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Elementary inequalities for a quotient of Gamma function

In honour of Professor Zenon Moszner on his 70th birthday

Abstract. Three elementary estimations of the ratio $\frac{\Gamma(x+1)}{\Gamma(x+s)}$, for $x > 0$ and $0 < s < 1$ are derived.

Kershaw [2] gave the inequalities

$$\left(x+\frac{1}{2}s\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{1}{2}(s+1)\right)\right] \qquad \text{(K)}$$

for $x > 0$ and $0 < s < 1$. The bibliography in [3] lists several articles containing inequalities of this kind, nearly all of which involve the digamma function ψ ($=\Gamma'/\Gamma$). In this article I present similar inequalities in which the approximants (extreme members) involve only elementary functions.

Lemma

The inequalities

$$\frac{1}{2y} < \log y - \psi(y) < \frac{1}{y}$$

hold for $y > 0$.

Proof. By [1] 1.7 (27),

$$\log y - \psi(y) = \frac{1}{2y} + 2\int_0^\infty \frac{1}{y^2+t^2}\frac{t}{e^{2\pi t}-1}\,dt.$$

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The inequality on the left is obvious from this. That one on the right follows from

$$2\int_0^\infty \frac{1}{y^2+t^2}\frac{t}{e^{2\pi t}-1}\,dt < \frac{1}{\pi}\int_0^\infty \frac{1}{y^2+t^2}\,dt = \frac{1}{\pi y}\left[\arctan\frac{t}{y}\right]_{t=0}^{\infty} = \frac{1}{2y},$$

which was to be proved.

First inequality. For $y > 0$ the Lemma gives

$$\log y - y^{-1} < \psi(y) < \log y - (2y)^{-1}$$

so that, for $s < 1$,

$$\left(y\exp\left(\frac{-1}{y}\right)\right)^{1-s} < (\exp\psi(y))^{1-s} < \left(y\exp\left(\frac{-1}{2y}\right)\right)^{1-s}. \tag{1}$$

Now put $y = x + \frac{1}{2}(s+1)$, and suppose that $x > 0$ and $s > -1$. The middle member in (1) becomes

$$\exp\left[(1-s)\psi\left(x+\frac{1}{2}(s+1)\right)\right]$$

and so (K) gives

$$\left(x+\frac{1}{2}s\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(\left(x+\frac{1}{2}(s+1)\right)\exp\left(\frac{1}{2x+s+1}\right)\right)^{1-s} \tag{I}$$

for $x > 0$ and $0 < s < 1$. This is my first elementary inequality.

REMARKS

Te inequality on the right of (I) is of course less close than that in (K), because it is derived from (1). The fractional increase so entailed is, by (1), less than

$$\frac{\left(y\exp\left(-\frac{1}{2y}\right)\right)^{1-s}}{\left(y\exp\left(-\frac{1}{y}\right)\right)^{1-s}} = \exp\left(\frac{1-s}{2y}\right) = \exp\left(\frac{1-s}{2x+s+1}\right).$$

As x and s increase this decreases. For $x \geqslant 1$ and $s \geqslant \frac{1}{1}$ it is less than 1.260; for $x \geqslant 5$ and $s \geqslant \frac{1}{2}$ it is less than 1.045. As $x \to \infty$ it tends to 1, uniformly on $s \in [0,1]$.

Second Inequality. Again suppose that $x > 0$ and $0 < s < 1$, the conditions assumed for (I). In (I) replace x by $x+s$ and s by $1-s$; this is permissible because $x+s > 0$ and $0 < 1-s < 1$. Thus

$$\left(x+\frac{1}{2}(s+1)\right)^s < (x+s)\frac{\Gamma(x+s)}{\Gamma(x+1)} < \left(\frac{x+\frac{1}{2}s+1}{\exp\left(\frac{1}{2x+s+2}\right)}\right)^s;$$

now by taking reciprocals,

$$(x+s)\left(\frac{\exp\left(\frac{1}{2x+s+2}\right)}{x+\frac{1}{2}s+1}\right)^s < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{x+s}{\left(x+\frac{1}{2}(s+1)\right)^s}. \tag{II}$$

This is my second elementary inequality, proved for $x > 0$ and $0 < s < 1$.

Third Inequality. Inequalities (I) and (II) together give the simpler elementary inequality

$$\left(x+\frac{1}{2}s\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{x+s}{\left(x+\frac{1}{2}(s+1)\right)^s}. \tag{III}$$

for which the conditions $x > 0$ and $0 < s < 1$ are again sufficient.

Remarks

To assess how close the extremes in (III) are to the middle member, consider their ratio

$$f(x,s) := \frac{x+s}{x+\frac{1}{2}s}\left(\frac{x+\frac{1}{2}s}{x+\frac{1}{2}(s+1)}\right)^s. \tag{2}$$

Differentation gives

$$\frac{\partial f}{\partial x} = \frac{\frac{1}{2}s}{\left(x+\frac{1}{2}s\right)^2}\left(1+\frac{\frac{1}{2}}{x+\frac{1}{2}s}\right)^{-s-1}\left(\left(1+\frac{\frac{1}{2}s}{x+\frac{1}{2}s}\right)-\left(1+\frac{\frac{1}{2}}{x+\frac{1}{2}s}\right)\right).$$

Since $0 < s < 1$, we have $\frac{\partial f}{\partial x} < 0$. So $f(\cdot, s)$ decreases as x increases in $(0,\infty)$, for any fixed $s \in (0,1)$. By (2),

$$f(0+,s) = 2\left(1+\frac{1}{s}\right)^{-s} \tag{3}$$

and $f(x,s) \to 1$ as $x \to \infty$. Thus the range of $f(\cdot, s)$ is the interval

$$(1, 2\left(1+\tfrac{1}{s}\right)^{-s}).$$

Further

$$\frac{d}{ds}\log\left(1+\frac{1}{s}\right)^{-s} = -\left(\log\left(1+\frac{1}{s}\right) - \frac{1}{s+1}\right).$$

Writting $v = \frac{1}{s}$, this is equal to

$$\frac{v}{1+v} - \int_0^v \frac{dt}{1+t} = \int_0^v \left(\frac{1}{1+v} - \frac{1}{1+t}\right) dt < 0;$$

so that $f(0+, s)$ in (3) decreases as s increases. Its minimum, at $s = 1$, is 1, by (3); and its maximum (or rather, upper bound), at $s = 0+$, is

$$2 \lim_{v \to \infty} (1+v)^{-\frac{1}{v}} = 2.$$

So the range of (III) is the interval $(1, 2)$.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vol. I., McGraw-Hill, New York – Toronto – London, 1953.

[2] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607-611.

[3] J. Pecarić, G. Allasia, C. Giordano, *Convexity and the gamma function*, Indian J. Math. **41** (1999), 79-93.

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