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Approximation by a nonlinear convolution operator in modular function spaces

Dedicated to Professor Zenon Moszner on the occasion of His 70th birth anniversary

Abstract. There is investigated a nonlinear convolution operator

$$Tf = \widehat{K} * f + Pf$$

in modular function spaces $L^0_\rho(G)$, with application to approximation of functions f in $L^0_\rho(G)$.

1. One of the leading problems in the theory of Fourier series is the problem of approximation of 2π-periodic functions, Lebesgue integrable in the interval $[-\pi, \pi]$ by means of linear, singular integral operators of the form

$$\int_{-\pi}^{\pi} \widehat{K}_n(x-y) f(y)\, dy.$$

For example, in case of the first arithmetic means of partial sums of Fourier series of a function f, $(\widehat{K}_n)_{n=1}^{\infty}$ is the Fejér kernel

$$\widehat{K}_n(u) = \frac{1}{2n\pi}\left(\frac{\sin \frac{1}{2}nu}{\sin \frac{1}{2}u}\right)^2,$$

and the singularity conditions mean that

$$\int_{-\pi}^{-\delta} K_n(u)\, du + \int_{\delta}^{\pi} K_n(u)\, du \to 0 \quad \text{as } n \to \infty$$

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for every $0 < \delta < \pi$ and

$$\int_{-\pi}^{\pi} K_n(u)\,du = 1 \quad \text{for } n = 1, 2, \ldots$$

(see e.g. [3]). In further investigations, these problems were extended in three directions:

(a) The interval $[-\pi, \pi]$ was replaced by a locally compact abelian group G with Haar measure μ; the most important examples besides the interval $[-\pi, \pi]$ are $G = \mathbb{R} = (-\infty, \infty)$ with the operation of addition, $G = \mathbb{R}_+^0 = [0, \infty)$ with the operation of multiplication and $d\mu = \frac{dt}{t}$, and $G = \mathbb{Z}$ — the group of all integers with the operation of addition.

(b) The set $\mathbb{N} = \{1, 2, \ldots\}$ of indices was replaced by an abstract set W of indices with convergence $\xrightarrow{\mathcal{W}}$ generated by a filter $\mathcal{W}$ of subsets of W; this enables a unique treatment of the "discrete" and the "continuous" case.

(c) The kernel $(\widehat{K}_n)_{n=1}^{\infty}$ is replaced by a kernel $(\widehat{K}_w)_{w \in W}$, where $\widehat{K}_w\colon G \times \mathbb{R} \to \mathbb{R}$ do not need to be linear with respect to the second variable (see [1], [4]).

In this paper we shall investigate the case when the integral operators consist of a linear part and of a nonlinear perturbation.

Let G be an abelian, locally compact Hausdorff topological group with operation $+\colon G \times G \to G$ and with Haar measure μ. The σ-algebra of measurable sets will be denoted by Σ. Let $L^0(G)$ be the space of all measurable, finite μ-a.e. functions $f\colon G \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the extended real line, and let $L^1(G)$ be the subspace of all μ-integrable functions $f \in L^0(G)$ with equality μ-a.e.; we write $\|f\|_1 = \int_G |f(t)|\,d\mu(t)$. In the theory of linear operators a special role is played by the convolution operator $\widehat{K} * f$, defined by

$$(\widehat{K} * f)(x) = \int_G \widehat{K}(x - y) f(y)\,d\mu(y) = \int_G K(t) f(t + x)\,d\mu(t)$$

for $x \in G$, $f, K \in L^1(G)$, where $\widehat{K}(x) = K(-x)$. Recently, there was considered also a nonlinear version of the convolution operator of the form

$$(Tf)(x) = \int_G k(t, f(t + x))\,d\mu(t)$$

(see, e.g. [1], [4]). We shall investigate here the case when the nonlinear operator T is a sum of a linear convolution operator $\widehat{K} * f$ and a perturbation operator P of the form

$$(Pf)(x) = \int_G k(t, f(t+x))\, d\mu(t),$$

i.e.

$$(Tf)(x) = (\widehat{K} * f)(x) + (Pf)(x) \tag{1}$$

for μ-a.e. $x \in G$. We suppose that $K\colon G \to \mathbb{R}$ and $k\colon G \times \mathbb{R} \to \mathbb{R}$ are kernel functions, i.e. $K \in L^0(G)$, $K(0) = 0$ and $k(\cdot, u) \in L^0(G)$ for $u \in \mathbb{R}$, $k(t, \cdot)$ is continuous for $t \in G$, $k(t, 0) = 0$ for $t \in G$. We call K the *linear kernel function*, and k — the *perturbated kernel function*.

Let $0 < L \in L^1(G)$, $p(t) = \frac{L(t)}{\|L\|_1}$. Let $\psi\colon G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, \infty)$, $\psi(\cdot, u)$ is measurable on G for all $u \geqslant 0$, $\psi(t, \cdot)$ is continuous and nondecreasing, $\psi(t, 0) = 0$, $\psi(t, u) > 0$ for $u > 0$, $\psi(t, u) \to \infty$ as $u \to \infty$ for all $t \in G$. The perturbated kernel function k is called $(L, \psi)_0$*-Lipschitz*, if

$$|k(t, u)| \leqslant L(t)\psi(t, u) \quad \text{for all } t \in G,\ u \geqslant 0.$$

The *domain* $\operatorname{Dom} T$ of the operator T is defined as the set of all functions $f \in L^0(G)$ such that $(Tf)(x)$ exists for μ-a.e. $x \in G$ and Tf is a measurable function on G.

2. Let ρ be a *modular* on $L^0(G)$, i.e. a functional $\rho\colon L^0(G) \to \overline{\mathbb{R}}^* = [0, \infty]$ satisfying the conditions:

$$\rho(0) = 0, \qquad \rho(f) > 0 \ \text{ for } f \neq 0, \qquad \rho(-f) = \rho(f)$$

and

$$\rho(\alpha f + \beta g) \leqslant \rho(f) + \rho(g) \quad \text{for } f, g \in L^0(G),\ \alpha, \beta \geqslant 0,\ \alpha + \beta = 1.$$

If, additionally,

$$\rho(\alpha f + \beta g) \leqslant \alpha\rho(f) + \beta\rho(g) \quad \text{for } f, g \in L^0(G) \text{ and } \alpha, \beta \geqslant 0,\ \alpha + \beta = 1,$$

we say that ρ is a *convex modular*.

Every norm $\|\cdot\|$ on a linear subspace L' of $L^0(G)$ defines a convex modular ρ on $L^0(G)$ by means of the conditions $\rho(f) = \|f\|$ if $f \in L'$, $\rho(f) = \infty$ if $f \in L^0(G) \setminus L'$.

The standard example of a modular (convex modular) is provided by the formula $\rho(f) = \int_G \varphi(|f(t)|)\, d\mu(t)$, where $\varphi(0) = 0$ and φ is a homeomorphism (a convex homemorphism) of $\mathbb{R}_0^+$ onto itself, and leads to an Orlicz space $L^\varphi(G)$.

Turning back to the general case, a convex modular ρ on $L^0(G)$ defines a *modular space*

$$L^0_\rho(G) = \{f \in L^0(G)\colon\ \rho(\lambda f) < \infty \text{ for some } \lambda > 0\},$$

which is a linear normed space with the norm

$$\|f\|_\rho = \inf\left\{u > 0\colon\ \rho\left(\frac{f}{u}\right) \leqslant 1\right\},$$

generated by the modular ρ. In the case of a normed space with norm $\|\cdot\|$, the modular $\rho(\cdot) = \|\cdot\|$ defines again the norm $\|\cdot\|_\rho = \|\cdot\|$. In the case of the above mentioned Orlicz convex modular, there holds $L^0_\rho(G) = L^\varphi(G)$. In a general modular space, convergence $\|f_n\|_\rho \to 0$ is equivalent to the condition $\rho(\lambda f_n) \to 0$ for every $\lambda > 0$. If $\rho(\lambda f_n) \to 0$ for some $\lambda > 0$, we say that (f_n) is *ρ-convergent* to 0 and we write $f_n \xrightarrow{\rho} 0$. For further information concerning modulars, see e.g. [5].

There will be needed some definitions concerning modulars. We say that a modular ρ on $L^0(G)$ is:

(a) *monotone*, if from $f, g \in L^0(G)$, $|f| \leqslant |g|$, follows $\rho(f) \leqslant \rho(g)$,

(b) *finite*, if for every set $A \in \Sigma$ of finite measure μ there holds $\chi_A \in L^0_\rho(G)$, where χ_A is the characteristic function of the set A,

(c) *absolutely finite*, if it is finite and if for every $\varepsilon > 0$ and every $\lambda > 0$ there exists a $\delta > 0$ such that for every set $B \in \Sigma$ of measure $\mu(B) < \delta$ there holds the inequality $\rho(\lambda\chi_B) < \varepsilon$,

(d) *J-convex*, if the inequality

$$\rho\left(\int_G p(t)h(t,\cdot)\,d\mu(t)\right) \leqslant \int_{G_1} p(t)\rho[h(t,\cdot)]\,d\mu(t)$$

holds for every nonnegative $p \in L^1(G)$ such that $\|p\|_1 = 1$ and every measurable $h\colon G \times G \to \mathbb{R}^+_0$,

(e) *absolutely continuous*, if for every $f \in L^0(G)$ such that $\rho(f) < \infty$ there hold the following conditions:

1) for every $\varepsilon > 0$ there exists $A \in \Sigma$ such that $\mu(A) < \infty$ and $\rho(f\chi_{G\setminus A}) < \varepsilon$,

2) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $B \in \Sigma$ with $\mu(B) < \delta$ there holds $\rho(f\chi_B) < \varepsilon$,

(f) *subbounded*, if there exist constants $C \geqslant 1$ and $h_0 \geqslant 0$ such that $\rho(f(t+\cdot)) \leqslant \rho(Cf) + h_0$ for all $f \in L^0(G)$ and $t \in G$,

(g) *τ-subbounded*, if there exist a constant $C \geqslant 1$ and a function $h \in L^\infty(G)$ tending to 0 as $t \to 0$ in the sense of the topology in G, satisfying the inequality $\rho(f(t+\cdot)) \leqslant \rho(Cf) + h(t)$ for all $f \in L^0(G)$ and $t \in G$.

Let ρ and η be two modulars on $L^0(G)$ and let ψ be the function from the $(L,\psi)_0$-Lipschitz condition.

We say that $\{\rho,\psi,\eta\}$ is a *properly directed triple*, if there is a set $G_0 \in \Sigma$ of measure $\mu(G_0) = 0$ such that for every number $\lambda \in (0,1)$ there exists a number $C_\lambda \in (0,1)$ satisfying the inequality

$$\rho[C_\lambda \psi(t, |F(\cdot)|)] \leqslant \eta(\lambda F(\cdot))$$

for all $t \in G \setminus G_0$ and $F \in L^0(G)$.

The following example explains the motivation. Let ρ be the Orlicz modular defined by means of a convex homeomorphism $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and let $\psi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the inverse of φ. Since ψ is concave, it follows $\rho(\lambda\psi(|F(\cdot)|) \leqslant \eta(\lambda F(\cdot))$ for $0 < \lambda < 1$, $F \in L^0(G)$, if we take $\eta(F) = \int_G |F(t)|\, d\mu(t)$. Hence $\{\rho,\psi,\eta\}$ is a properly directed triple with $C_\lambda = \lambda$. Let us remark that for a given λ, the set of numbers C_λ is an interval I_λ of the form $I_\lambda = (0,\delta_\lambda)$ with $0 < \delta_\lambda \leqslant 1$ or $I_\lambda = (0,\delta]$ with $0 < \delta_\lambda < 1$. Writing C_λ we shall have in mind an arbitrary number $C_\lambda \in I_\lambda$.

3. We start with the following

EMBEDDING THEOREM

Let ρ and η be two convex modulars on $L^0(G)$, subbounded with constants C', h_0' and C'', h_0'', respectively; moreover, let ρ be monotone and J-convex. Let the linear kernel function K belong to $L^1(G)$ and let the perturbated kernel function k be $(L,\psi)_0$-Lipschitz with $L \in L^1(G)$. Suppose that $\{\rho,\psi,\eta\}$ is a properly directed triple, and let the operator T be given by (1). *Let $0 < \lambda < 1$ and $0 < \alpha \leqslant \frac{1}{2}\min\left(\frac{\lambda^2}{\|K\|_1}, \frac{C_{\lambda^2}}{\|L\|_1}\right)$. Then for every function $f \in L^0_{\rho+\eta}(G) \cap \operatorname{Dom} T$ there holds the inequality*

$$\rho(\alpha T f) \leqslant \frac{1}{2}\lambda[\rho(\lambda C' f) + \eta(\lambda C'' f) + h_0' + h_0'')].$$

Proof. Since ρ is monotone, convex and subbounded with constants C', h_0', so taking $\alpha > 0$ so small that $2\alpha\|K\|_1 \leqslant \lambda^2$ with $0 < \lambda \leqslant 1$ we obtain

$$\begin{aligned}\rho(2\alpha(\widehat{K} * f)) &\leqslant \int_G \frac{|K(t)|}{\|K\|_1}\rho(2\alpha\|K\|_1|f(t+\cdot)|)\, d\mu(t)\\ &\leqslant \frac{\lambda}{\|K\|_1}\int_G |K(t)|\rho(\lambda|f(t+\cdot)|)\, d\mu(t)\\ &\leqslant \lambda[\rho(\lambda C' f) + h_0'].\end{aligned}$$

Since k is $(L,\psi)_0$-Lipschitz, $\{\rho,\ \psi,\eta\}$ is properly directed and η is convex and subbounded with constants C'', h_0'', we get

$$\rho(2\alpha Pf) \leqslant \int_G \frac{L(t)}{\|L\|_1}\rho[2\alpha\|L\|_1\psi(t,|f(t+\cdot)|)\,d\mu(t)$$
$$\leqslant \int_G \frac{L(t)}{\|L\|_1}\eta(\lambda^2 f(t+\cdot))\,d\mu(t)$$
$$\leqslant \lambda[\eta(\lambda C'' f) + h_0''],$$

if $2\alpha\|L\|_1 \leqslant C_{\lambda^2}$. By convexity of ρ, we obtain

$$\rho(\alpha Tf) \leqslant \frac{1}{2}[\rho(2\alpha(\widehat{K} * f) + \rho(2\alpha Pf)]$$
$$\leqslant \frac{1}{2}\lambda[\rho(\lambda C' f) + \eta(\lambda C'' f) + h_0' + h_0''].$$

From the above Theorem one may conclude, immediately:

COROLLARY

Under assumptions of the Embedding Theorem, T maps $L^0_{\rho+\eta}(G) \cap$ Dom T into $L^0_\rho(G)$ and is continuous at 0 in sense of the $(\rho+\eta)$-convergence.

4. Let ρ be a modular on $L^0(G)$ and let $\mathcal{U}$ be a basis of Σ-measurable neighbourhoods of the zero element $0 \in G$. Then the map

$$\omega_\rho\colon L^0(G) \times \mathcal{U} \to \overline{\mathbb{R}}^+$$

defined by

$$\omega_\rho(f,U) = \sup_{t\in U}\rho(f(t+\cdot) - f(\cdot)) \quad \text{for } f \in L^0(G),\ U \in \mathcal{U}$$

is called the *ρ-modulus of continuity.* There holds the following Lemma (see [2], Theorem 2):

LEMMA

Let ρ be a monotone, absolutely finite, absolutely continuous and τ-subbounded modular on $L^0(G)$. Then for every $f \in L^0_\rho(G)$ there exists a constant $\lambda > 0$ such that

$$\omega_\rho(\lambda f, U) \to 0$$

in the sense of $\mathcal{U}$.

APPROXIMATION THEOREM

Let ρ be a convex, monotone, subbounded with constants C', h_0' and J-convex modular on $L^0(G)$ and let η be a τ-subbounded with constant C'' and function h'' modular on $L^0(G)$. Let the linear kernel function K belong to $L^1(G)$ and let the perturbated kernel function k be $(L,\psi)_0$-Lipschitz. Moreover, let $\{\rho,\psi,\eta\}$ be a properly directed triple. Let $0 < \lambda < 1$, $0 < 2\alpha\|L\|_1 \leqslant C_\lambda$, $U \in \mathcal{U}$ and $f \in L^0_{\rho+\eta}(G) \cap \mathrm{Dom}\,T$. Then there holds the inequality

$$\begin{aligned}\rho(\alpha(Tf-f)) \leqslant \frac{1}{4}\omega_\rho(4\alpha\|K\|_1 f, U) \\ + \frac{1}{4}[\rho(8\alpha C'\|K\|_1 f) + h_0'] \int_{G\setminus U} \frac{|K(t)|}{\|K\|_1}\, d\mu(t) \\ + \frac{1}{2}[\eta(\lambda C'' f) + h_U''] + \frac{1}{2} h_G'' \int_{G\setminus U} \frac{L(t)}{\|L\|_1}\, d\mu(t) \\ + \frac{1}{4}\rho[4\alpha(\|K\|_1 - 1)f],\end{aligned} \tag{2}$$

where $h_U'' = \operatorname*{supess}_{t\in U} h''(t)$.

Proof. By convexity of ρ, we have for every $\alpha > 0$

$$\rho(\alpha(Tf-f)) \leqslant \frac{1}{2}\rho[2\alpha(\widehat{K} * f - f)] + \frac{1}{2}\rho(2\alpha Pf). \tag{3}$$

Applying the identity

$$(\widehat{K} * f - f)(x) = \int_G K(t)(f(t+x) - f(x))\, d\mu(t) + (\|K\|_1 - 1)f(x),$$

convexity and monotonicity of ρ, we obtain

$$\rho[2\alpha(\widehat{K} * f - f)] \leqslant \frac{1}{2}J_1 + \frac{1}{2}J_2, \tag{4}$$

where

$$J_1 = \rho\left(4\alpha \int_G |K(t)|\,|f(t+\cdot) - f(\cdot)|\, d\mu(t)\right), \quad J_2 = \rho(4\alpha(\|K\|_1 - 1)f). \tag{5}$$

Since ρ is J-convex and subbounded, we obtain for every $U \in \mathcal{U}$

$$\begin{aligned} J_1 &\leqslant \int_G \frac{|K(t)|}{\|K\|_1}\rho[4\alpha\|K\|_1|f(t+\cdot) - f(\cdot)|]\, d\mu(t) \\ &\leqslant \int_U \frac{|K(t)|}{\|K\|_1}\rho[4\alpha\|K\|_1|f(t+\cdot) - f(\cdot)|]\, d\mu(t) \\ &\quad + \frac{1}{2}\int_{G\setminus U} \frac{|K(t)|}{\|K\|_1}\rho[8\alpha\|K\|_1 f(t+\cdot)]\, d\mu(t) \\ &\quad + \frac{1}{2}\int_{G\setminus U} \frac{|K(t)|}{\|K\|_1}\rho(8\|K\|_1 f) d\mu(t) \\ &\leqslant \omega_\rho(4\alpha\|K\|_1 f, U) + \left[\rho(8\alpha C'\|K\|_1 f) + \frac{1}{2}h_0'\right]\int_{G\setminus U} \frac{|K(t)|}{\|K\|_1}\, d\mu(t). \end{aligned} \tag{6}$$

Moreover, since ρ is monotone and J-convex, the permutated kernel k is $(L,\psi)_0$-Lipschitz, $\{\rho,\psi,\eta\}$ is a directed triple and η is τ-subbounded with constant C'' and function h'', so we obtain for $U \in \mathcal{U}$, $0 < \lambda < 1$ and $0 < 2\alpha\|L\|_1 \leqslant C_\lambda$

$$\begin{aligned} g(2\alpha Pf) &\leqslant \int_G \frac{L(t)}{\|L\|_1}\rho[2\alpha\|L\|_1\psi(t,|f(t+\cdot)|)]\,d\mu(t) \\ &\leqslant \eta(\lambda C''f) + \int_U \frac{L(t)}{\|L\|_1}h''(t)\,d\mu(t) + \int_{G\setminus U}\frac{L(t)}{\|L\|_1}h''(t)\,d\mu(t) \\ &\leqslant \eta(\lambda C''f) + h''_U + h''_G\int_{G\setminus U}\frac{L(t)}{\|L\|_1}\,d\mu(t). \end{aligned}$$

Taking together the last inequality and inequalities (3), (4), (5) and (6) we obtain the inequality (2).

5. We take now a family $\mathcal{T} = (T_w)_{w\in W}$ of operators T_w, filtered by means of a filter $\mathcal{W}$ of subsets of the set W of indices w. Thus we have

$$(T_w f)(x) = (\widehat{K}_w * f)(x) + (P_w f)(x)$$

for μ-a.e. $x \in G$ (the exceptional set of measure zero is supposed to be independent of the index w).

The family $\mathcal{K} = (K_w)_{w\in W}$ of linear kernel functions is called a *linear kernel*, and the family $\mathbb{K} = (k_w)_{w\in W}$ is called a *perturbated kernel.* The *domain* $\operatorname{Dom}\mathcal{T}$ of $\mathcal{T}$ is defined as $\operatorname{Dom}\mathcal{T} = \bigcap_{w\in W}\operatorname{Dom}T_w$. We say that $\mathbb{K}$ is $(\mathcal{L},\psi)_0$*-Lipschitz* for $\mathcal{L} = (L_w)_{w\in W}$, if k_w are $(L_w,\psi)_0$-Lipschitz for all $w \in W$. Convergence in the sense of the filter $\mathcal{W}$ will be denoted by $\xrightarrow{\mathcal{W}}$.

We say that the linear kernel K is *singular*, if the set $\{\|K_w\|_1\colon\ w\in W\}$ is bounded and

$$\frac{1}{\|K_w\|_1}\int_{G\setminus U}|K_w(t)|\,d\mu(t) \xrightarrow{\mathcal{W}} 0$$

for every $U \in \mathcal{U}$. $\mathbb{K}$ is called *strongly singular*, if it is singular and satisfies the condition

$$\|K_w\|_1 \xrightarrow{\mathcal{W}} 1.$$

We say that the perturbated $(\mathbb{L},\psi)_0$-Lipschitz kernel $\mathbb{K}$ is *singular*, if the set $\{\|L_w\|_1\colon\ w\in W\}$ is bounded and

$$\frac{1}{\|L_w\|_1}\int_{G\setminus U}L_w(t)\,d\mu(t) \xrightarrow{\mathcal{W}} 0.$$

Now, we formulate a theorem concerning the ρ-convergence $T_w f \xrightarrow{\mathcal{W}} f$.

CONVERGENCE THEOREM

Let ρ be a convex, monotone, absolutely finite, absolutely continuous, τ-subbounded and J-convex modular on $L^0(G)$. Let η be a τ-subbounded modular on $L^0(G)$. Let $\mathcal{K} = (K_w)_{w\in W}$ be a strongly singular linear kernel and let $\mathbb{K} = (k_w)_{w\in W}$ be an $(L,\psi)_0$-Lipschitz, singular perturbated kernel. Let $\{\rho,\psi,\eta\}$ be a properly directed triple and let $f \in L^0_{\rho+\eta}(G) \cap \operatorname{Dom}\mathcal{T}$. Then there exists a $\lambda > 0$ such that

$$\rho[\lambda(T_w f - f)] \xrightarrow{\mathcal{W}} 0.$$

Proof. We apply inequality (2) with T_w, K_w and L_w in place of T, K and L, respectively. Let $\overline{K} = \sup_{w\in W}\|K_w\|_1$, $\overline{L} = \sup_{w\in W}\|L_w\|_1$. Let $\varepsilon > 0$ be arbitrary. Since $f \in L^0_\eta(G)$, we may find a λ such that $0 < \lambda < 1$ and $\eta(\lambda C'' f) < \frac{1}{8}\varepsilon$. Since $f \in L^0_\rho(G)$, we have $\rho(8\alpha C'\overline{K} f) < \infty$ for sufficiently small $\alpha > 0$; we may also assume that $2\alpha\overline{L} \leqslant C_\lambda$. Moreover, by the Lemma, we may take α so small that $\omega_\rho(4\alpha\overline{K}f, U) \to 0$ in the sense of $\mathcal{U}$, i.e. there exists a $U \in \mathcal{U}$ depending on ε such that

$$\omega_\rho(4\alpha\overline{K}f, U) < \varepsilon.$$

Since η is τ-subbounded, we may choose the set U so small that $h''_U < \frac{1}{8}\varepsilon$. We fix the set U. Now, since $\mathcal{K}$ and $\mathbb{K}$ are singular, there exists a set $W_1 \in \mathcal{W}$ such that

$$[\rho(8\alpha C'\overline{K}f) + h'_0]\int_{G\setminus U} \frac{|K_w(t)|}{\|K_w\|_1}\,d\mu(t) < \varepsilon$$

and

$$h''_G \int_{G\setminus U} \frac{L_w(t)}{\|L_w\|_1}\,d\mu(t) < \frac{1}{4}\varepsilon$$

for $w \in W_1$. We obtain

$$\rho(\alpha(T_w f - f)) \leqslant \frac{3}{4}\varepsilon + \frac{1}{4}\rho[4\alpha(\|K_w\|_1 - 1)f]$$

for $w \in W_1$. Since $f \in L^0_\rho(G)$, there exists a $\lambda_0 > 0$ such that $\rho(\lambda_0 f) < \varepsilon$. Since $\mathcal{K}$ is strongly singular, we have $\|K_w\|_1 - 1 \xrightarrow{\mathcal{W}} 0$. Hence there exists a set $W_2 \in \mathcal{W}$ such that $4\alpha|\,\|K_w\|_1 - 1| \leqslant \lambda_0$ for $w \in W_2$, and so $\rho[4\alpha(\|K_w\|_1 - 1)f] < \varepsilon$ for $w \in W_2$. Taking $W = W_1 \cap W_2$, we thus obtain $\rho(\alpha(T_w f - f)) < \varepsilon$ for $w \in W$. This ends the proof.

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