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Boundary properties of solutions of the iterated Helmholtz equation

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. Some theorems about boundary properties of solutions of the equation $(\Delta - c^2)^2 u(x,y) = 0$ in Hölder spaces are given. This is a continuation of E. Wachnicki's study of the above equation, cf. [6]. Similar theorems referring to Picard's singular integral may be found in [1], [2].

1. Definitions and notation

1.1. In the paper [6] E. Wachnicki gives the solution of the Dirichlet problem for the equation

$$(\Delta - c^2)^2 u(x,y) = 0, \quad c > 0, \tag{1}$$

in the upper half-plane $\mathbb{R}^2_+ = \{(x,y) : |x| < \infty, y > 0\}$, with the conditions

$$u(x,y)|_{y=0} = f(x), \quad \Delta u(x,y)|_{y=0} = g(x),$$

where Δ denotes the Laplace operator, that is

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and f and g are given functions defined on $\mathbb{R}$.

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The solution presented by E. Wachnicki is of the form

$$u(x,y;f,g) = \frac{1}{2\pi}\int_{\mathbb{R}} f(s)\left[2cr^{-1}yK_1(cr) + c^2yK_0(cr)\right] ds \\ - \frac{1}{2\pi}\int_{\mathbb{R}} g(s)yK_0(cr)\, ds, \tag{2}$$

where $r^2 = (x-s)^2 + y^2$ and K_v is the v-th MacDonald's function.

1.2. Let $L^p \equiv L^p(\mathbb{R})$, $1 < p < \infty$, be the classical space of real functions whose p-th power is Lebesgue integrable. The symbol $L^\infty \equiv L^\infty(\mathbb{R})$ will denote the space of uniformly continuous and bounded functions. The space L^p, $1 \leqslant p \leqslant \infty$ is equipped with the usual norm

$$\|f\|_{L^p} := \begin{cases} \left(\int_{-\infty}^{+\infty} |f(x)|^p\, dx\right)^{\frac{1}{p}} & \text{for } 1 \leqslant p < \infty, \\ \sup\limits_{s\in\mathbb{R}} |f(s)| & \text{for } p = \infty. \end{cases} \tag{3}$$

For any given function $f \in L^p$, $1 \leqslant p \leqslant \infty$ and $x, h \in \mathbb{R}$ we introduce the modulus of continuity:

$$\omega(t; f, L^p) := \sup_{|h|\leqslant t} \|\Delta_h f\|_{L^p}, \tag{4}$$

where $\Delta_h f$ is the classical difference:

$$\Delta_h f(x) = f(x+h) - f(x).$$

By Ω (cf. [5]) we denote the set of those functions $f \in L^p$ which have modulus of continuity, satisfying the conditions

a) f is defined and continuous in $[0, +\infty]$,

b) $f(t)$ is increasing for $t > 0$,

c) $\frac{f(t)}{t}$ is decreasing for $t > 0$.

Following [3] we define, for given $\omega \in \Omega$ and $1 \leqslant p \leqslant \infty$, the generalized Hölder space (a subspace of L^p-space), which we denote by $H^{p,\omega}$, as the set of all functions $f \in L^p$ for which the quantity

$$\|f\|_{p,\omega} := \sup_{0<h<1} \frac{\|\Delta_h f\|_{L^p}}{\omega(h)}, \tag{5}$$

is finite. The norm in $H^{p,\omega}$ is defined by

$$\|f\|_{H^{p,\omega}} := \|f\|_{L^p} + \|f\|_{p,\omega}. \tag{6}$$

Like in [3], for given $1 \leqslant p \leqslant \infty$ and $\omega \in \Omega$, we define $\widetilde{H}^{p,\omega}$ as the set of all functions $f \in H^{p,\omega}$ for which

$$\lim_{h\to 0^+} \frac{\|\Delta_h f\|_{L^p}}{\omega(h)} = 0.$$

The norm in $\widetilde{H}^{p,\omega}$ is given by the formula

$$\|f\|_{\widetilde{H}^{p,\omega}} := \|f\|_{L^p} + \|f\|_{p,\omega}\,.$$

In the whole paper the symbols $M_{a,b}$ will denote positive constants, dependent only on the indicated parametres a, b.

2. Auxiliary results

In this part of the paper we will give some auxiliary inequalities which will be used in the proofs of main theorems.

It is known (cf. [6]) that for $y > 0$, $r^2 = (x-s)^2 + y^2$, $\rho^2 = s^2 + y^2$, we have

$$\int_0^{\infty} yK_0(c\rho)\,ds = \frac{\pi}{2c} y e^{-cy}, \tag{7}$$

$$\frac{1}{2\pi}\left\| c^2 \int_{\mathbb{R}} yf(s)yK_0(cr)\,ds \right\|_{L^p} \leqslant \frac{1}{2} c\|f\|_{L^p}\, y, \tag{8}$$

$$\frac{1}{2\pi}\left\| \int_{\mathbb{R}} g(s)yK_0(cr)\,ds \right\|_{L^p} \leqslant \frac{1}{2c}\|g\|_{L^p}\, y, \tag{9}$$

$$\frac{1}{\pi}\int_{\mathbb{R}} cyr^{-1}K_1(cr)\,ds = e^{-cy}. \tag{10}$$

We will show some other properties of the solution $u(\cdot, y; f, g)$ defined by (2).

LEMMA 1

If $f \in L^p$, $1 \leqslant p \leqslant \infty$, then

$$\|u(\cdot, y; f, g)\|_{L^p} \leqslant \frac{3}{2}\|f\|_{L^p} + \frac{1}{4c^2}\|g\|_{L^p} \quad \textit{for } y > 0. \tag{11}$$

Proof. By the definition of the norm (3) from (2) we have

$$\|u(\cdot, y; f, g)\|_{L^p}$$
$$\leqslant \|f\|_{L^p} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\frac{2cy}{\sqrt{t^2+y^2}} K_1\left(c\sqrt{t^2+y^2}\right) + c^2 y \left| K_0\left(c\sqrt{t^2+y^2}\right)\right| \right] dt$$
$$+ \|g\|_{L^p} \frac{1}{2\pi}\int_{\mathbb{R}} y\left|K_0\left(c\sqrt{t^2+y^2}\right)\right| dt$$

Since $K_v\left(c\sqrt{t^2+y^2}\right) \geqslant 0$ for $v = 0, 1$, hence by (7) and (10) we get

$$\|u(\cdot,y;f,g)\|_{L^p} \leqslant \|f\|_{L^p}\frac{1}{2\pi}\left\{2\pi e^{-cy}+2c^2\frac{\pi}{2c}ye^{-cy}\right\}+\|g\|_{L^p}\frac{1}{2\pi}\frac{\pi}{2c}ye^{-cy}$$
$$\leqslant \|f\|_{L^p}\left\{e^{-cy}+\frac{c}{2}ye^{-cy}\right\}+\|g\|_{L^p}\frac{1}{4c}ye^{-cy}.$$

Taking into account the inequalities

$$ye^{-cy}\leqslant\frac{1}{c}\quad\text{and}\quad e^{-cy}<1\quad\text{for } y>0,\ c>0,$$

we finally get (11).

LEMMA 2

If $f,g\in H^{p,\omega}$ *with fixed* $1\leqslant p\leqslant\infty$ *and* $\omega\in\Omega$ *then*

$$\|u(\cdot,y;f,g)\|_{p,\omega}\leqslant\frac{3}{2}\|f\|_{p,\omega}+\frac{1}{4c^2}\|g\|_{p,\omega} \tag{12}$$

for $y>0$, *i.e.* $u(\cdot,y;f,g)\in H^{p,\omega}$ *for any fixed* $y>0$.

Proof. From (1) it follows that for $h,x\in\mathbb{R}$ and $y>0$:

$$\|\Delta_h u(x,y;f,g)\|_{L^p}=u(x,y;\Delta_h f,\Delta_h g). \tag{13}$$

Hence by (3) and Lemma 1 we obtain

$$\|\Delta_h u(\cdot,y;f,g)\|_{L^p}\leqslant\|u(\cdot,y;\Delta_h f,\Delta_h g)\|_{L^p}\leqslant\frac{3}{2}\|\Delta_h f\|_{L^p}+\frac{1}{4c^2}\|\Delta_h g\|_{L^p}.$$

Finally, by the definition (5) we can write

$$\|u(\cdot,y;f,g)\|_{p,\omega}\leqslant\sup_{0<h<1}\frac{\frac{3}{2}\|\Delta_h f\|_{L^p}+\frac{1}{4c^2}\|\Delta_h g\|_{L^p}}{\omega(h)}$$
$$\leqslant\frac{3}{2}\sup_{0<h<1}\frac{\|\Delta_h f\|_{L^p}}{\omega(h)}+\frac{1}{4c^2}\sup_{0<h<1}\frac{\|\Delta_h g\|_{L^p}}{\omega(h)}$$
$$=\frac{3}{2}\|f\|_{p,\omega}+\frac{1}{4c^2}\|g\|_{p,\omega}$$

for $y>0$, which ends the proof of the lemma.

From the definition of the norm in $H^{p,\omega}$ (defined by (6)) and Lemmas 1 and 2 we have the following corollaries:

COROLLARY 1

If $f,g\in H^{p,\omega}$, $1\leqslant p\leqslant\infty$, $\omega\in\Omega$, *then*

$$\|u(\cdot,y;f,g)\|_{H^{p,\omega}}\leqslant\frac{3}{2}\|f\|_{H^{p,\omega}}+\frac{1}{4c^2}\|g\|_{H^{p,\omega}}\quad\text{for } y>0,$$

i.e. for any fixed $y>0$ *the functions* $u(\cdot,y;f,g)\in H^{p,\omega}$.

COROLLARY 2

If $f, g \in \widetilde{H}^{p,\omega}$, $1 \leqslant p \leqslant \infty$, $\omega \in \Omega$, then for any fixed $y > 0$

$$u(\cdot, y; f, g) \in \widetilde{H}^{p,\omega}.$$

3. Main results

3.1. In this part of the paper we will prove two boundary theorems for the solution $u(\cdot, y; f, g)$ of (1) in the case of functions f, g belonging to given L_p-space, $1 \leqslant p \leqslant \infty$.

THEOREM 1

If $f, g \in L^p$, $1 \leqslant p \leqslant \infty$ and the modulus of continuity of the function f fulfills the condition

$$\omega(f, t) \leqslant M t^{\alpha}, \quad 0 < \alpha < 1,\ t > 0,\ M = \text{const.} \tag{14}$$

then there exists a positive constant M_α such that for $0 < y \leqslant 1$ the following inequality holds:

$$\|u(\cdot, y; f, g) - f\|_{L^p} \leqslant M_a y^{\alpha}. \tag{15}$$

Proof. Let $r^2 = (x-s)^2 + y^2$, $\rho^2 = s^2 + y^2$. From the integral form (2) of the function u; (3), (7) and (10) it follows, that (for $y > 0$)

$$\begin{aligned}
\|u(\cdot, y; f, g) - f\|_{L^p} \leqslant & \frac{1}{\pi} \left\| \int_{\mathbb{R}} [f(s) - f(\cdot)]\, c y r^{-1} K_1(cr)\, ds \right\|_{L^p} \\
& + \frac{c^2}{2\pi} \left\| \int_{\mathbb{R}} f(s) y K_0(cr)\, ds \right\|_{L^p} \\
& + \frac{1}{\pi} \left\| \int_{\mathbb{R}} f(s) y K_0(cr)\, ds \right\|_{L^p} + \|f\|_{L^p} |e^{-cy} - 1| \\
\leqslant & A + \frac{3}{2} c \|f\|_{L^p} + \frac{1}{2c} \|g\|_{L^p} y,
\end{aligned}$$

where

$$A = \frac{1}{\pi} \left\| \int_{\mathbb{R}} [f(s) - f]\, c y r^{-1} K_1(cr)\, ds \right\|_{L^p}.$$

Having put

$$\varphi(t, s) = f(t-s) - 2f(t) + f(t+s); \quad t, s > 0,$$

from the definition of modulus of continuity and (14) we can write

$$A = \frac{1}{\pi}\left\|\int_0^\infty \varphi(\cdot,s)cy\rho^{-1}K_1(c\rho)\,ds\right\|_{L^p}$$
$$\leqslant \frac{2Mc}{\pi}\int_0^\infty \|s\|^\alpha y\rho^{-1}K_1(c\rho)\,ds.$$

It is known ([6], [7]) that

$$K_1(c\rho) \leqslant (c\rho)^{-1}.$$

Since $\rho^2 = s^2 + y^2$, this implies

$$A + \frac{2M}{\pi}y\int_0^\infty \frac{s^a}{s^2+y^2}\,ds = \frac{2My}{\pi}\left[\int_0^y \frac{s^2}{s^2+y^2}\,ds + \int_y^\infty \frac{s^a}{s^2+y^2}\,ds\right]$$
$$\leqslant \frac{2My}{\pi}\left[\frac{y^\alpha}{y^2}\int_0^y ds + \int_y^\infty s^{\alpha-2}\,ds\right]$$

and by a direct calculation we finally get

$$A \leqslant M\alpha y^\alpha \quad \text{for } y > 0,$$

and also

$$\|u(\cdot,y;f,g)\|_{L^p} \leqslant M_\alpha y^\alpha + \left(\frac{3}{2}c\|f\|_{L^p} + \frac{1}{2c}\|g\|_{L^p}\right)y \leqslant M_\alpha y^\alpha$$

for $0 < y \leqslant 1$. The proof of (15) is completed.

The following theorem can be proved analogously.

THEOREM 2
If $f,g \in L^p$, $1 \leqslant p \leqslant \infty$ and the modulus of continuity of g fulfills the condition

$$\omega(g,t) \leqslant Mt^\alpha, \quad 0<\alpha<1,\ t>0,\ M = const. \tag{16}$$

then there exists a positive constant M_α such that for $0 \leqslant y \leqslant 1$ the inequality

$$\|\Delta u(\cdot,y;f,g) - g\|_{L^p} \leqslant M_\alpha y^\alpha \tag{17}$$

holds true.

Theorems 1 and 2 imply the following corollaries:

COROLLARY 3
Under the assumptions of Theorem 1 we have

$$\lim_{y\to 0+}\|u(\cdot,y;f,g) - f\|_{L^p} = 0.$$

COROLLARY 4

Under the assumptions of Theorem 2 we have

$$\lim_{y\to 0^+} \|\Delta u(\cdot, y; f, g) - g\|_{L^p} = 0.$$

3.2. Using the estimations obtained in part 3.1 we will give next two boundary theorems for the solution $u(\cdot, y; f, g)$ in Hölder norms.

THEOREM 3

Let $f, g \in H^{p,\omega}$, and $\omega(t) = t^\alpha$, $0 < \alpha < 1$ where $1 \leqslant p \leqslant \infty$ are fixed. Let also $\mu(t) = t^\beta$ with fixed β, $0 < \beta < \alpha < 1$. Then there is a constant $M_\alpha > 0$ such that for $0 < y \leqslant 1$ the inequality

$$\|u(\cdot, y; f, g) - f(\cdot)\|_{H^{p,\omega}} \leqslant M_\alpha y^{\alpha-\beta} \tag{18}$$

holds true.

Proof. Denote (for short):

$$W(x, y; f, g) = u(x, y; f, g) - f(x) \quad \text{for } x \in \mathbb{R},\ y > 0.$$

Then by (6) we have

$$\|W(\cdot, y; f, g)\|_{H^{p,\omega}} = \|W(\cdot, y; f, g)\|_{L^p} + \|W(\cdot, y; f, g)\|_{p,\mu}$$

and moreover, by Theorem 1,

$$\|W(\cdot, y; f, g)\|_{L^p} \leqslant M_\alpha y^\alpha \quad \text{for } 0 < y \leqslant 1,$$

which further gives

$$\|W(\cdot, y; f, g)\|_{L^p} \leqslant M_\alpha y^{\alpha-\beta} \quad \text{for } 0 < y \leqslant 1.$$

Taking into account the definition (5) we can write

$$\|W(\cdot, y; f, g)\|_{p,\mu} = \sup_{h>0} \frac{\|W(\cdot, y; f, g)\|_{L^p}}{h^\beta} \leqslant T(y) + S(y) \quad \text{for } y \in (0, 1],$$

where

$$T(y) := \sup_{h>y} \frac{\|\Delta_h W(\cdot, y; f, g)\|_{L^p}}{h^\beta}, \quad S(y) := \sup_{0<h\leqslant y} \frac{\|\Delta_h W(\cdot, y; f, g)\|_{L^p}}{h^\beta}.$$

Since for any $F \in L^p$ we have

$$\|\Delta_h F\|_{L^p} \leqslant 2\|F\|_{L^p},$$

hence

$$T(y) \leqslant \frac{1}{h^\beta} \sup_{h>y} \|\Delta_h W(\cdot, y; f, g)\|_{L^p} \leqslant \frac{1}{y^\beta} 2\|W(\cdot, y; f, g)\|_{L^p}$$

and by Theorem 1 we get

$$T(y) \leqslant \frac{2}{y^\beta} M_\alpha y^\alpha = 2M_\alpha y^{\alpha-\beta} \quad \text{for } 0 < y \leqslant 1.$$

Using identity (13) we have

$$\Delta_\mu W(x,y;f,g) = u(x,y;\Delta_h f, \Delta_h g) - \Delta_\mu f(x),$$

which by the formula (3) and Lemma 1 yields

$$\begin{aligned}\|\Delta_h W(\cdot,y;f,g)\|_{L^p} &\leqslant \|u(\cdot,y;\Delta_\mu f,\Delta_h)\|_{L^p} + \|\Delta_h f\|_{L^p} \\ &\leqslant \frac{5}{2}\|\Delta_h f\|_{L^p} + \frac{1}{4c^2}\|\Delta_h g\|_{L^p}.\end{aligned}$$

Further, from the assumption $f, g \in H^{p,\omega}$ with $\omega(t) = t^\alpha$, that is $\omega(t,f,L^p) \leqslant M_\alpha t^\alpha$ and $\omega(t,g,L^p) \leqslant M_\alpha t^\alpha$ it follows that

$$\|\Delta_\mu W(\cdot,y;f,g)\|_{L^p} \leqslant M_\alpha h^\alpha.$$

Consequently

$$S(y) \leqslant M_\alpha \sup_{0<h\leqslant y} \frac{\|\Delta_\mu f\|_{L^p} + \|\Delta_h g\|_{L^p}}{\mu^\beta} \leqslant M_\alpha \sup_{0<h\leqslant y} \frac{\mu^\alpha}{h^\beta} \leqslant M_\alpha y^{\alpha-\beta}$$

for $0 < y \leqslant 1$. Adding the above estimations we obtain (18). This ends the proof of Theorem 3.

Three following theorems can be proved analogously.

THEOREM 4

Let f, g, ω, μ and α, β fulfill the assumptions of Theorem 3. Then there exists a constant M_α such that

$$\|\Delta u(\cdot,y;f,g) - g(\cdot)\|_{H^{p,\mu}} \leqslant M_\alpha y^{\alpha-\beta} \quad \textit{for } 0 < y \leqslant 1. \tag{19}$$

THEOREM 5

Let $f, g \in \tilde{H}^{p,\omega}$ and $\omega(t) = t^\alpha$, with fixed $1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$. Let also $\mu(t) = t^\beta$ with fixed β, $0 < \beta < \alpha < 1$. Then

$$\|W(\cdot,y;f,g)\|_{\tilde{H}^{p,\mu}} = O\left(y^{\alpha-\beta}\right), \quad y \to 0^+.$$

THEOREM 6

Under the assumptions of Theorem 5 we have

$$\|\Delta u(\cdot,y;f,g) - g(\cdot)\|_{\tilde{H}^{p,\mu}} = O\left(y^{\alpha-\beta}\right), \quad y \to 0^+.$$

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