalent to

$$\varphi^{-1}(u) = \psi^{-1}(cu + d) \quad \text{for } u \in J \tag{7}$$

and

$$\psi^{-1}(u) = a\varphi^{-1}(u) + b \quad \text{for } u \in J, \tag{8}$$

respectively.

(iii) Assume that (φ, ψ) satisfies (4). Then

$$\lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} \psi(x) \quad \text{iff} \quad ax_0 + b = x_0$$

for each $x_0 \in \operatorname{cl} I$. Observe also that if I is a halfline then a has to be positive and the finite endpoint of I is a fixed point of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$. It follows from (5) that f has a fixed point in $\operatorname{cl} I$ also when I is a bounded interval.

Assume that (φ, ψ) satisfies (3). Then, by (7),

$$\lim_{u \to u_0} \varphi^{-1}(u) = \lim_{u \to u_0} \psi^{-1}(u) \quad \text{iff} \quad cu_0 + d = u_0$$

for each $u_0 \in \operatorname{cl} J$. If J is a halfline then c is positive and the finite endpoint of

J is a fixed point of the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = cx + d$. It follows from (6) that g has a fixed point in $\operatorname{cl} J$ also in the case of bounded J.

According to Corollary 1 and Remark 2 (iii) the following is obvious.

COROLLARY 2

Let φ and ψ be homeomorphisms mapping I onto J such that (φ, ψ) satisfies (1). *Then*

(i) *either $\varphi = \psi \circ T$ with a translation $T : I \to I$, or there exists exactly one point $\xi \in \operatorname{cl} I$ such that $\lim_{x\to\xi} \varphi(x) = \lim_{x\to\xi} \psi(x)$; if I is a halfline and the second case holds then ξ is the finite endpoint of I.*

(ii) *either $\psi = T \circ \varphi$ with a translation $T : J \to J$, or there exists exactly one point $\eta \in \operatorname{cl} J$ such that $\lim_{u\to\eta} \varphi^{-1}(u) = \lim_{u\to\eta} \psi^{-1}(u)$; if J is a halfline and the second case holds then η is the finite endpoint of J.*

In the sequel, given a pair (φ, ψ) of homeomorphisms satisfying (1), ξ and η will always denote the points which have been uniquely described in Corollary 2.

3. Trivial solutions

As follows from Corollary 1 among pairs (φ, ψ) of homeomorphisms satisfying (1) there are very special ones, namely solutions of the system

$$\psi = T_2 \circ \varphi \quad \text{and} \quad \varphi = \psi \circ T_1 \tag{9}$$

where $T_1 : I \to I$ and $T_2 : J \to J$ are translations, and solutions of the system

$$\psi = S_2 \circ \varphi \quad \text{and} \quad \varphi = \psi \circ S_1 \tag{10}$$

where $S_1 : I \to I$ and $S_2 : J \to J$ are point symmetries. In what fol ows such pairs are said to be *trivial*.

THEOREM 1

Let $T_1 : I \to I$ and $T_2 : J \to J$ be translations. Then homeomorphic solutions of (9) *depend on arbitrary function:*

if both T_1 and T_2 are the identity functions then every pair (φ, ψ), where φ is a homeomorphism mapping I onto J, satisfies (9);

if both T_1 and T_2 are non-trivial then $I = J = \mathbb{R}$ and for every $x_0 \in \mathbb{R}$ any homeomorphism φ_0 mapping the interval with endpoints x_0 and $T_1(x_0)$ onto the interval with the endpoints $\varphi_0(x_0)$ and $T_2^{-1}(\varphi_0(x_0))$ can be uniquely extended to a homeomorphism φ mapping $\mathbb{R}$ onto $\mathbb{R}$ such that $(\varphi, T_2 \circ \varphi)$ satisfies (9).

Conversely: if (9) *has a homeomorphic solution then either both* T_1 *and* T_2 *are the identity functions, or both of them are non-trivial and, moreover, every homeomorphic solution of* (9) *can be obtained in the above way.*

Proof. The case when T_1 and T_2 are the identity functions is obvious, so assume that both of them are non-trivial. Since by Remark 1 (i) we have $T_1(I) = I$ and $T_2(J) = J$ this means that $I = J = \mathbb{R}$.

Let $b, d \in \mathbb{R} \setminus \{0\}$ be such that

$$T_1(x) = x + b \quad \text{and} \quad T_2(u) = u + d \quad \text{for } x, u \in \mathbb{R}.$$

Then (9) is equivalent to the system

$$\psi(x) = \varphi(x) + d$$

and

$$\varphi(x) = \varphi(x + b) + d. \tag{11}$$

It is clear that if $x_0 \in \mathbb{R}$ then every homeomorphism φ_0 mapping the interval with the endpoints x_0 and $x_0 + b$ onto the interval with the endpoints $\varphi_0(x_0)$ and $\varphi_0(x_0) - d$ can be uniquely extended to a homeomorphism φ mapping $\mathbb{R}$ onto $\mathbb{R}$ and satisfying (11). Consequently, the pair $(\varphi, T_2 \circ \varphi)$ is a homeomorphic solution of (9).

The remaining part is quite clear.

THEOREM 2

Let $S_1 : I \to I$ *and* $S_2 : J \to J$ *be point symmetries. Then homeomorphic solutions of* (10) *depend on an arbitrary function:*

if ξ *and* η *denote the unique fixed points of* S_1 *and* S_2, *respectively, then any homeomorphism* φ_0 *mapping one of the intervals* $I \cap (-\infty, \xi\,]$ *and* $I \cap [\,\xi, \infty)$ *onto one of the intervals* $J \cap (-\infty, \eta]$ *and* $J \cap [\eta, \infty)$ *and fulfilling* $\varphi_0(\xi) = \eta$ *can be uniquely extended to a homeomorphism* φ *mapping* I *onto* J *such that* $(\varphi, S_2 \circ \varphi)$ *satisfies* (10).

Conversely: every homeomorphic solution of (10) *can be obtained in the above way.*

Proof. We proceed similarly as in the main part of the proof of Theorem 1. Let $b, d \in \mathbb{R}$ be such that

$$S_1(x) = b - x \quad \text{and} \quad S_2(u) = d - u \quad \text{for } x \in I \text{ and } u \in J.$$

Then (10) is equivalent to the system

$$\psi(x) = d - \varphi(x)$$

and

$$\varphi(x) = d - \varphi(b - x). \tag{12}$$

Since $S_1(\xi) = \xi$ and $S_2(\eta) = \eta$ we have $\xi = \frac{b}{2}$ and $\eta = \frac{d}{2}$. Thus it is clear that homeomorphism φ_0 mapping, for instance, the interval $I \cap (-\infty, \xi]$ onto $J \cap (-\infty, \eta]$ and fulfilling $\varphi_0(\xi) = \eta$ can be uniquely extended to a homeomorphism φ mapping I onto J and satisfying (12). Consequently, $(\varphi, S_2 \circ \varphi)$ is a homeomorphic solution of (10).

In view of Corollary 1 we can roughly rewrite Theorems 1 and 2 in the following form.

COROLLARY 3

Trivial solutions of (1) *depend on an arbitrary function.*

However, some assumptions imposed on trivial solutions give the uniqueness. The result below shows that sometimes they lead to the arithmetic mean A.

COROLLARY 4

Let φ and ψ be different homeomorphisms mapping I onto J such that (φ, ψ) is a trivial solution of (1). *If one of the functions φ and ψ is convex or concave then they are the restrictions of affine mappings and*

$$M_\varphi = M_\psi = M_{\varphi^{-1}} = M_{\psi^{-1}} = A.$$

Proof. Assume that (φ, ψ) satisfies (9) and one of the functions φ and ψ is, for instance, convex. Then, clearly, so is the second one. Therefore φ is a convex solution of (11) with some $b, d \in \mathbb{R} \setminus \{0\}$, that is $\gamma : I \to \mathbb{R}$ defined by

$$\gamma(x) = \varphi(x) + \frac{d}{b} x$$

is a convex b-periodic function. Consequently, γ is constant and the assertion follows.

If (φ, ψ) satisfies (10) and one of φ and ψ is, say, convex then φ is a convex solution of (12) with some $b, d \in \mathbb{R}$. Since the function $I \ni x \to d - \varphi(b - x)$ is concave it follows from (12) that φ, being simultaneously convex and concave, is the restriction of an affine function. This completes the proof.

We finish this section with the following simple observation.

LEMMA 1

Let $a, b, c, d \in \mathbb{R}$ and let φ and ψ be homeomorphisms mapping I onto J and satisfying (3) *and* (4). *If one of the numbers a, c is -1 then so is the other one.*

Proof. Assume that $a = -1$. By (4) we get

$$\varphi(x) = \psi(b - x) \quad \text{for } x \in I, \tag{13}$$

whence, by (3) and (4), we have

$$\varphi(x) = c\varphi(b - x) + d = c\psi(x) + d$$

for each $x \in I$. Therefore, taking into account (3), we obtain

$$\varphi(x) - \psi(x) = c[\psi(x) - \varphi(x)] \quad \text{for } x \in I.$$

If $c \neq -1$ then $\varphi = \psi$ contrary to (13) and the invertibility of ψ. Consequently, $c = -1$.

If $c = -1$ then, according to Remark 2 (ii), we may apply the just proved part of the lemma to the system of (7) and (8) and infer that $a = -1$.

4. Bounded domain

Here we prove that if I is bounded then (1) has only trivial solutions.

THEOREM 3

Assume that the interval I is bounded. Then every homeomorphic solution of (1) *is trivial. More exactly: if φ and ψ are homeomorphisms mapping I onto J then (φ, ψ) satisfies* (1) *if and only if either $\varphi = \psi$ or there are point symmetries $S_1 : I \to I$ and $S_2 : J \to J$ such that (φ, ψ) satisfies* (10).

Proof. Assume that (φ, ψ) satisfies (1). Then, by Corollary 1, there are reals a, b, c, d such that conditions (3) and (4) are fulfilled.

If I is a singleton then $\varphi = \psi$. So we can assume that I is not degenerated. Since I is bounded, the identity function is the unique increasing affine function mapping I onto itself. Therefore, putting $f(x) = ax + b$ for $x \in I$, we have

$$f^2(x) = a^2x + b(1 + a) = x \quad \text{for } x \in I,$$

whence either $a = 1$ and $b = 0$, or $a = -1$. In the first case $\varphi = \psi$ on account of (4). In the second case it is enough to use Lemma 1.

The converse is clear due to Corollary 1.

5. Unbounded open domain

By Corollary 2, given homeomorphisms φ and ψ mapping I onto J such that (φ, ψ) is a non-trivial solution of (1), we have only two possible symmetric cases: either

(ξ) $\varphi = \psi \circ T$ for no translation $T : I \to I$ and there exists exactly one point $\xi \in \operatorname{cl} I$ such that $\lim_{x\to\xi} \varphi(x) = \lim_{x\to\xi} \psi(x)$,

or

(η) $\psi = T \circ \varphi$ for no translation $T : J \to J$ and there exists exactly one point $\eta \in \operatorname{cl} J$ such that $\lim_{u\to\eta} \varphi^{-1}(u) = \lim_{u\to\eta} \psi^{-1}(u)$.

Moreover, in case (ξ): if I is a halfline then ξ is its finite endpoint, and in case (η): if J is a halfline then η is its finite endpoint.

In our further considerations we confine ourselves to case (ξ) only. The obvious analogues of results below for case (η) can be formulated by changing the role of I and J as well as of ξ and η, and by replacing the pair (φ, ψ) by $(\psi^{-1}, \varphi^{-1})$.

THEOREM 4

Let φ and ψ be homeomorphisms mapping I onto J such that (φ, ψ) is a non-trivial solution of (1) *and case* (ξ) *holds.*

(i) *Let $I = (-\infty, \xi)$ or $I = (\xi, \infty)$ and assume that there exists a real number r such that the function*

$$I \ni x \to \frac{\varphi(x) - \psi(x)}{|x - \xi|^r}$$

has a finite positive limit at one of the points of I. Then r is non-zero and there are numbers $\eta \in \mathbb{R}$ and $p, q \in \mathbb{R} \setminus \{0\}$ such that

$$\varphi(x) = p\,|x - \xi|^r + \eta \quad \textit{and} \quad \psi(x) = q\,|x - \xi|^r + \eta \quad \textit{for } x \in I.$$

(ii) *Let $I = \mathbb{R}$ and assume that φ and ψ are of the same type of monotonicity and there exist real numbers r, s such that the function*

$$(-\infty, \xi) \ni x \to \frac{\varphi(x) - \psi(x)}{|x - \xi|^s}$$

has a finite positive limit at one of the points $-\infty$, ξ and the function

$$(\xi, \infty) \ni x \to \frac{\varphi(x) - \psi(x)}{|x - \xi|^r}$$

has a finite positive limit at one of the points ξ, ∞. Then r, s are positive and there are numbers $\eta \in \mathbb{R}$ and $p_-, p_+, q_-, q_+ \in \mathbb{R} \setminus \{0\}$ such that

$$\varphi(x) = p_-\,|x - \xi|^s + \eta \quad \textit{and} \quad \psi(x) = q_-\,|x - \xi|^s + \eta \quad \textit{for } x \in (-\infty, \xi)$$

and

$$\varphi(x) = p_+\,|x - \xi|^r + \eta \quad \textit{and} \quad \psi(x) = q_+\,|x - \xi|^r + \eta \quad \textit{for } x \in [\xi, \infty).$$

(iii) *Let $I = \mathbb{R}$ and assume that φ and ψ are of different types of monotonicity and there exists a real number r such that the function*

$$I \ni x \to \frac{\varphi(x) - \psi(x)}{|x-\xi|^r}$$

has a finite positive limit at one of the points $-\infty$, ξ, ∞. *Then* r *is positive and there are numbers* $\eta \in \mathbb{R}$ *and* $p, q \in \mathbb{R} \setminus \{0\}$ *such that*

$$\varphi(x) = p\,|x-\xi|^r \operatorname{sgn}(x-\xi) + \eta \quad \textit{and} \quad \psi(x) = q\,|x-\xi|^r \operatorname{sgn}(x-\xi) + \eta$$

for $x \in I$.

Proof. According to Corollary 1, Remark 2 (i), and Lemma 1 there are reals a, b, c, d such that conditions (3) and (4) are satisfied and $a, c \neq 0, -1$. By virtue of (ξ) we have $a \neq 1$. If $c = 1$ then, by (3), we would have $\varphi - \psi = -d$ contrary to each of the asymptotic conditions imposed on φ and ψ in the statement. Consequently, also $c \neq 1$.

It follows from Remark 2 (iii) that

$$\xi = \frac{b}{1-a}. \tag{14}$$

Put

$$\eta := \frac{d}{1-c} \tag{15}$$

and define a function $\phi : (I - \xi) \to \mathbb{R}$ by

$$\phi(x) := \varphi(x+\xi) - \eta. \tag{16}$$

By (3) and (4) we have

$$\varphi(x) = c\varphi(ax+b) + d \quad \text{for } x \in I.$$

Thus, if $x \in I - \xi$ then, by (14) and (15),

$$\begin{aligned} \phi(x) &= \varphi(x+\xi) - \eta = c\varphi(ax+\xi) + d - \eta \\ &= c[\varphi(ax+\xi) - \eta] + d - \eta(1-c) \\ &= c\phi(ax), \end{aligned}$$

that is ϕ satisfies the Schröder equation

$$\phi(x) = c\phi(ax). \tag{17}$$

At first assume that either $I = (\xi, \infty)$, or $I = \mathbb{R}$ and φ and ψ are of the same type of monotonicity. Then (see Remarks 2 (iii) and 2 (i) in the first case and (3) and (4) in the second one) the numbers a and c are positive. Let $r \in \mathbb{R}$ be such that the function $(\xi, \infty) \ni x \to \frac{\varphi(x)-\psi(x)}{(x-\xi)^r}$ has a finite non-zero limit at one of the points ξ and ∞. If $x \in (0, \infty)$ then, by (15) and (3),

$$\frac{\phi(x)}{x^r} = \frac{\varphi(x+\xi) - \eta}{x^r} = \frac{(1-c)\varphi(x+\xi) - d}{(1-c)x^r} = \frac{\varphi(x+\xi) - \psi(x+\xi)}{(1-c)x^r}.$$

Thus the function $(\xi,\infty)\ni x\to\frac{\phi(x)}{x^r}$ has a finite non-zero limit p at one of the points 0 and ∞. By (17)

$$\frac{\phi(x)}{x^r}=ca^r\frac{\phi(ax)}{(ax)^r}\quad\text{for } x\in(0,\infty),$$

whence $p=ca^rp$ and, consequently, $r=-\frac{\log c}{\log a}$. Therefore, applying [1; Theorem 9] by J. Aczél and M. Kuczma or [2; Proposition 4 (i)] by N. Brillouët-Bellout and J. Gapaillard to (17) or to the equivalent equation

$$\phi(x)=\frac{1}{c}\phi\left(\frac{1}{a}x\right)$$

we infer that

$$\phi(x)=px^r\quad\text{for } x\in(0,\infty),\tag{18}$$

that is

$$\varphi(x)=p(x-\xi)^r+\eta\quad\text{for } x\in(\xi,\infty)$$

and, by (3) and (15),

$$\psi(x)=cp(x-\xi)^r+c\eta+d=q(x-\xi)^r+\eta\quad\text{for } x\in(\xi,\infty),$$

where $q:=cp$. Since φ is invertible, r is non-zero.

This proves (i) in the case $I=(\xi,\infty)$ and one of two parts of (ii). The rest of (i) and (ii) can be easily obtained by applying what we have just proved to the functions

$$2\xi-I\ni x\to\varphi(2\xi-x)\quad\text{and}\quad 2\xi-I\ni x\to\psi(2\xi-x)\tag{19}$$

and noticing that the continuity of φ at zero in (ii) implies positivity of r and s.

Now we pass to the proof of (iii). Since φ and ψ are of different types of monotonicity it follows from (3) and (4) that the numbers a and c are negative. Using (17) we infer that

$$\phi(x)=c^2\phi(a^2x).\tag{20}$$

Similarly as in the first part we prove that the function $\mathbb{R}\setminus\{0\}\ni x\to\frac{\phi(x)}{x^r}$ has a finite non-zero limit p at one of the points $-\infty$, 0, ∞. Replacing, if necessary, φ and ψ by functions (19) we may additionally assume that this is the case either for 0, or for ∞. By (20)

$$\frac{\phi(x)}{x^r}=c^2a^{2r}\frac{\phi(a^2x)}{(a^2x)^r}\quad\text{for } x\in(0,\infty),$$

whence $p=c^2a^{2r}p$, that is

$$-c(-a)^r=1\tag{21}$$

and $r = -\frac{\log(-c)}{\log(-a)}$. Using [1; Theorem 9] or [2; Proposition 4 (i)] again we have (18). If $x \in (-\infty, 0)$ then $ax \in (0, \infty)$, so, by (17), (18), and (21),

$$\phi(x) = c\phi(ax) = cp(ax)^r = c(-a)^r p(-x)^r = -p(-x)^r.$$

The continuity of ϕ at zero gives $r > 0$. Consequently,

$$\phi(x) = p\,|x|^r \operatorname{sgn} x \quad \text{for } x \in \mathbb{R},$$

whence the forms of φ and ψ follow.

REMARK 3

The power function occuring in Theorem 4 generates a mean M which, for instance, in the case $I = (\xi, \infty)$ is given by

$$M(x, y) = \left(\frac{(x-\xi)^r + (y-\xi)^r}{2}\right)^{\frac{1}{r}} + \xi \quad \text{for } x, y \in I,$$

where r is a non-zero real.

The next two results deal with geometrically convex functions. Given an interval $X \subset (0, \infty)$ a function $F : X \to (0, \infty)$ is said to be *geometrically convex* if

$$F(x^\lambda y^{1-\lambda}) \leqslant F(x)^\lambda F(y)^{1-\lambda} \quad \text{for } x, y \in X \text{ and } \lambda \in [0, 1].$$

Observe that F is geometrically convex if and only if the function $\log \circ F \circ \exp_{|\log X}$ is convex.

LEMMA 2

Let $s, t \in (0, \infty) \setminus \{1\}$ and let $F : (0, \infty) \to (0, \infty)$ be a solution of the equation

$$F(x) = sF(tx). \tag{22}$$

If F is geometrically convex then there exist numbers $r \in \mathbb{R} \setminus \{0\}$ and $p \in (0, \infty)$ such that

$$F(x) = px^r \quad \textit{for } x \in (0, \infty).$$

Proof. Since F is geometrically convex, the function $\log \circ F \circ \exp$ is convex. Thus (see [5; Theorem VII.4.1]) $\log \circ F \circ \exp$ and, consequently, F has the right derivative at every point. Moreover, the function

$$\mathbb{R} \ni x \to (\log \circ F \circ \exp)'_+(x)$$

is increasing, that is

$$(0, \infty) \ni x \to x\frac{F'_+(x)}{F(x)} \tag{23}$$

is an increasing function.

Fix an $x_0 \in (0,\infty)$. Since $t \in (0,\infty)\setminus\{1\}$, it follows that the sequence $(t^n x_0 :\ n \in \mathbb{Z})$ is strictly monotonic and 0 and ∞ are its limit points. By (22) and a simple induction we get

$$F'_+(t^n x_0) = stF'_+(t^{n+1}x_0),$$

whence, in view of (22) again,

$$\frac{F'_+(t^n x_0)}{F(t^n x_0)} = \frac{stF'_+(t^{n+1}x_0)}{sF(t^{n+1}x_0)}$$

or

$$t^n x_0 \frac{F'_+(t^n x_0)}{F(t^n x_0)} = t^{n+1} x_0 \frac{F'_+(t^{n+1}x_0)}{F(t^{n+1}x_0)}$$

for every $n \in \mathbb{Z}$. Thus the function (23) is constant. Let $r \in \mathbb{R}$ be such that

$$x\frac{F'_+(x)}{F(x)} = r \quad \text{for } x \in (0,\infty). \tag{24}$$

Define $p : (0,\infty) \to (0,\infty)$ by

$$p(x) := \frac{F(x)}{x^r}.$$

Then

$$(\log \circ p \circ \exp)(x) = (\log \circ F \circ \exp)(x) - rx \quad \text{for } x \in \mathbb{R}$$

and, by (24),

$$(\log \circ p \circ \exp)'_+(x) = 0 \quad \text{for } x \in \mathbb{R}.$$

Thus $\log \circ p \circ \exp$, as a convex function with the vanishing right derivative is (see [5: Theorem VII.4.5]) a constant function. Consequently, p is constant and F has the required form. It is clear that $r \neq 0$ and $p \in (0,\infty)$.

The Lemma above can be also deduced from a theorem of W. Krull (cf. [4; Theorem 5.11]).

THEOREM 5

Let φ and ψ be homeomorphisms mapping I onto J such that (φ,ψ) is a non-trivial solution of (1) *and case* (ξ) *holds. Assume that the limit $\eta := \lim_{x\to\xi}\varphi(x)$ is finite and one of the following conditions is satisfied* [1]*:*

(i) *$I = (\xi,\infty)$, φ is strictly increasing, and the function $I \ni x \to \varphi(x+\xi)-\eta$ is geometrically convex;*

[1]Clearly, due to a symmetry of the problem one can also impose anologous assumptios on ψ.

(ii) $I = (\xi, \infty)$, φ *is strictly decreasing, and the function* $I \ni x \to \eta - \varphi(x+\xi)$ *is geometrically convex;*

(iii) $I = (-\infty, \xi)$, φ *is strictly increasing, and the function* $I \ni x \to \eta - \varphi(-x+\xi)$ *is geometrically convex;*

(iv) $I = (-\infty, \xi)$, φ *is strictly decreasing, and the function* $I \ni x \to \varphi(-x + \xi) - \eta$ *is geometrically convex.*

Then there exist numbers $r, p, q \in \mathbb{R} \setminus \{0\}$ *such that*

$$\varphi(x) = p\,|x-\xi|^r + \eta \quad \textit{and} \quad \psi(x) = q\,|x-\xi|^r + \eta \quad \textit{for } x \in I.$$

Proof. According to Corollary 1 and Remarks 2 (iii) and 2 (i) we can find $a, b, c, d \in \mathbb{R}$ such that conditions (3) and (4) are satisfied and $a, c \in (0, \infty)$. In view of (ξ) we have $a \neq 1$ and, by the assumption,

$$\lim_{x\to\xi} \varphi(x) = \lim_{x\to\xi} \psi(x) = \eta \in \mathbb{R}. \tag{25}$$

Thus if $c = 1$ then, by (3), we would have $\psi = \varphi + d$, whence $d = 0$ and $\psi = \varphi$, contrary to (ξ). Consequently, $c \neq 1$ and (cf. (3) and (25))

$$\eta = \frac{d}{1-c}. \tag{26}$$

Assume condition (i). Then the function $\phi : (0, \infty) \to (0, \infty)$, given by (16), is geometrically convex. Moreover, using equalities (4), (3), and (26), one can easily verify that ϕ satisfies (17). Thus, by virtue of Lemma 2, there exist numbers $r \in \mathbb{R} \setminus \{0\}$ and $p \in (0, \infty)$ such that (18) is satisfied, that is

$$\varphi(x) = p(x-\xi)^r + \eta \quad \text{for } x \in (\xi, \infty)$$

and, consequently, by (3) and (26),

$$\psi(x) = q(x-\xi)^r + \eta \quad \text{for } x \in (\xi, \infty),$$

where $q := cp$.

Similar arguments prove the assertion under any of conditions (ii) - (iv).

6. Closed halfline

The last theorem deals with functions defined on halflines containing their finite endpoints. In its proof we will use the following obvious fact (cf. also [4; Chapter II]).

LEMMA 3

Let X be a real interval such that $0 \in \operatorname{cl} X$ and let $s, t \in \mathbb{R}$. If $|s| < 1$ and $|t| < 1$ then the zero function is the unique solution $F : X \to \mathbb{R}$ of (22) which is bounded in a vicinity of 0.

Proof. It is enough to observe that

$$F(x) = s^n F(t^n x) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}$$

and let n tend to infinity.

THEOREM 6

Let φ and ψ be homeomorphisms mapping I onto J such that (φ, ψ) is a non-trivial solution of (1) *and case* (ξ) *holds. Assume that $I = (-\infty, \xi\,]$, or $I = [\,\xi, \infty)$, or $I = \mathbb{R}$.*

If one of the functions φ and ψ is of class C^∞ in a neighbourhood of ξ then there exists a positive integer n and numbers $\eta \in \mathbb{R}$ and $p, q \in \mathbb{R} \setminus \{0\}$ such that

$$\varphi(x) = p(x-\xi)^n + \eta \quad \textit{and} \quad \psi(x) = q(x-\xi)^n + \eta \quad \textit{for } x \in I.$$

Proof. Assume for instance that φ is of class C^∞ in a neighbourhood of ξ.

By Corollary 1 there are $a, b, c, d \in \mathbb{R}$ such that conditions (3) and (4) are satisfied. According to Remark 2 (i) and Lemma 1 we have $a, c \neq 0, -1$. By virtue of (ξ) we have $a \neq 1$. If $c = 1$ then, by (3), we would have $\psi = \varphi + a$, whence (cf. (ξ)) $d = 0$ which is impossible (see (ξ) again). Therefore also $c \neq 1$. Clearly ξ satisfies (14). Define η and ϕ: $I - \xi \to \mathbb{R}$ by equalities (15) and (16), respectively. As in the proof of Theorem 4 one can easily verify that ϕ is a solution of equation (17). It follows from (16) that ϕ is of class C^∞ in a neighbourhood of 0. Hence, by (17), we infer that ϕ is of class C^∞.

Assume that $|a| < 1$. (Otherwise consider the equation

$$\phi(x) = \frac{1}{c}\phi\left(\frac{1}{a}x\right)$$

which is equivalent to (17).)

If $|c| < 1$ then, by Lemma 3, we would have $\phi = 0$ contrary to the invertibility of ϕ. Therefore $|c| \geqslant 1$. Choose a non-negative integer n such that

$$\left|ca^{n+1}\right| < 1 \leqslant |ca^n|. \tag{27}$$

Suppose that $n = 0$. Then $|ca| < 1$. Since

$$\phi'(x) = ca\phi'(ax) \quad \text{for } x \in I - \xi$$

it follows from Lemma 3 that $\phi' = 0$ and, consequently, ϕ would be constant. This contradiction proves that $n \in \mathbb{N}$.

Since $\phi^{(n+1)}$ is a continuous solution of the equation

$$\phi^{(n+1)}(x) = ca^{n+1}\phi^{(n+1)}(ax)$$

it follows from (27) and Lemma 3 that $\phi^{(n+1)} = 0$. Thus there exist $p_0, p_1, \ldots, p_n \in \mathbb{R}$, such that

$$\phi(x) = p_0 + p_1 x + \cdots + p_n x^n \quad \text{for } x \in I - \xi.$$

Taking into account the condition

$$\phi^{(k)}(x) = ca^k \phi^{(k)}(ax) \quad \text{for } x \in I - \xi \text{ and } k \in \{0, \ldots, n\},$$

we obtain (cf. (27)) that

$$ca^n = 1 \quad \text{and} \quad p_0 = p_1 = \ldots = p_{n-1} = 0$$

and, consequently,

$$\phi(x) = p_n x^n \quad \text{for } x \in I - \xi.$$

Since ϕ is one-to-one we have $p_n \neq 0$. Hence, setting $p = p_n$, we get

$$\phi(x) = px^n \quad \text{for } x \in I - \xi$$

and the forms of φ and ψ follow by (16) and (3).

Remark 4

If $I = \mathbb{R}$ in Theorem 6 then, since φ is one-to-one, n must be odd.

Theorem 6 can be also deduced from [1; Theorem 22] or [2; Proposition 9 and Theorem 11]. However, we believe that the immediate argument presented above can be of interest for the reader.

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References

[1] J. Aczél, M. Kuczma, *Generalizations of a "folk-theorem" on simple functional equations in a single variable*. Results Math. **19** (1991), 5-21.

[2] N. Brillouët-Belluot, J. Gapaillard, *On a simple linear iterative functional equation*, Demonstratio Math. **31** (1998), 735-752.

[3] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[4] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Matematyczne **46**, Polish Scientific Publishers, Warszawa, 1968.

[5] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Polish Scientific Publishers and Silesian University, Warszawa – Kraków – Katowice, 1985.

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