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Mazur's criterion for continuity of convex set-valued maps

Dedicated to Professor Zenon Moszner on his 70th birthday

Abstract. The following result is proved: If a set-valued map $F: D \to \mathrm{cc}(Y)$ is K-midconvex and for every function $x: [0,1] \to D$ there exists a Lebesgue measurable set-valued map $G: [0,1] \to \mathrm{c}(Y)$ such that $G(t) \subset F(x(t)) + K$, $t \in [0,1]$, then F is K-continuous. It is a generalization of some results obtained by I. Labuda and R.D. Mauldin (for additive functionals) and by R. Ger (for Jensen-convex functionals) related to a problem posed by S. Mazur.

About 1935 S. Mazur posed the following question (cf. [5], Problem 24): In a Banach space E there is given an additive functional f such that, for every continuous function $x: [0,1] \to E$, the superposition $f \circ x$ is Lebesgue measurable. Is f continuous?

The answer to that question, in the affirmative, was given almost fifty years after by I. Labuda and R.D. Mauldin [3] (cf. also Z. Lipecki [4]). In 1995 R. Ger [1] showed that the same remains true in the case where f is a Jensen-convex functional defined on an open convex subset D of E. More precisely, he proved that each Jensen-convex functional $f: D \to E$ such that for every continuous function $x: [0,1] \to D$, the superposition $f \circ x$ admits a Lebesgue measurable majorant, is continuous (cf. [1], Theorem 1′). The aim of this note is to present a set-valued generalization of this result.

Let us begin with some definitions. Let X and Y be topological vector spaces (real and Hausdorff in the whole paper), D be a convex subset of X

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and K be a convex cone in Y (i.e. $K+K\subset K$ and $tK\subset K$ for all $t\geqslant 0$). Denote by $\mathrm{n}(Y)$, $\mathrm{c}(Y)$ and $\mathrm{cc}(Y)$ the families of all nonempty, nonempty compact and nonempty compact convex subsets of Y, respectively.

A set-valued map $F: D\to \mathrm{n}(Y)$ is called *K-Jensen convex* (or *K-midconvex*) if

$$\frac{F(x_1)+F(x_2)}{2}\subset F\left(\frac{x_1+x_2}{2}\right)+K, \tag{1}$$

for all $x_1,x_2\in D$. Note that if F is single-valued and Y is endowed with the relation $\leqslant_K$ of partial order defined by $x\leqslant_K y :\Longleftrightarrow y-x\in K$, then condition (1) reduces to the following one

$$F\left(\frac{x_1+x_2}{2}\right)\leqslant_K \frac{F(x_1)+F(x_2)}{2}.$$

In particular, if $Y=\mathbb{R}$ and $K=[0,\infty)$, we obtain the standard definition of Jensen-convex functions.

We say that a set-valued map $F: D\to \mathrm{n}(Y)$ is *K-continuous* at a point $x_0\in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0)\subset F(x)+W+K \quad\text{and}\quad F(x)\subset F(x_0)+W+K$$

for every $x\in(x_0+U)\cap D$. In the case where $K=\{0\}$, the K-continuity means continuity with respect to the Hausdorff topology on $\mathrm{n}(Y)$. It is known that if F is single-valued and K is a normal cone (i.e. if there exists a base $\mathcal{W}$ of neighbourhoods of zero in Y such that $W=(W-K)\cap(W+K)$ for every $W\in\mathcal{W}$), then K-continuity means continuity.

Finally, recall that a set-valued map $G:[0,1]\to \mathrm{n}(Y)$ is *Lebesgue measurable* if for every open set $W\subset Y$ the set

$$F^+(W)=\{t\in[0,1]:\ F(t)\subset W\}$$

is Lebesgue measurable.

The main result of this note reads as follows:

THEOREM 1

Let E be a real Banach space, D — an open convex subset of E, Y — a locally convex topological vector space and K — a convex cone in Y. Moreover, assume that there exist bounded sets $B_n\subset Y$, $n\in\mathbb{N}$, such that

$$\bigcup_{n\in\mathbb{N}}(B_n-K) \;=\; Y. \tag{2}$$

If a set-valued map $F: D\to \mathrm{cc}(Y)$ is K-midconvex and for every continuous *function* $x : [0,1] \to D$ *there exists a Lebesgue measurable set-valued map* $G : [0,1] \to \mathrm{c}(Y)$ *such that*

$$G(t) \subset F(x(t)) + K, \quad t \in [0,1],$$

then F *is* K*-continuous on* D.

In the proof of this theorem we will use the following result giving a condition under which K-midconvex set-valued maps are K-continuous. We denote by K^* the conjugate cone of K, i.e. the set of all continuous linear functionals on Y which are nonnegative on K.

LEMMA 1 ([6], Theorem 1)
Let E, D, Y *and* K *be such as in Theorem 1. If a set-valued map* $F : D \to \mathrm{cc}(Y)$ *is* K*-midconvex and for every* $y^* \in K^*$ *the functional* $f_{y^*}(x) = \inf y^*(F(x))$, $x \in D$, *is continuous on* D, *then* F *is* K*-continuous on* D.

Proof of Theorem 1. Fix an arbitrary $y^* \in K^*$. Note first that the functional f_{y^*} is Jensen-convex. Indeed, by the K-midconvexity of F, we have

$$\begin{aligned}\frac{f_{y^*}(x_1) + f_{y^*}(x_2)}{2} &= \frac{\inf y^*(F(x_1)) + \inf y^*(F(x_2))}{2}\\ &= \inf y^*\left(\frac{F(x_1) + F(x_2)}{2}\right)\\ &\geqslant \inf y^*\left(F\left(\frac{x_1 + x_2}{2}\right) + K\right)\\ &= \inf y^*\left(F\left(\frac{x_1 + x_2}{2}\right)\right) + \inf y^*(K)\\ &\geqslant f_{y^*}\left(\frac{x_1 + x_2}{2}\right).\end{aligned}$$

Now, take an arbitrary continuous function $x : [0,1] \to D$. By assumption, there exists a Lebesgue measurable set-valued map $G : [0,1] \to \mathrm{c}(Y)$ such that

$$G(t) \subset F(x(t)) + K, \quad t \in [0,1]. \tag{3}$$

The set-valued map $y^* \circ G : [0,1] \to \mathrm{c}(\mathbb{R})$ is also measurable and so, by the Kuratowski and Ryll-Nardzewski theorem (cf. [2]), it has a measurable selection $h : [0,1] \to \mathbb{R}$. Hence, using (3), we get

$$h(t) \in y^*(G(t)) \subset y^*(F(x(t))) + [0,\infty) = [f_{y^*}(x(t)), \infty), \quad t \in [0,1].$$

This means that h is a measurable majorant of the superposition $f_{y^*} \circ x$. Consequently, by the result of Ger mentioned above, the functional f_{y^*} is continuous. Since y^* was fixed arbitrarily in K^*, it follows from Lemma 1 that F is K-continuous. This completes the proof.

Remark

The assumption (2) was not used explicitely in the above proof, but it is needed in Lemma 1. Note that it is trivially satisfied if Y is a normed space (and K is an arbitrary convex cone in Y). It is also fulfilled if there exists an order unit in Y, i.e. such an element $e \in Y$ that for every $y \in Y$ we have $y \leqslant_K ne$ with some $n \in \mathbb{N}$ (we put then $B_n = \{ne\}$). In particular, if $\operatorname{int} K \neq \emptyset$, then every element of $\operatorname{int} K$ is an order unit.

As an immediate consequence of the above theorem (under the same assumptions on E, D, Y and K) we obtain the following corollaries.

Corollary 1

If a set-valued map $F : D \to \operatorname{cc}(Y)$ is K-midconvex and for every continuous function $x : [0,1] \to D$ the superposition $F \circ x$ is Lebesgue measurable, then F is K-continuous on D.

Corollary 2

If a set-valued map $F : D \to \operatorname{cc}(Y)$ is K-midconvex and for every continuous function $x : [0,1] \to D$ the superposition $F \circ x$ has a Lebesgue measurable selection, then F is K-continuous on D.

Corollary 3

If a function $f : D \to Y$ is K-midconvex and for every continuous function $x : [0,1] \to D$ there exists a Lebesgue measurable function $g : [0,1] \to Y$ such that

$$f(x(t)) \leqslant_K g(t), \quad t \in [0,1],$$

then F is K-continuous on D.

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