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Functional equations and characterizations of classes of particular function series

Dedicated to Professor Zenon Moszner on the occasion of his 70th birthday

Abstract. We present numerous new characterizations of $F[\varphi] : \mathbb{R} \to \mathbb{R}$, given by

$$F[\varphi](x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, 1-periodic and even. For these characterizations we use a set of functional equations for $F[\varphi]$ together with appropriate regularity assumptions. Many of those series $F[\varphi]$ of Knopp, Behrend and Mikolás type are prominent continuous nowhere differentiable functions.

1. Historical background

Let $\psi : \mathbb{R} \to \mathbb{R}$ continuous and 1-periodic, α, β positive real numbers and let $G[\psi : \alpha, \beta] : \mathbb{R} \to \mathbb{R}$ be defined by

$$G[\psi; \alpha, \beta](x) := \sum_{k=0}^{\infty} \alpha^k \psi(\beta^k x).$$

Functions of this type have been investigated by many authors. For instance, in the works of Baouche and Dubuc [1] 1994, Behrend [2] 1949, Billingsley [3] 1982, Cater [4] 1984, Hardy [8] 1916, Hata and Yamaguti [9] 1984, Knopp [13] 1918, Mikolás [14] 1956, Schnitzer [16] 1995, Takagi [17] 1903, van der

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Waerden [18] 1930, Weierstrass [19] 1872, various functions $G[\psi;\alpha,\beta]$ for particular generating functions ψ have been proved to be continuous but nowhere differentiable (*cnd*).

By the last property we understand that $G[\psi;\alpha,\beta]$ has no *finite* derivative at any point. Note that some authors like Knopp [13] consider functions which have no derivative, finite or infinite, at any point.

First Knopp, later Behrend and Mikolás presented large classes of generating functions ψ for which they succeeded to state sufficient conditions on α, β guaranteeing the *cnd* property of $G[\psi;\alpha,\beta]$.

The particular choice $\psi(x)=\Delta(x):=\operatorname{dist}(x,\mathbb{Z})$ yields

$$G[\Delta;\alpha,\beta](x)=\sum_{k=0}^{\infty}\alpha^k\Delta(\beta^k x).$$

$G[\Delta;\alpha,\beta]$ is *cnd* for $0<\alpha<1$, $\beta\in\mathbb{N}$, $\alpha\beta\geqslant 1$.

A further specification, namely $\alpha=\frac{1}{2}$, $\beta=2$, gives the Takagi function [17]

$$T(x):=G[\Delta;\tfrac{1}{2},2](x)=\sum_{k=0}^{\infty}\frac{1}{2^k}\Delta(2^k x),$$

which is treated in more or less detail in most of the papers [1] to [17], sometimes without an explicit reference to Takagi. Van der Waerden's [18] famous *cnd* function V is as well generated by the distance function Δ: $V=G[\Delta;\frac{1}{10},10]$.

Another particular choice of ψ, namely $\psi_1(x)=\cos 2\pi x$ or $\psi_2(x)=\sin 2\pi x$ produces Weierstrass [19] type functions

$$G[\psi_1;\alpha,\beta](x)=\sum_{k=0}^{\infty}\alpha^k\cos(2\pi\beta^k x),$$

$$G[\psi_2;\alpha,\beta](x)=\sum_{k=0}^{\infty}\alpha^k\sin(2\pi\beta^k x).$$

By a well known result of Hardy [8], these functions are *cnd* for $0<\alpha<1$ and $\alpha\beta\geqslant 1$ (β not necessarily an integer).

The main subject of this paper are characterizations by functional equations for a suitable class of functions $G[\psi;\alpha,\beta]$ of Knopp, Behrend and Mikolás type.

As we shall see in Section 2, we get an especially rich system of those functional equations if we put $\alpha=\frac{1}{2}$, $\beta=2$ and assume that ψ belongs to the real vector space

$$\mathcal{H} := \{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ continuous, 1-periodic, even}\}.$$

Now let $F[\varphi] := G[\varphi; \frac{1}{2}, 2]$, that is

$$F[\varphi](x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

and define $\mathcal{K}$ to be the vector space of all functions $F[\varphi]$, $\varphi \in \mathcal{H}$.

In particular, for $\varphi = \Delta$ we get $F[\varphi] = T$, for $\varphi(x) = \cos 2\pi x$ we get $F[\varphi](x) = W(x) = \sum_{k=0}^{\infty} 2^{-k} \cos(2\pi 2^k x)$ and for $\varphi(x) = \sin 2\pi x$ we get $F[\varphi](x) = W^*(x) = \sum_{k=0}^{\infty} 2^{-k} \sin(2\pi 2^k x)$.

The Takagi function T and the Weierstrass functions W, W^* are used in the sequel as important examples and counterexamples.

In Section 2 we derive a system $\underline{E}$ of functional equations for $F[\varphi] \in \mathcal{K}$. For a systematical treatment of characterizations of $F[\varphi]$ (the main purpose of this paper) it is helpful to exhibit *all* dependences within the system $\underline{E}$. This is done in Section 2 as well.

In Section 3 we first present *all* possible characterizations of a given $F[\varphi] \in \mathcal{K}$ as a bounded respectively a continuous respectively a bounded and continuous solution of suitable subsets of $\underline{E}$. Finally we give a large number of new characterizations with more subtle regularity assumptions and verify non-characterizations by providing counterexamples.

2. Functional equations for $F[\varphi] \in \mathcal{K}$

It is quite straightforward to check that every $F[\varphi]$ satisfies the following *system* $\underline{E}$ of functional equations on the whole real line:

($\underline{1}$) $F(x) - 2F\left(\frac{x}{2}\right) = -2\varphi\left(\frac{x}{2}\right)$,

($\underline{2}$) $F(x) - 2F\left(\frac{x+1}{2}\right) = -2\varphi\left(\frac{x+1}{2}\right)$,

($\underline{3}$) $F\left(\frac{x}{2}\right) - F\left(\frac{x+1}{2}\right) = \varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{x+1}{2}\right)$,

($\underline{4}$) $F(x) - F\left(\frac{x}{2}\right) - F\left(\frac{x+1}{2}\right) = -\varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{x+1}{2}\right)$,

($\underline{5}$) $F(x+1) - F(x) = 0$,

($\underline{6}$) $F(-x) - F(x) = 0$,

($\underline{7}$) $F(1-x) - F(x) = 0$.

If some $f : \mathbb{R} \to \mathbb{R}$ other than $F[\varphi]$ satisfies equations ($\underline{1}$) to ($\underline{7}$) then $g := f - F[\varphi]$ satisfies on $\mathbb{R}$ the corresponding homogeneous *system* E:

(1) $g(x) - 2g\left(\frac{x}{2}\right) = 0$,

(2) $g(x) - 2g\left(\frac{x+1}{2}\right) = 0$,

(3) $g\left(\frac{x}{2}\right) - g\left(\frac{x+1}{2}\right) = 0,$
(4) $g(x) - g\left(\frac{x}{2}\right) - g\left(\frac{x+1}{2}\right) = 0,$
(5) $g(x+1) - g(x) = 0,$
(6) $g(-x) - g(x) = 0,$
(7) $g(1-x) - g(x) = 0.$

Conversely, if g satisfies some equation (n) out of (1) to (7), then $f := g + F[\varphi]$ satisfies $(\underline{n})$. Therefore, the discussion of the homogeneous system gives full insight in structural properties of $\mathcal{K}$ without referring to fixed elements $F[\varphi]$ of $\mathcal{K}$.

Particular equations of $(\underline{1})$-$(\underline{7})$, respectively (1)-(7), have been thoroughly investigated. (1) and (2) are Schröder type equations, $(\underline{1})$ was examined by de Rham [15] for $\varphi = \Delta$, $(\underline{3})$ is a Nörlund type difference equation, (4) a replicativity equation (Kairies [10]), the geometrical meaning of $(\underline{5})$, $(\underline{6})$ and $(\underline{7})$ is obvious. The pair $(\underline{1})$, $(\underline{2})$ is a system which was discussed in detail by Girgensohn [6], [7]. The system $(\underline{1})$ to $(\underline{4})$ has been investigated for $\varphi = \Delta$ by Kairies [11], a pre-form already by Darsow, Frank and Kairies [5].

Our next goal is a systematic investigation of the whole system (1) to (7): E. The equations of E are not independent from each other. For example, it is easily seen that any two of the equations (5), (6), (7) imply the remaining one. We want to exhibit all such dependences within E. To do so, we introduce a well adapted special notation:

Assume that g satisfies some "subset" S of equations from E and a prescribed regularity property ρ for which we allow four choices: a (no regularity assumption at all), b (bounded), c (continuous), d (bounded and continuous). Then we denote by $S\rho$ the largest "subset" of equations from E which is as well satisfied by g. To keep notations short we accept abbreviations as $S \subset E$ or $\rho \in \{a, b, c, d\}$ and write, say, $S = (1, 2, 3)$ instead of $S = \{(1), (2), (3)\}$.

In the subsequent Propositions 1 to 6 we describe $S\rho$ for any given nonempty $S \subset E$ and any $\rho \in \{a, b, c, d\}$. This complete and explicit collection is given here for the convenience of the reader.

PROPOSITION 1

a) $(1)a = (1)$, $(2)a = (2)$, $(3)a = (3,5)$, $(4)a = (4)$,
$(5)a = (5)$, $(6)a = (6)$, $(7)a = (7)$.

b) $(1)b = (2)b = E$, $(3)b = (3,5)$, $(4)b = (4)$,
$(5)b = (5)$, $(6)b = (6)$, $(7)b = (7)$.

c) $(1)c = (1)$, $(2)c = (2)$, $(3)c = (3,5)$, $(4)c = (4)$,
$(5)c = (5)$, $(6)c = (6)$, $(7)c = (7)$.

d) $(1)d = (2)d = E, \quad (3)d = (3,5), \quad (4)d = (4,5),$
$(5)d = (5), \quad (6)d = (6), \quad (7)d = (7).$

PROPOSITION 2

a) $(1,2)a = (1,3)a = (1,4)a = (1,5)a = (1,2,3,4,5),$
$(1,6)a = (1,6),$
$(1,7)a = E,$
$(2,3)a = (2,4)a = (2,5)a = (1,2,3,4,5),$
$(2,6)a = (2,7)a = E,$
$(3,4)a = (1,2,3,4,5),$
$(3,5)a = (3.5),$
$(3,6)a = (3,7)a = (3,5,6,7),$
$(4,5)a = (4,5),$
$(4,6)a = (4,5,6,7),$
$(4,7)a = (4,7),$
$(5,6)a = (5,7)a = (6,7)a = (5,6,7).$

b) $(1,k)b = (2,m)b = E$ *for* $2 \leqslant k \leqslant 7, \ 3 \leqslant m \leqslant 7,$
$(3,4)b = E,$
$(3,5)b = (3,5),$
$(3,6)b = (3,7)b = (3,5,6,7),$
$(4,5)b = (4,5),$
$(4,6)b = (4,5,6,7),$
$(4,7)b = (4,7),$
$(5,6)b = (5,7)b = (6,7)b = (5,6,7).$

c) $(1,k)c = E$ *for* $2 \leqslant k \leqslant 5,$
$(1,6)c = (1,6),$
$(1,7)c = E,$
$(2,m)c = E$ *for* $3 \leqslant m \leqslant 7,$
$(3,4)c = E,$
$(3,5)c = (3,5),$
$(3,6)c = (3,7)c = (3,5,6,7),$
$(4,5)c = (4,5),$
$(4,6)c = (4,7)c = (4,5,6,7),$
$(5.6)c = (5,7)c = (6,7)c = (5,6,7).$

d) $(1,k)d = (2,m)d = E$ *for* $2 \leqslant k \leqslant 7, \ 3 \leqslant m \leqslant 7,$
$(3,4)d = E,$
$(3,5)d = (3,5),$
$(3,6)d = (3,7)d = (3,5,6,7),$
$(4,5)d = (4,5),$

$$(4,6)d = (4,7)d = (4,5,6,7),$$
$$(5,6)d = (5,7)d = (6,7)d = (5,6,7).$$

PROPOSITION 3

a) $$\begin{aligned}(1,2,6)a &= (1,2,7)a = (1,3,6)a = (1,3,7)a = (1,4,6)a = (1,4,7)a \\ &= (1,5,6)a = (1,5,7)a = (1,6,7)a = (2,3,6)a = (2,3,7)a \\ &= (2,4,6)a = (2,4,7)a = (2,5,6)a = (2,5,7)a = (2,6,7)a \\ &= (3,4,6)a = (3,4,7)a = E,\end{aligned}$$
$$\begin{aligned}(1,2,3)a &= (1,2,4)a = (1,2,5)a = (1,3,4)a = (1,3,5)a = (1,4,5)a \\ &= (2,3,4)a = (2,3,5)a = (2,4,5)a = (3,4,5)a = (1,2,3,4,5),\end{aligned}$$
$$(3,5,6)a = (3,5,7)a = (3,6,7)a = (3,5,6,7),$$
$$(4,5,6)a = (4,5,7)a = (4,6,7)a = (4,5,6,7),$$
$$(5,6,7)a = (5,6,7).$$

b) $$\begin{aligned}(1,2,3)\rho &= (1,2,4)\rho = (1,2,5)\rho = (1,3,4)\rho = (1,3,5)\rho = (1,4,5)\rho \\ &= (2,3,4)\rho = (2,3,5)\rho = (2,4,5)\rho = (3,4,5)\rho = E,\end{aligned}$$
$$(3,5,6)\rho = (3,5,7)\rho = (3,6,7)\rho = (3,5,6,7),$$
$$(4,5,6)\rho = (4,5,7)\rho = (4,6,7)\rho = (4,5,6,7),$$
$$(5,6,7)\rho = (5,6,7)$$
for every $\rho \in \{b,c,d\}$.

PROPOSITION 4

a) $$\begin{aligned}(1,2,3,4)a &= (1,2,3,5)a = (1,2,4,5)a = (1,3,4,5)a = (2,3,4,5)a \\ &= (1,2,3,4,5),\end{aligned}$$
$(3,5,6,7)a = (3,5,6,7)$, $(4,5,6,7)a = (4,5,6,7)$;
for every other S *with four elements we have* $Sa = E$.

b) $$(1,2,3,4)\rho = (1,2,3,5)\rho = (1,2,4,5)\rho = (1,3,4,5)\rho = (2,3,4,5)\rho = E,$$
$$(3,5,6,7)\rho = (3,5,6,7),$$
$$(4,5,6,7)\rho = (4,5,6,7)$$
for every $\rho \in \{b,c,d\}$.

PROPOSITION 5

a) $(1,2,3,4,5)a = (1,2,3,4,5)$;
for every other S *with five elements we have* $Sa = E$.

b) $(1,2,3,4,5)\rho = E$ *for every* $\rho \in \{b,c,d\}$.

PROPOSITION 6

Let card $S \geqslant 6$. *Then* $S\rho = E$ *for every* $\rho \in \{a,b,c,d\}$.

In the following Lemma 1 we collect some facts which are useful for proving the above stated propositions.

LEMMA 1

a) *Let* $e(\lambda,\mu) := (\lambda+\mu)g(x) - 2\lambda g\left(\frac{x}{2}\right) - 2\mu g\left(\frac{x+1}{2}\right)$. *Assume that* $g:\mathbb{R}\to\mathbb{R}$ *satisfies* $e(\lambda_1,\mu_1) = 0$ *and* $e(\lambda_2,\mu_2) = 0$ *for linearly independent* $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in \mathbb{R}^2$. *Then* g *satisfies* $e(\lambda,\mu) = 0$ *for every* $(\lambda,\mu)\in\mathbb{R}^2$. *In particular, any two equations out of* (1) *to* (4) *imply the remaining two.*

b) *Let* $h(x) := g(x+1)$. *Then* h *satisfies equation* (1) *if and only if* g *satisfies equation* (2).

c) *Assume that* G *satisfies equation* (4) *on* $\mathbb{R}$ *and let* $u(x) := G(x+1) - G(x)$. *Then* $u(x) = u\left(\frac{x}{2}\right)$ *holds for every* $x\in\mathbb{R}$.

d) *Assume that* $u:\mathbb{R}\to\mathbb{R}$ *satisfies* $u(x) = u\left(\frac{x}{2}\right)$ *for every* $x\in\mathbb{R}$ *and assume that* G *is a solution of* (4) *on the interval* $[0,1)$ *which is extended to the whole real line by* $G(x+1) = G(x) + u(x)$. *Then* G *satisfies* (4) *for every* $x\in\mathbb{R}$.

e) *For any* $S\subset E$ *and any* $\rho\in\{a,b,c,d\}$ *the solution set*

$$L(S,\rho) := \{g:\mathbb{R}\to\mathbb{R};\ g \text{ has property } \rho \text{ and satisfies } S\}$$

is a real vector space.

The proof of Propositions 1 to 6 and of Lemma 1 shall not be given here. Instead of this we refer the reader to the work of Kairies [12] 1999, where a part of the results is proved and provide just a few examples:

$(4)d = (4,5)$. Assume that g is a continuous and bounded solution of (4) and let $u(x) := g(x+1) - g(x)$. By Lemma 1c, $u(x) = u\left(\frac{x}{2}\right)$ for every $x\in\mathbb{R}$. As u is continuous and bounded as well, necessarily u is a constant function with value γ: $u = \gamma\mathbf{1}$. A summation of the equations $g(x+k+1) - g(x+k) = u(x+k)$, $0\leqslant k\leqslant n-1$, gives $g(x+n) - g(x) = n\gamma$ for any $n\in\mathbb{N}$. The boundedness of g implies $\gamma = 0$. Hence $g(x+1) - g(x) = 0$ for every $x\in\mathbb{R}$. On the other hand, the replicative function given by $W^*(x) = \sum_{k=0}^{\infty} 2^{-k}\sin(2\pi 2^k x)$ provides a bounded and continuous solution of (4) and (5) but it satisfies none of the remaining equations out of E.

$(1,5)a = (1,2,3,4,5)$. (1) and (5) imply (2): $g(x) = g(x+1) = 2g(\frac{x+1}{2})$. (1) and (2) imply (1,2,3,4) by Lemma 1a. On the other hand, it is routine to check that any discontinuous additive function $A:\mathbb{R}\to\mathbb{R}$ with $A(1) = 0$ satisfies (1) to (5) but not (6), (7).

$(4,7)a = (4,7)$. Let $\bar g(x) := W(x) = \sum_{k=0}^{\infty} 2^{-k}\cos(2\pi 2^k x)$ for $0\leqslant x < 1$ and $\bar g(x+1) - \bar g(x) = \operatorname{sgn} x$ for $x\in\mathbb{R}$. The replicative function W satisfies (4) on $[0,1)$. By Lemma 1d, the extension via $\bar g(x+1) - \bar g(x) = \operatorname{sgn} x$, $x\in\mathbb{R}$, satisfies (4) on the whole real line. We have $\bar{g}(x) = W(x)$ on $[0,1]$, $\bar{g}(x) = W(x)+n$ on $(n, n+1]$ and $\bar{g}(x) = W(x)+n$ on $[-n, -n+1)$, $n \in \mathbb{N}$. Therefore $\bar{g}$ also satisfies (7) but none of the remaining equations from E.

The knowledge of the *ρ-generating* sets (sets S with $S\rho = E$) is basic for the first part of Section 3. These sets can be collected from Propositions 1 to 6. We list the minimal ones in the following

THEOREM 1

a) *All minimal a-generating sets are* $(1,7)$, $(2,6)$, $(2,7)$, $(1,3,6)$, $(1,4,6)$, $(1,5,6)$, $(3,4,6)$, $(3,4,7)$.
b) *All minimal b-generating sets are* (1), (2), $(3,4)$.
c) *All minimal c-generating sets are* $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,7)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,7)$, $(3,4)$.
d) *All minimal d-generating sets are* (1), (2), $(3,4)$.

REMARK 1

a) No proper subset of a minimal generating set is generating. The generating sets are the supersets of the listed minimal generating sets.
b) A detailed analysis of the proof of Theorem 1b and c shows:
   (I) Every bounded solution of either (1) or (2) or (3,4) is necessarily the zero function $\mathbf{o}$.
   (II) Every continuous solution of either (1,2) or (1,3) or (1,4) or (1,5) or (1,7) or (2,3) or (2,4) or (2,5) or (2,6) or (2,7) or (3,4) is necessarily $\mathbf{o}$.
c) Let $A : \mathbb{R} \to \mathbb{R}$ be an additive, discontinuous function with $A(1) = 0$. Then $|A|$ satisfies all the equations (1) to (7) [I owe this observation to J. Brzdęk, oral communication]. Consequently, the solution space $L(S,a)$ from Lemma 1e has infinite dimension for every $S \subset E$. However, our results also show that many of the $L(S,\rho)$ degenerate to the zero space if $\rho \in \{b,c,d\}$. For example, $L(S,b) = \{\mathbf{o}\}$ for every b-generating set S, $L(S,c) = \{\mathbf{o}\}$ for every c-generating set S.

3. Characterizations of $F[\varphi] \in \mathcal{K}$

A characterization of some $F[\varphi] \in \mathcal{K}$ by functional equations from (1) to (7) without any regularity assumption is impossible by Remark 1c.

So we start with the previously considered regularity types $\rho \in \{b,c,d\}$ and give as an introductory example characterizations of Takagi's $T(x) = \sum_{k=0}^{\infty} 2^{-k}\Delta(2^k x)$ and of the Weierstrass function W given by

$$W(x) = \sum_{k=0}^{\infty} 2^{-k} \cos(2\pi 2^k x).$$

Recall that these functions are particular $F[\varphi] \in \mathcal{K}$ with $\varphi(x) = \Delta(x)$, respectively $\varphi(x) = \cos 2\pi x$.

PROPOSITION 7

I) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is bounded and satisfies* $f(x) - 2f\left(\frac{x+1}{2}\right) = 2\Delta\left(\frac{x}{2}\right) - 1$, *respectively* $f(x) - 2f\left(\frac{x+1}{2}\right) = 2\cos \pi x$ *for every* $x \in \mathbb{R}$. *Then* $f = T$, *respectively* $f = W$.

II) *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* 1*-periodic and satisfies* $f(x) - 2f\left(\frac{x}{2}\right) = -2\Delta\left(\frac{x}{2}\right)$, *respectively* $f(x) - 2f\left(\frac{x}{2}\right) = -2\cos \pi x$ *for every* $x \in \mathbb{R}$. *Then* $f = T$, *respectively* $f = W$.

Proof. I) $g := f - T$, respectively $g := f - W$ is bounded and satisfies (2). By Remark 1b(I), g is necessarily the zero function $\mathbf{o}$.

II) $g := f - T$, respectively $g := f - W$ is continuous and satisfies (1) and (5). By Remark 1b(II), g is necessarily the zero function $\mathbf{o}$.

The same reasoning applies to any arbitrary $F[\varphi] \in \mathcal{K}$ and to every statement from the exhaustive collection in Remark 1b and yields

THEOREM 2

Assume that $f : \mathbb{R} \to \mathbb{R}$ *is bounded and* $\varphi \in \mathcal{H}$. *Then* $f = F[\varphi]$ *iff either* $(\underline{1})$ *or* $(\underline{2})$ *or* $(\underline{3}.\underline{4})$ *holds.*

Any superset of either $(\underline{1})$ *or* $(\underline{2})$ *or* $(\underline{3}, \underline{4})$ *is characteristic as well.*

No other subset of $(\underline{1})$ *to* $(\underline{7})$ *is b-characteristic.*

Proof. $g := f - F[\varphi]$ is bounded and satisfies either (1) or (2) or (3,4). By Remark 1b(I), necessarily $g = \mathbf{o}$. Clearly any superset of (1) or (2) or (3,4) has $\mathbf{o}$ as its only bounded solution. Every other subset of E has a nonzero bounded solution, a fact which can be documented by means of counterexamples.

THEOREM 3

Assume that $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $\varphi \in \mathcal{H}$. *Then* $f = F[\varphi]$ *iff either* $(\underline{1}, \underline{2})$ *or* $(\underline{1}, \underline{3})$ *or* $(\underline{1}, \underline{4})$ *or* $(\underline{1}, \underline{5})$ *or* $(\underline{1}, \underline{7})$ *or* $(\underline{2}, \underline{3})$ *or* $(\underline{2}, \underline{4})$ *or* $(\underline{2}, \underline{5})$ *or* $(\underline{2}, \underline{6})$ *or* $(\underline{2}, \underline{7})$ *or* $(\underline{3}, \underline{4})$ *holds.*

Any superset of either $(\underline{1}, \underline{2})$ *or* ... *or* $(\underline{3}, \underline{4})$ *is characteristic as well.*

No other subset of $(\underline{1})$ *to* $(\underline{7})$ *is c-characteristic.*

Proof. $g := f - F[\varphi]$ is continuous and satisfies (1,2) or (1,3) or ... or (3,4).

By Remark 1b(II), necessarily $g = \mathbf{o}$. Any superset of (1,2) or (1,3) or ... or (3,4) has $\mathbf{o}$ as its only continuous solution. Every other subset of E has a nonzero continuous solution, a fact which can be documented by means of counterexamples.

REMARK 2

a) Some of the above characterizations are well known for particular choices of φ. For instance, Theorem 2 with equation (1) and $\varphi = \Delta$ gives the famous de Rham [15] characterization of Takagi's T.

b) If there is a characterization of some $F[\varphi] \in \mathcal{K}$ as a *bounded and continuous* solution of an appropriate subset of (1) to (7) then this characterization remains true if the *continuity* assumption is dropped. This follows from a comparison of statements b) and d) of Theorem 1 (continuity on top of boundedness has *no* characterizing power).

c) A replacement of "$f : \mathbb{R} \to \mathbb{R}$ bounded" in Theorem 2 by "f is bounded on $(-\infty, a]$ for some $a \in \mathbb{R}$ and $f'(0)$ exists" would immediately rule out the *cnd* functions $f = F[\varphi] \in \mathcal{K}$ and therefore never lead to a general characterization theorem for all functions $F[\varphi] \in \mathcal{K}$. However, the situation changes, if we start from the previously considered statements (I) and (II) from Remark 1b, which were fundamental for our characterizations. We reformulate them in an explicit version which is useful for subsequent considerations.

(B) a) g bounded on $\mathbb{R}$ and (1) imply $g = \mathbf{o}$.
b) g bounded on $\mathbb{R}$ and (2) imply $g = \mathbf{o}$.
c) g bounded on $\mathbb{R}$ and (3,4) imply $g = \mathbf{o}$.

(C) g continuous on $\mathbb{R}$ and either a) (1,2), b) (1,3), c) (1,4), d) (1,5), e) (2,3), f) (2,4), g) (2,5), h) (3,4), i) (1,7), j) (2,6), k) (2,7) imply $g = \mathbf{o}$.

Now a replacement of "g bounded on $\mathbb{R}$" in (B), a) by "g is bounded on $(-\infty, a]$ for some $a \in \mathbb{R}$ and $g'(0)$ exists" makes sense and in fact, as we shall see in a moment, the implication "$g = \mathbf{o}$" remains true after this specific replacement.

d) The following considerations are concerned with the problem: To what extent is it possible to modify or to weaken the assumptions "g bounded on $\mathbb{R}$" in (B) respectively "g continuous on $\mathbb{R}$" in (C) such that the implication "$g = \mathbf{o}$" still remains true? Here are some answers.

PROPOSITION 8

a) *"g bounded on* $\mathbb{R}$*" in* (B)-a) *can be replaced by any statement out of*

1) *g bounded on* $(-\infty, a] \cup [b, \infty)$ *for some* $a, b \in \mathbb{R}$,
2) *g bounded on* $(-\infty, a]$ *for some* $a \in \mathbb{R}$ *and* $g'(0) \in \mathbb{R}$,
3) *g bounded on* $[b, \infty)$ *for some* $b \in \mathbb{R}$ *and* $g'(0) \in \mathbb{R}$,
4) $g'(0) = 0$.

However, it cannot be replaced by any one of

5) *g bounded in* $(-\infty, a]$ *for some* $a \in \mathbb{R}$ *and g continuous on* $\mathbb{R}$,
6) *g bounded in* $[b, \infty)$ *for some* $b \in \mathbb{R}$ *and g continuous on* $\mathbb{R}$,
7) *g bounded on* $(-\infty, a]$ *and bounded a.e. on* $[a, \infty)$ *for some* $a \in \mathbb{R}$,
8) *g bounded on* $[b, \infty)$ *and bounded a.e. on* $(-\infty, b]$ *for some* $b \in \mathbb{R}$,
9) *g analytic and monotonic and convex.*

b) *"g bounded on* $\mathbb{R}$*" in* (B)-b) *can be replaced by any one of 1),*

2′) *g bounded on* $(-\infty, a]$ *for some* $a \in \mathbb{R}$ *and* $g'(1) \in \mathbb{R}$,
3′) *g bounded on* $[b, \infty)$ *for some* $b \in \mathbb{R}$ *and* $g'(1) \in \mathbb{R}$,
4′) $g'(1) = 0$.

However, it cannot be replaced by any one of 5), 6), 7), 8) *or* 9).

c) *"g bounded on* $\mathbb{R}$*" in* (B)-c) *can be replaced by any one of* 1), 2), 2'), 3), 3'), 4), 4'), 5), 6), 7), 8) *or* 9) *and by any one of*

10) $|g|$ *attains it maximum at some* $\xi \in \mathbb{R}$,
11) *g is bounded on some nonvoid open set* $T \subset \mathbb{R}$,
12) *g is continuous in some* $\zeta \in \mathbb{R}$.

However, it cannot be replaced by any one of

13) *g is bounded below,*
14) *g is bounded above.*

Proof. a) 1). (1) implies $g(2^n x) = 2^n g(x)$ for every $x \in \mathbb{R}, n \in \mathbb{N}$ and $g(0) = 0$. By hypothesis, there is a $\gamma \in \mathbb{R}_+$ such that $|g(z)| \leqslant \gamma$ for $z \in (-\infty, a]$ and for $z \in [b, \infty)$. Consequently, for every $x \neq 0$ there is an $n_0 \in \mathbb{N}$ such that $|g(x)| = |g(2^n x)| \cdot 2^{-n} \leqslant \gamma \cdot 2^{-n}$ holds, whenever $n \geqslant n_0$. Hence $g(x) = 0$.

4). If $g(x) = \alpha$ for some $x \neq 0$, all the difference quotients $\frac{g(2^{-n}x) - g(0)}{2^{-n}x - 0}$, $n \in \mathbb{N}$, have the value $\frac{\alpha}{x}$, as $g(2^{-n}x) = 2^{-n}g(x)$ and $g(0) = 0$. Hence $\alpha = g'(0)x = 0$ and g has to be the zero function.

2). As in 1) we have $g(x) = 0$ for $x \leqslant 0$. So necessarily $g'(0) = 0$ and by 4) we get $g = \mathbf{o}$.

3). Use the same argument as in 2).

5). Counterexample: $\tilde{g}(x) := 0$ for $x \leqslant 0$, $\tilde{g}(x) := x$ for $x > 0$.
6). Counterexample: $\tilde{g}(x) := x$ for $x \leqslant 0$, $\tilde{g}(x) := 0$ for $x > 0$.
7). Counterexample: $\tilde{g}(2^m) := 2^m$ for $m \in \mathbb{Z}$, $\tilde{g}(x) := 0$ otherwise.
8). Counterexample: $\tilde{g}(-2^m) := 2^m$ for $m \in \mathbb{Z}$, $\tilde{g}(x) := 0$ otherwise.
9). Counterexample: $\tilde{g}(x) := x$.

b) Use Lemma 1b and a).

c) Recall that (3,4) imply (1) to (5). By a) and b), any of the conditions 1) to 9),
2′) to 4′) together with (3,4) implies $g = \mathbf{o}$.

10). Assume that $\gamma := \max\{|g(x)|;\ x \in \mathbb{R}\}$ is attained at $\xi \in \mathbb{R}$. Since g is $\frac{1}{2}$-periodic, there is an $\xi_0 \in [0, \frac{1}{2})$ with $|g(\xi_0)| = \gamma$. Because of (3) and (4) we have $|g(2\xi_0)| = |g(\xi_0) + g\left(\xi_0 + \frac{1}{2}\right)| = 2\gamma$, which is only possible for $\gamma = 0$. Hence $g = \mathbf{o}$.

11). Assume that $|g(x)| \leqslant \beta$ for $x \in T$. As g is $\frac{1}{2}$-periodic, we may assume that $|g(x)| \leqslant \beta$ holds for $x \in I_1 \subset [0,1)$, I_1 a closed interval of length 2^{-k} (with some $k \in \mathbb{N}$). Because of $g(2x) = g(x) + g\left(x + \frac{1}{2}\right)$ we get $|g(x)| \leqslant 2\beta$ for $x \in I_2$, I_2 a closed interval of length $2^{-(k-1)}$. Continuing this way, we conclude that $|g(x)| \leqslant 2^{k-1}\beta$ for $x \in I_k$, I_k a closed interval of length $\frac{1}{2}$. By $\frac{1}{2}$-periodicity, g is bounded on $\mathbb{R}$, hence $g = \mathbf{o}$.

12). Note that g is bounded on some open neighbourhood of ζ and apply 11).

13, 14). Let $A : \mathbb{R} \to \mathbb{R}$ denote some discontinuous additive function with $A(1) = 0$. Then the functions $|A|$, $-|A|$ are bounded below respectively bounded above and satisfy (3) as well as (4).

PROPOSITION 9

"g continuous on $\mathbb{R}$" in (C) *can be replaced for every of the cases* a) *to* k) *by any one of the statements* 1), 2), 2'), 3), 3'), 4), 4'), 5), 6), 7), 8), 9), 10), 11) *or* 12) *but in none of these cases by* 13) *or by* 14).

Proof. Note that the pairs in (C) a) to h) each imply the equations (1) to (5). So in any of these cases we are exactly in the same situation as in Proposition 8, Part c) and the assertions are proved as before. The pairs in (C) i), j) and k) each imply even the full set (1) to (7). Therefore the replacement of "g continuous on $\mathbb{R}$" by one of 1) or ... or 12) is possible as well for i), j) and k). The counterexamples $|A|$ and $-|A|$ given at the end of the proof of Proposition 8 work as well in the cases i), j) and k), because $|A|$ and $-|A|$ both satisfy the full set (1) to (7).

All the conditions 1), 5), 6), 7), 8), 11), 12), 13), 14) from Proposition 8 for g can be reformulated for $f = g + F[\varphi]$, $F[\varphi] \in \mathcal{K}$, yielding several modifications and improvements of Theorems 2 and 3 (together with some additional non-characterization statements). We finish our paper with just one example, namely the substitution of condition 11) into Theorem 3. This gives numerous new characterizations of the elements $F[\varphi]$ of the vector space $\mathcal{K}$ being Knopp, Behrend and Mikolás type functions.

THEOREM 4

Assume that $f : \mathbb{R} \to \mathbb{R}$ *is bounded on some nonvoid open set* $U \subset \mathbb{R}$ *and* $\varphi \in \mathcal{H}$. *Then* $f = F[\varphi]$ *iff either* $(\underline{1}, \underline{2})$ *or* $(\underline{1}, \underline{3})$ *or* $(\underline{1}, \underline{4})$ *or* $(\underline{1}, \underline{5})$ *or* $(\underline{1}, \underline{7})$ *or* $(\underline{2}, \underline{3})$ *or* $(\underline{2}, \underline{4})$ *or* $(\underline{2}, \underline{5})$ *or* $(\underline{2}, \underline{6})$ *or* $(\underline{2}, \underline{7})$ *or* $(\underline{3}, \underline{4})$ *holds.*

To show the reader at least one of these new characterizations in its explicit form, we restate the essential part of the very last statement of Theorem 4:

Let $\varphi \in \mathcal{H}$ *and assume that* $f : \mathbb{R} \to \mathbb{R}$ *is bounded on some nonvoid open set* $U \subset \mathbb{R}$ *and satisfies*

$(\underline{3})$ $f\left(\frac{x}{2}\right) - f\left(\frac{x+1}{2}\right) = \varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{x+1}{2}\right)$ *as well as*

$(\underline{4})$ $f(x) - f\left(\frac{x}{2}\right) - f\left(\frac{x+1}{2}\right) = -\varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{x+1}{2}\right)$

for every $x \in \mathbb{R}$.

Then necessarily $f(x) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$ *for every* $x \in \mathbb{R}$.

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