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Peter Kahlig, Janusz Matkowski and
Aleksander N. Sharkovsky

Dido's functional equation revisited

Dedicated to Professor Zenon Moszner
on his 70th birthday

Abstract. Applying the method of invariants ("first integrals"), an explicit form of the general solution of Dido's functional equation $2f(2x) = f(x) + \sqrt{f(x)^2 + \frac{1}{x^2}}$, related to the ancient isoperimetric problem, is determined. The general solution depends on an arbitrary positive and bounded 1-periodic function. There is a unique solution having a given finite and positive limit at infinity. This improves a result proved by a different method in a paper of the first two authors.

1. Introduction

The Dido functional equation

$$2f(2x) = f(x) + \sqrt{f(x)^2 + \frac{1}{x^2}}, \quad x > \alpha, \tag{1}$$

where $f : (\alpha, \infty) \to (0, \infty)$ and $\alpha > 0$ is fixed, appears in connection with the ancient isoperimetric problem of Dido (cf. [1]). According to the legend, Queen Dido of Carthage could get such an amount of land that a bull's hide would cover. Smartly cutting the hide into one long strip, she intuitively solved the isoperimetric problem: starting with a regular triangle, say, she observed that the area would increase noticeably upon doubling the number of vertices by halving the sides of the initial triangle (obtaining a regular hexagon). Repeating this process of halving the sides of consecutive regular n-

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gons, she enlarged the area at each step. This procedure leads to the functional equation (1). (For details cf. [1].)

To find the explicit form of the general solution of equation (1), we apply the *method of invariants* described in [2]. According to this method, we adjoin to the functional equation (1) a self-mapping, the so-called "characteristic map" (cf. [2]), of the first quadrant of the plane. The graph of every solution, as a subset of the first quadrant, is invariant with respect to the characteristic map, i.e. the graph consists of a family of trajectories of this map. In order to get all invariant sets which "give" (i.e. correspond to) solutions of the functional equation, it is sufficient to find a "complete collection" of independent invariant functions — in our case, two such functions. This allows to determine the general solution in explicit form (cf. Theorem 1).

As an immediate corollary we obtain Theorem 2 which says that every solution ϱ of equation (1) satisfying the condition

$$\lim_{x\to\infty} \varrho(x) = a \quad (a > 0,\ \text{fixed})$$

must be of the form

$$\varrho(x) = \frac{1}{x}\cot\frac{1}{ax}, \quad x \in (\alpha, \infty).$$

This improves the main result of [1] where the solution ϱ is assumed to satisfy a stronger asymptotic type regularity condition at infinity.

2. Motivation for the Dido equation and preliminary remarks

We present a sketch of the geometrical reasoning showing how the Dido functional equation appears in connection with the oldest variational problem. (For more details cf. [1].)

Remark 1

Consider a right-angled triangle with sides of length a, b, c, such that $\max(a, b) < c$. Construct an isosceles triangle in the following way: modify the sides of length b and c to an arbitrary common value $b' = c'$; denote the length of the new base side by a', and by m the length of the height of the isosceles triangle, orthogonal to the base side of length a'. Then (Lemma 1 in [1])

$$m = \frac{b+c}{2} \quad \text{iff} \quad a' = a.$$

Considering regular polygons of order $n \in \mathbb{N}_2 := \mathbb{N}\setminus\{1\}$ with the same fixed perimeter P, and denoting by r_n the radius of the inner circle, by R_n the radius of the outer circle, and by $s_n (= \frac{P}{n})$ the length of any segment, we get

$$r_{2n} = \frac{r_n + R_n}{2} \quad \text{and} \quad R_n = \sqrt{r_n^2 + \left(\frac{s_n}{2}\right)^2} = \sqrt{r_n^2 + s_{2n}^2}\,.$$

These two relations imply that

$$2r_{2n} = r_n + \sqrt{r_n^2 + \left(\frac{P}{2n}\right)^2}, \quad n \in \mathbb{N}_2\,.$$

Setting

$$f(n) := \frac{2r_n}{P},$$

we can write this recurrence in the form

$$2f(2n) = f(n) + \sqrt{f(n)^2 + \frac{1}{n^2}}, \quad n \in \mathbb{N}_2\,. \tag{2}$$

Extending the domain of f to (α, ∞) where $\alpha > 0$ is a suitably chosen number, and replacing f by the corresponding function $\varrho : (\alpha, \infty) \to (0, \infty)$, we get the Dido functional equation

$$2\varrho(2x) = \varrho(x) + \sqrt{\varrho(x)^2 + \frac{1}{x^2}}, \quad x \in (\alpha, \infty).$$

In the proof of Theorem 1, for the construction of an invariant function we will need a trigonometric identity for the function arccot presented in the following

REMARK 2

The function cot is a one-to-one mapping of $(0, \pi)$ onto $(-\infty, \infty)$, and, in particular, a mapping of $(0, \frac{\pi}{2})$ onto $(0, \infty)$. Consequently, its inverse, arccot : $(-\infty, \infty) \to (0, \pi)$, maps $(0, \infty)$ onto $(0, \frac{\pi}{2})$. Since the function sin is positive in $(0, \pi)$, we have

$$\cot \frac{x}{2} = \frac{\cos x + 1}{\sin x} = \cot x + \frac{1}{\sin x} = \cot x + \sqrt{\cot^2 x + 1}\,, \quad x \in (0, \pi).$$

Taking into account that $\operatorname{arccot}((0, \infty)) = (0, \frac{\pi}{2})$, we get the identity

$$\operatorname{arccot}\left(w + \sqrt{w^2 + 1}\right) = \frac{1}{2} \operatorname{arccot} w, \quad w \in (0, \infty).$$

3. General solution

The general solution of the Dido functional equation is described by the following

THEOREM 1

Let $\alpha > 0$ be arbitrarily fixed. Then a function $\varrho : (\alpha, \infty) \to (0, \infty)$ satisfies the Dido functional equation (1) *if and only if*

$$\varrho(x) = \frac{1}{x} \cot \left[\frac{1}{x} p\left(\frac{\ln x}{\ln 2}\right) \right], \quad x \in (\alpha, \infty), \tag{3}$$

where $p : \mathbb{R} \to (0, \infty)$ *is a positive* 1*-periodic function such that*

$$p(x) < \frac{\pi}{2} \cdot 2^x \quad \textit{for } x \in (x_0, x_0 + 1], \; x_0 = \frac{\ln \alpha}{\ln 2}. \tag{4}$$

Proof. Suppose that $\varrho : (\alpha, \infty) \to (0, \infty)$ satisfies the functional equation (1). Multiplying this equation by x gives

$$2x\varrho(2x) = x\varrho(x) + \sqrt{[x\varrho(x)]^2 + 1}, \quad x > \alpha.$$

Hence, putting $w(x) := x\varrho(x)$, we obtain

$$w(2x) = w(x) + \sqrt{[w(x)]^2 + 1}, \quad x > \alpha. \tag{5}$$

Define a map $S : (0, \infty)^2 \to (0, \infty)^2$ by

$$S(x, w) = \left(2x, w + \sqrt{w^2 + 1}\right), \quad x, w \in (0, \infty), \tag{6}$$

and put

$$S_1(x) = 2x, \quad S_2(w) = w + \sqrt{w^2 + 1}.$$

The map S is one-to-one, $(0, \infty) \times (0, \infty) \to (0, \infty) \times (1, \infty)$. If the function $x \mapsto w(x)$, $x > \alpha$, is a solution of equation (5), then its graph, as a curve in $(0, \infty)^2$, is invariant with respect to the map $S = (S_1, S_2)$, i.e. it has to consist of the trajectories $\{S^n(x, w(x)) : x \in (\alpha, \infty), \; n \in \mathbb{N}\}$.

Define a two-place function $J_1 : (0, \infty)^2 \to \mathbb{R}$ by

$$J_1(x, w) = x \cdot \operatorname{arccot} w, \quad x, w > 0.$$

Since, in view of Remark 2 and the definition of S_2,

$$\operatorname{arccot} \circ S_2 = \frac{1}{2} \operatorname{arccot},$$

we have, for all $x, w > 0$,

$$(J_1 \circ S)(x, w) = J_1(S_1(x), S_2(w)) = 2x \cdot \operatorname{arccot}(S_2(w)) = J_1(x, w),$$

which means that the function J_1 is S-invariant.

Thus, the set $J_1(x, w) = c$ is a curve invariant with respect to the map S (briefly, S-invariant) which "gives" the solution $w(x) = \cot \frac{c}{x}$ of equation (5), for any constant $c \in (0, \frac{\pi}{2}\alpha)$ (so that $w > 0$). But not every S-invariant curve in $(0, \infty)^2$ can be obtained as a level curve of the function J_1.

As a second invariant function for the map S, it is possible to take any function J_2 depending only on the first variable. Then, according to the definition of S,

$$J_2(2x) = J_2(x), \quad x > 0,$$

and this implies that J_2 is S-invariant if and only if

$$J_2(x, w) = p_0\left(\frac{\ln x}{\ln 2}\right), \quad x, w > 0, \tag{7}$$

where $p_0 : \mathbb{R} \to (0, \infty)$ is some 1-periodic function.

It is obvious that for an arbitrary function $\Phi : (0, \infty)^2 \to \mathbb{R}$ the function $\Phi \circ (J_1, J_2)$ is also S-invariant. It can be shown that every S-invariant curve can be represented in the form $\Phi \circ (J_1, J_2)(x, w) = 0$ (in the same way, every trajectory of S can be given by the intersection of a level curve $J_1(x, w) = c_1$ and a level curve $J_2(x) = c_2$ where c_1, c_2 are some constants).

"Resolving" the equation $\Phi(z_1, z_2) = 0$ with respect to the first argument, we get

$$J_1(x, w) = \varphi \circ J_2(x)$$

where $\varphi : (0, \infty) \to (0, \infty)$ is an arbitrary function. We can rewrite the last equation in the following form

$$x \cdot \operatorname{arccot} w = p\left(\frac{\ln x}{\ln 2}\right)$$

where $p = \varphi \circ p_0$ is an arbitrary 1-periodic function. In this manner we get a representation for an arbitrary invariant curve of the map S.

This allows us to give the corresponding representation for solutions of equation (5). If $w(x)$ is a solution of equation (5) then

$$w(x) = \cot\left[\frac{1}{x} p\left(\frac{\ln x}{\ln 2}\right)\right], \quad x > \alpha$$

where p is a 1-periodic function given by the formula

$$p(x) = 2^x \cdot \operatorname{arccot} w(2^x) \quad \text{for } x \in (x_0, x_0 + 1], \; x_0 = \frac{\ln \alpha}{\ln 2}.$$

The relation $w(x) = x\varrho(x)$ gives the representation formula for the solution ϱ of the Dido equation (1). Since $w(x) > 0$ for all $x > \alpha$, the function p must be such that

$$p\left(\frac{\ln x}{\ln 2}\right) < \frac{\pi}{2}\, x, \quad x > \alpha,$$

and consequently (4) holds. This ends the "only if" part of the proof. Since the "if" part is easy to verify, the proof is completed.

REMARK 3

For each $\alpha > 0$ there exists a lower bound for solutions of the Dido equation (1) on the domain $x > \alpha$, namely, the piecewise continuous function $\varrho_*(x) = \frac{1}{x}\cot\frac{\pi}{2^n}$ for $x \in (2^{n-1}\alpha, 2^n\alpha]$, $n = 1, 2, \dots$.

For any solution $\varrho : (\alpha, +\infty) \to (0, \infty)$ of (1) we have

$$\varrho(x) > \varrho_*(x).$$

for all $x > \alpha$.

REMARK 4

Both the Dido equation (1) and equation (5) have no positive solutions defined on $(0, \infty)$. This is explained by the fact that the map S has no "full" trajectories, i.e. sets of the form $\{S^n(x, w) : n \in \mathbb{Z}\}$, contained in $(0, \infty)^2$; namely, for every $\alpha > 0$ there exists an integer $m > 0$ such that $S^{-m}((\alpha, \infty) \times (0, \infty)) \not\subset (0, \infty)^2$.

REMARK 5

There is full analogy between the notion of invariant function of a map and the notion of first integral (as a function preserving a fixed value along every solution) from the theory of ordinary differential equations. The invariant functions J_1 and J_2 may be also referred to as first integrals of the map S (cf. [2]). This analogy, as is seen from the proof of Theorem 1, can be continued: the relation between a functional equation and its characteristic map is similar to the relation between a partial differential equation and the system of ordinary differential equations for its characteristics.

4. Regular solutions at infinity

Applying Theorem 1 we obtain the following

THEOREM 2

Let $\alpha > 0$ be arbitrarily fixed. Suppose that $\varrho : (\alpha, \infty) \to (0, \infty)$ satisfies the Dido functional equation (1). *If for some $a > 0$*

$$\lim_{x\to\infty} \varrho(x) = a, \tag{8}$$

then

$$\varrho(x) = \frac{1}{x}\cot\frac{1}{ax}, \quad x \in (\alpha, \infty). \tag{9}$$

This function actually satisfies equation (1) *in the interval (α, ∞) if*

$$a \geqslant \frac{2}{\pi\alpha}.$$

Moreover, if

$$a = \frac{2}{\pi\alpha}$$

then

$$\varrho(\alpha+) := \lim_{x\to\alpha+} \varrho(x) = 0.$$

Proof. Taking for the periodic function p the constant $\frac{1}{a}$ in Theorem 1 gives formula (9). The remaining statements are easy to verify.

REMARK 6

Theorem 2 improves the main result of [1] where, instead of (8), it is assumed that for some $a > 0$ and $q > 0$,

$$\varrho(x) = a + o(x^{-q}) \quad \text{as } x \to \infty.$$

5. Remarks about the Dido recurrence

Modifying in an obvious way the arguments used in the proof of Theorem 1, we get the following

THEOREM 3

Let $f : \mathbb{N}_2 \to [0,\infty)$ satisfy the recurrence (2). *If, for some $a > 0$,*

$$\lim_{n\to\infty} f(n) = a,$$

then

$$f(n) = \frac{1}{n}\cot\frac{1}{an}, \qquad n \in \mathbb{N}_2 .$$

Moreover, f satisfies (2) *in $\mathbb{N}_2$ if $a \geqslant \frac{1}{\pi}$, and, if $a = \frac{1}{\pi}$ then $f(2) = 0$. Furthermore, if $f(2) = 0$, then*

$$f(n) < \frac{1}{\pi}, \qquad n \in \mathbb{N}_2 ,$$

and

$$f(0) = \lim_{n\to\infty} f(n) = \frac{1}{\pi}.$$

REMARK 7

According to the geometrical interpretation of the values $f(n)$ in Remark 1, the condition $f(2) = 0$ means that a regular polygon of order 2 has, by definition, a vanishing inner radius.

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P. Kahlig
Institut für Meteorologie und Geophysik
Universität Wien
UZA2, Althanstr. 14
A-1090 Wien
Austria
E-mail: peter.kahlig@univie.ac.at

J. Matkowski
Institute of Mathematics
Pedagogical University
Pl. Słowiański 9
PL-65-069 Zielona Góra
Poland
and
Institute of Mathematics
Silesian University
Bankowa 14
PL-40-007 Katowice
Poland
E-mail: matkow@omega.im.wsp.zgora.pl

A.N. Sharkovsky
Institute of Mathematics
National Academy of Sciences of Ukraine
UA-252004 Kiev
Ukraine
E-mail: asharkov@imath.kiev.ua

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