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The generalized associativity equation revisited

Dedicated to Professor Zenon Moszner
on his 70th birthday

Abstract. In this note we solve the generalized associativity equation in the real case supposing that the unknown functions are continuous and strictly monotonic in each variable. We do not suppose surjectivity of any kind.

1. Introduction

In Aczél [1] and also in Aczél–Belousov–Hosszú [3] the generalized associativity equation

$$F(G(x,y),z) = H(x,K(y,z)) \tag{1}$$

is solved on quasigroups and the result is specialised, by using the classical result of Aczél [2], [1] on the real continuous cancellative semigroups (see also Craigen–Páles [4]), to real continuous case. Because of the quasigroup technics surjectivity assumptions were made. In Taylor [7] equation (1) is solved on generalized groupoids supposing surjectivity and other solvability conditions. Therefore some important functions (like simple addition on restricted domain and some mean values) are excluded from the investigations.

In this note we do not suppose surjectivity of any kind. We suppose however that the functions in (1) are real-valued, defined on the cartesian product of real intervals and that they are continuous and strictly monotonic in each variable. We follow the method used in Maksa [5] for solving the generalized bisymmetry equation, namely first we solve the simpler associativity equation

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$$A(u+v,w)=B(u,v+w) \tag{2}$$

and next we reduce (1) to (2) by using a result of von Stengel [6].

The set of all real numbers will be denoted by $\mathbb{R}$. By a real interval we shall mean a subinterval of positive length of $\mathbb{R}$. A real valued function f will be called *CM function* or simply $f \in CM$ if f is defined on the Cartesian product of finitely many real intervals and if it is continuous and strictly monotonic in each real variable (not necesseraly in the same direction). If U and V are real intervals then, obviously, $U+V=\{u+v: \ u \in U, v \in V\}$ is a real interval again.

2. A simple asssociativity equation

In this section we solve equation (2) by proving the following

LEMMA 1

Let U, V, and W be real intervals, $A:(U+V)\times W \to \mathbb{R}$, $B: U\times(V+W) \to \mathbb{R}$, and suppose that (2) *holds for all $u \in U$, $v \in V$, and $w \in W$. Then there exists a $\varphi: U+V+W \to \mathbb{R}$ such that*

$$A(p,q)=\varphi(p+q) \tag{3}$$

for all $p \in U+V$, $q \in W$ and

$$B(t,s)=\varphi(t+s) \tag{4}$$

for all $t \in U$, $s \in V+W$. Furthermore, if $A \in CM$ or $B \in CM$ then $\varphi \in CM$, too.

Proof. First we suppose that U, V, and W are compact real intervals: $U=[u_1,u_2]$, $V=[v_1,v_2]$, and $W=[w_1,w_2]$. Choose the intervals

$$W^{(k)}=[w_{1\,k},w_{2\,k}], \quad k=1,\dots,N \ (1<N, \ N \text{ is an integer})$$

such that

$$\max_{1\leqslant k\leqslant N}(w_{2\,k}-w_{1\,k})<v_2-v_1, \tag{5}$$

$$\bigcup_{k=1}^{N} W^{(k)}=W, \tag{6}$$

and

$$w_{1\,k}<w_{1\,k+1}<w_{2\,k}<w_{2\,k+1} \quad (k=1,\dots,N-1). \tag{7}$$

For fixed $k \in \{1,\dots,N\}$ and $i \in \{1,2\}$ define

$$\varphi_{ik}(\xi)=A(\xi-w_{ik},w_{ik}), \quad \xi \in U+V+w_{ik}.$$

Then, by (2)

$$\varphi_{ik}(t+s) = A(t+s-w_{ik}, w_{ik}) = B(t,s) \tag{8}$$

for all $t \in U$ and $s \in V + w_{ik}$. This shows that

$$\varphi_{1\,k}(t+s) = \varphi_{2\,k}(t+s) \tag{9}$$

if $t \in U$ and $s \in (V+w_{1\,k}) \cap (V+w_{2\,k})$. Since (5) holds $(V+w_{1\,k}) \cap (V+w_{2\,k}) = [v_1 + w_{2\,k}, v_2 + w_{1\,k}]$ and so

$$U + ((V+w_{1\,k}) \cap (V+w_{2\,k})) = (U+V+w_{1\,k}) \cap (U+V+w_{2\,k}).$$

Thus, by (9),

$$\varphi_{1\,k}(\xi) = \varphi_{2\,k}(\xi) \quad \text{if } \xi \in (U+V+w_{1\,k}) \cap (U+V+w_{2\,k})$$

and $k \in \{1, \ldots, N\}$. Therefore the definition of the function φ_k (k is fixed in $\{1, \ldots, N\}$) given by

$$\varphi_k(\xi) = \begin{cases} \varphi_{1\,k}(\xi) & \text{if } \xi \in U+V+w_{1\,k}, \\ \varphi_{2\,k}(\xi) & \text{if } \xi \in U+V+w_{2\,k} \end{cases}$$

is correct. The function φ_k is defined on $U+V+W^{(k)}$ since, by (5), $(U+V+w_{2\,k}) \cup (U+V+w_{2\,k}) = U+V+W^{(k)}$. Furthermore (9) implies that

$$\varphi_k(t+s) = B(t,s) \tag{10}$$

for all $t \in U$, $s \in V + W^{(k)}$, and $k \in \{1, \ldots, N\}$.

This shows that $\varphi_k(\xi) = \varphi_{k+1}(\xi)$ if

$$\xi \in \left(U+V+W^{(k)}\right) \cap \left(U+V+W^{(k+1)}\right) \quad \text{and} \quad k \in \{1, \ldots, N-1\}.$$

This, (6) and (7) imply that the function φ is well-defined by

$$\varphi(\xi) = \varphi_k(\xi) \quad \text{if } \xi \in U+V+W^{(k)} \quad \text{for some } k \in \{1, \ldots, N\},$$

$\varphi : U+V+W \to \mathbb{R}$ and, by (10), we obtain (4).

To prove (3) let $p \in U+V$ and $q \in W$. Then $p = u+v$ for some $u \in U$ and $v \in V$. Thus, by (2) and (4) we get

$$\begin{aligned} A(p,q) &= A(u+v,q) = B(u,v+q) \\ &= \varphi(u+(v+q)) = \varphi((u+v)+q) \\ &= \varphi(p+q), \end{aligned}$$

i.e. (3) holds, too.

Finally we drop the requirement of compactness of U, V, and W. There are sequences (U_n), (V_n), and (W_n) of compact real intervals such that

$$U_n \subset U_{n+1}, \quad V_n \subset V_{n+1}, \quad W_n \subset W_{n+1}$$

for all natural numbers n and

$$U = \bigcup_{n=1}^{\infty} U_n, \quad V = \bigcup_{n=1}^{\infty} V_n, \quad \text{and} \quad W = \bigcup_{n=1}^{\infty} W_n.$$

We have proved that, for all fixed n

$$A(p,q) = \Phi_n(p+q) \quad \text{and} \quad B(t,s) = \Phi_n(t+s) \tag{11}$$

whenever $p \in U_n + V_n$, $q \in W_n$, $t \in U_n$, $s \in V_n + W_n$ for some $\Phi_n : U_n + V_n + W_n \to \mathbb{R}$. By (11) the restriction of Φ_{n+1} to $U_n + V_n + W_n$ is Φ_n therefore the function φ is well-defined by $\varphi = \bigcup_{n=1}^{\infty} \Phi_n$ (i.e; $\varphi(\xi) = \Phi_n(\xi)$ if $\xi \in U_n + V_n + W_n$), $\varphi : U + V + W \to \mathbb{R}$ and (3) and (4) follow from (11).

It is clear from the construction of φ that $\varphi \in CM$ if $A \in CM$ or $B \in CM$.

It is obvious that the functions A and B given by (3) and (4), respectively, with any $\varphi : U + V + W \to \mathbb{R}$, satisfy (2).

3. The main result

In this section we prove our main result by using our lemma and a consequence of Theorem 21 in von Stengel [6] (see also Maksa [5]).

THEOREM 1

Let X, Y, Z be real intervals, $G : X \times Y \to \mathbb{R}$, $K : Y \times Z \to \mathbb{R}$ be CM functions with $G(X,Y) = J_1$, $K(Y,Z) = J_2$. Let furthermore $F : J_1 \times Z \to \mathbb{R}$ and $H : X \times J_2 \to \mathbb{R}$ be also CM functions and suppose that (1) holds for all $x \in X$, $y \in Y$, and $z \in Z$. Then there exist CM functions $\varphi : I \to \mathbb{R}$, $\varphi_i : J_i \to \mathbb{R}$ $(i = 1,2)$, $a : X \to \mathbb{R}$, $b : Y \to \mathbb{R}$, and $c : Z \to \mathbb{R}$ such that

$$F(\alpha, z) = \varphi(\varphi_1(\alpha) + c(z)), \tag{12a}$$

$$G(x,y) = \varphi_1^{-1}(a(x) + b(y)), \tag{12b}$$

$$H(x,\beta) = \varphi(a(x) + \varphi_2(\beta)), \tag{12c}$$

and

$$K(y,z) = \varphi_2^{-1}(b(y) + c(z)) \tag{12d}$$

hold for all $\alpha \in J_1$, $z \in Z$, $x \in X$, $y \in Y$, and $\beta \in J_2$.

Proof. It follows from von Stengel [6] (Theorem 21) that there exist CM functions $a : X \to \mathbb{R}$, $b : Y \to \mathbb{R}$, $c : Z \to \mathbb{R}$, and $d : a(X) + b(Y) + c(Z) \to \mathbb{R}$ such that

$$F(G(x,y),z) = H(x, K(y,z)) = d(a(x) + b(y) + c(z)) \tag{13}$$

for all $x \in X$, $y \in Y$, and $z \in Z$. Let first $z = z_0 \in Z$ and next $x = x_0 \in X$ be fixed in (13) and define the functions $f_i : J_i \to \mathbb{R}$ $(i = 1, 2)$ by

$$f_1(\alpha) = F(\alpha, z_0) \quad \text{and} \quad f_2(\beta) = H(x_0, \beta)$$

$(\alpha \in J_1,\ \beta \in J_2)$, respectively. Then f_1 and f_2 are CM functions and (13) implies (12b) and (12d) with the CM functions φ_1 and φ_2 defined by

$$\varphi_1(\alpha) = d^{-1}(f_1(\alpha)) - c(z_0) \quad \text{for } \alpha \in J_1$$

and

$$\varphi_1(\beta) = d^{-1}(f_2(\beta)) - a(x_0) \quad \text{for } \beta \in J_2,$$

respectively. To prove (12a) and (12c), write the form G and K given in (12b) and (12d), respectively into (1). Hence we get that

$$F(\varphi_1^{-1}(a(x) + b(y)), z) = H(x, \varphi_2^{-1}(b(y) + c(z))) \tag{14}$$

holds for all $x \in X$, $y \in Y$, and $z \in Z$. With the definitions $U = a(X)$, $V = b(Y)$, $W = c(Z)$,

$$A(p, q) = F(\varphi_1^{-1}(p), c^{-1}(q)) \quad \text{for } p \in U + V,\ q \in W$$

and

$$B(t, s) = H(a^{-1}(t), \varphi_2^{-1}(s)) \quad \text{for } t \in U,\ s \in V + W$$

equation (14) implies equation (2). Thus, applying Lemma 1, we have that there exists a CM function $\varphi : I \to \mathbb{R}$ $(I = U + V + W)$ such that

$$F(\varphi_1^{-1}(p), c^{-1}(q)) = \varphi(p + q) \quad \text{for } p \in U + V,\ q \in W$$

and

$$H(a^{-1}(t), \varphi_2^{-1}(s)) = \varphi(t + s) \quad \text{for } t \in U,\ s \in V + W$$

which obviously imply (12a) and (12c), respectively.

It can easily be checked that the CM functions F, G, H, and K defined by (12a-d) indeed are solutions of (1).

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