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A remark on a conditional cocycle equation

Dedicated to Professor Z. Moszner
on his 70th birthday

Abstract. A symmetric conditional cocycle need not be a coboundary.

1. Introduction

For a semigroup $(X, +)$ and a divisible abelian group $(G, +)$, we consider the functional equation, the so-called *cocycle equation*,

$$\begin{aligned} &F : X \times X \to G \\ &F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z) \quad (\forall\, x,y,z \in X). \end{aligned} \tag{Coc}$$

A solution of (Coc) is called a *cocycle* on X to G. In [1] and [2], where by the way many more references on (Coc) can be found, T.M.K. Davison and B.R. Ebanks gave sufficient conditions for every symmetric cocycle F to be represented in the form

$$F(x,y) = f(x) + f(y) - f(x+y) \quad (\forall\, x,y \in X) \tag{Cob}$$

with a suitable function $f : X \longrightarrow G$; F then is called a *coboundary* on X to G.

In [4], J. Sikorska asked whether the same kind of conclusion holds if X is a real inner product space with $\dim X \geqslant 3$ and if (Coc) and (Cob) are supposed only for pairwise orthogonal vectors from X, namely whether for $F : X \times X \longrightarrow G$ with

$$F(x,y) = F(y,x) \quad (\forall\, x,y \in X) \tag{S}$$

(symmetry of F), ($\mathrm{Coc}_\perp$) would imply ($\mathrm{Cob}_\perp$), where

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$$x, y, z \in X,$$
$$x \perp y,\ y \perp z,\ z \perp x \Longrightarrow F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z), \qquad (\mathrm{Coc}_\perp)$$
$$\exists f : X \to G \text{ such that}$$
$$x, y \in X,\ x \perp y \Longrightarrow F(x,y) = f(x) + f(y) - f(x+y). \qquad (\mathrm{Cob}_\perp)$$

The question remains unanswered here, but it is shown that the corresponding conclusion collapses for $\dim X = 2$ while it trivially holds for $\dim X \leqslant 1$ (Remark 4 below). This shows that there is an obstacle against the possibility of a Hahn-Banach type extension procedure as applied in [1].

2. Notations and preliminaries

$\mathbb{R}_+$, $\mathbb{R}^*$, $\mathbb{R}^*_+$ denote the sets of nonnegative, nonzero, positive real numbers, respectively. The symbol 0 is used for the zero vector, for the number zero, and for the identity element of the abelian group $(G,+)$; it will always be clear from the context what is meant. $\operatorname{lin} A$ stands for the linear hull (span) of the subset A of the $\mathbb{R}$-vector space X. $\langle \cdot, \cdot \rangle$ and $\perp$ denote an inner product on X and its associated (euclidean) orthogonality, respectively. $\underline{c}$ is the symbol for the constant function with value c.

In Remarks 1, 2, 3, let $\dim X$ be arbitrary.

REMARK 1

If $x, y, z \in X$, $x \perp y$, $y \perp z$, $z \perp x$, then by virtue of the symmetry and additivity of $\perp$, we have $x + y \perp z$ and $x \perp y + z$, so that in $(\mathrm{Coc}_\perp)$ only values of the restriction $F|\perp$ of F occur. The same is true for $(\mathrm{Cob}_\perp)$. This means that the values $F(x,y)$ for $(x,y) \in X \times X \backslash \perp$ are irrelevant for $(\mathrm{Coc}_\perp)$ and $(\mathrm{Cob}_\perp)$.

REMARK 2

In $(\mathrm{Coc}_\perp)$ and (S), $F : X \times X \longrightarrow G$ may be replaced by $F - \underline{F(0,0)}$ as $(G,+)$ is an abelian group. Therefore we may assume without loss of generality

$$F(0,0) = 0. \qquad (1)$$

If the $(\mathrm{Cob}_\perp)$ holds, (1) implies

$$f(0) = 0. \qquad (2)$$

REMARK 3

If F is a solution of $(\mathrm{Coc}_\perp)$, we have

$$F(x,0) = F(0,z) = F(0,0) =_{(1)}= 0 \quad (\forall\, x, z \in X). \qquad (3)$$

In fact, choose $x \in X$ arbitrarily and put $y = z = 0$ in $(\mathrm{Coc}_\perp)$. Then $F(x,0) = F(0,0)$ is immediate, and $F(0,z) = F(0,0)$ is obtained in the same way.

REMARK 4

Let be $\dim X \leqslant 1$, F a solution of $(\mathrm{Coc}_\perp)$ (not necessarily symmetric), and $(x,y) \in \perp$. Then we must have $x = 0$ and/or $y = 0$, and (3) implies $F(x,y) = F(0,0) = 0$. Therefore $F|\perp = \underset{\sim}{0}$, and $(\mathrm{Cob}_\perp)$ holds with $f = \underset{\sim}{0}$. So $(\mathrm{Coc}_\perp) \Longrightarrow (\mathrm{Cob}_\perp)$ for $\dim X \leqslant 1$.

3. The case $\dim \mathbf{X} = 2$

THEOREM

If $(X\langle\cdot,\cdot\rangle)$ is a real inner product space with $\dim X = 2$ and $(G,+)$ a uniquely 2-divisible nontrivial (i.e., $\operatorname{card} G > 1$) abelian group, then $(\mathrm{Coc}_\perp) \wedge (S)$ does not imply $(\mathrm{Cob}_\perp)$.

Proof. 1) Let $\{u,v\} \subset X$ be orthonormal and

$$S := \{(u,v),(v,u)\}, \quad T := S \cup (-S), \tag{4}$$

a an element of G distinct from the identity element 0 of G, and

$$F : X \times X \longrightarrow G, \quad F(x,y) := \begin{cases} a & (x,y) \in S, \\ -a & (x,y) \in (-S), \\ 0 & (x,y) \in X \times X \setminus T. \end{cases} \tag{5}$$

Then F is symmetric and odd. Let be $x,y,z \in X$, $x \perp y$, $y \perp z$, $z \perp x$. Since $\dim X = 2$, at least one of the vectors x,y,z must be 0.

The assumption $x = 0$ together with (5) turn both sides of the equation in $(\mathrm{Coc}_\perp)$ into $F(y,z)$; $y = 0$ or $z = 0$ have analogous consequences, so that F satifies $(\mathrm{Coc}_\perp)$.

2) Assume, contrary to the assertion of the Theorem, that there exist $f : X \longrightarrow G$ with

$$x,y \in X,\ x \perp y \Longrightarrow F(x,y) = f(x) + f(y) - f(x+y). \tag{6}$$

Let be $x,y \in X$, $x \perp y$. Then $(-x) \perp (-y)$, so by (6)

$$F(-x,-y) = f(-x) + f(-y) - f(-x-y). \tag{7}$$

$$\begin{aligned} F(x,y) &=_{(F\ \mathrm{odd})} = \tfrac{1}{2}(F(x,y) - F(-x,-y)) \\ &=_{(6),(7)} = \tfrac{1}{2}(f(x) + f(y) - f(x+y) - f(-x) - f(-y) + f(-x-y)) \\ &= \tfrac{1}{2}(f(x) - f(-x)) + \tfrac{1}{2}(f(y) - f(-y)) - \tfrac{1}{2}(f(x+y) - f(-x-y)). \end{aligned}$$

We define $h : X \longrightarrow G$, $h(z) := \frac{1}{2}(f(z) - f(-z))$ $(\forall z \in X)$ and just obtained

$$x,y \in X,\ x \perp y \Longrightarrow F(x,y) = h(x) + h(y) - h(x+y), \tag{8}$$

$$h \text{ is an odd function.} \tag{9}$$

3) For the two vectors

$$s := u + 2v, \quad t := 2u - v \tag{10}$$

we get $\langle s,t\rangle = \langle u+2v, 2u-v\rangle = 2\|u\|^2 - 2\|v\|^2 = 0$, so

$$s \perp t, \quad \|s\| = \|t\| = \sqrt{5}, \tag{11}$$

$$u = \frac{1}{5}s + \frac{2}{5}t, \quad v = \frac{2}{5}s - \frac{1}{5}t. \tag{12}$$

Now $\frac{1}{5}s, \frac{2}{5}s, \frac{3}{5}s \notin \{u, v, -u, -v\}$, i.e., by (4) and (11)

$$\left(\frac{1}{5}s, \frac{2}{5}t\right), \left(\frac{2}{5}s, -\frac{1}{5}t\right), \left(\frac{3}{5}s, \frac{1}{5}t\right) \in \perp \setminus T,$$

therefore by (5) $F\left(\frac{1}{5}s, \frac{2}{5}t\right) = F\left(\frac{2}{5}s, -\frac{1}{5}t\right) = F\left(\frac{3}{5}s, \frac{1}{5}t\right) = 0$, and by (8)

$$h\left(\frac{1}{5}s\right) + h\left(\frac{2}{5}t\right) = h\left(\frac{1}{5}s + \frac{2}{5}t\right) =_{(12)}= h(u), \tag{13}$$

$$h\left(\frac{2}{5}s\right) + h\left(-\frac{1}{5}t\right) = h\left(\frac{2}{5}s - \frac{1}{5}t\right) =_{(12)}= h(v), \tag{14}$$

$$h\left(\frac{3}{5}s\right) + h\left(\frac{1}{5}t\right) = h\left(\frac{3}{5}s + \frac{1}{5}t\right) =_{(12)}= h(u+v). \tag{15}$$

4) Let be $x \in \operatorname{lin}\{s\}$ and $\lambda \in \mathbb{R}_+$ arbitrary. For

$$x = \alpha s, \quad y := \sqrt{\lambda}\,\alpha t, \tag{16}$$

we have by (11) $x \perp y$ and $\langle x+y, \lambda x - y\rangle = \lambda\|x\|^2 - \|y\|^2 =_{(16)}= 0$, so

$$\begin{array}{l}\text{for all } x \in \operatorname{lin}\{s\},\ \lambda \in \mathbb{R}_+ \text{ there exists } y \in \operatorname{lin}\{t\}\\ \text{with } x \perp y \text{ and } x+y \perp \lambda x - y.\end{array} \tag{17}$$

Since $[\operatorname{lin}\{s\} \cup \operatorname{lin}\{t\}] \cap \{u, v, -u, -v\} = \emptyset$, we have moreover

$$(x,y), (\lambda x, -y) \in \perp \setminus T, \tag{18}$$

$$(x+y, \lambda x - y) \in \perp \setminus T. \tag{19}$$

To (19): If $x = 0$, then by (16) $\alpha = 0$, $y = 0$, $x + y = 0$, and (4) guarantees that $(x+y, \lambda x - y) \notin T$. Let in the following be $x \neq 0$.

Case 1: $\lambda \in \mathbb{R}_+ \setminus \{1\}$. Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, $\|\lambda x - y\|^2 = \lambda^2\|x\|^2 + \|y\|^2 \neq \|x+y\|^2$, and by (4) $(x+y, \lambda x - y) \notin T$.

Case 2: $\lambda = 1$. (16) and $x \neq 0$ imply $\alpha \in \mathbb{R}^*$, $y = \alpha t$, so $x+y = \alpha(s+t) =_{(10)}= \alpha(3u+v) \notin \{u, v, -u, -v\}$, and again by (4) $(x+y, \lambda x - y) \notin T$.

So (19) is established. Therefore, (17) can be sharpened and unified with (18), (19) to for all $x \in \operatorname{lin}\{s\}$, $\lambda \in \mathbb{R}_+$ there exists $y \in \operatorname{lin}\{t\} \subset X$ such that $(x, y), (\lambda x, -y), (x + y, \lambda x - y) \in \perp \setminus T$. (20)

Starting from $x \in \operatorname{lin}\{t\}$ instead of $x \in \operatorname{lin}\{s\}$, the same procedure is possible, and we get together with (20)

for all $x \in \operatorname{lin}\{s\} \cup \operatorname{lin}\{t\}$, $\lambda \in \mathbb{R}_+$ there exists $y \in X$ such that $(x, y), (\lambda x, -y), (x + y, \lambda x - y) \in \perp \setminus T$. (21)

5) Next we prove

$$x \in \operatorname{lin}\{s\} \cup \operatorname{lin}\{t\},\ \lambda \in \mathbb{R}_+ \Longrightarrow h(x + \lambda x) = h(x) + h(\lambda x). \tag{22}$$

In fact, for the given x choose $y \in X$ following (21). By (5) $F(x, y) = F(\lambda x, -y) = F(x + y, \lambda x - y) = 0$, so by (8)

$$h(x + y) = h(x) + h(y), \tag{23}$$

$$h(\lambda x - y) = h(\lambda x) + h(-y), \tag{24}$$

$$h(x + y + \lambda x - y) = h(x + y) + h(\lambda x - y). \tag{25}$$

Hence

$$\begin{aligned} h(x + \lambda y) &= h(x + y + \lambda x - y) =_{(25)}= h(x + y) + h(\lambda x - y) \\ &= _{(23),(24)} = h(x) + h(y) + h(\lambda x) + h(-y) =_{(9)}= h(x) + h(\lambda x), \end{aligned}$$

i.e., (22) holds.

6) Our final claim is

$$x_1, x_2 \in \operatorname{lin}\{s\} \text{ or } x_1, x_2 \in \operatorname{lin}\{t\} \Longrightarrow h(x_1 + x_2) = h(x_1) + h(x_2). \tag{26}$$

In fact, $x_1 = 0$ and (5) imply $F(x_1, x_2) = 0$, so by (8) $h(x_1 + x_2) = h(x_1) + h(x_2)$. Let in the following be $x_1 \neq 0$. Then there exists $\mu \in \mathbb{R}$ with

$$x_2 = \mu x_1. \tag{27}$$

Case 1: $\mu \geqslant 0$. Then $h(x_1 + x_2) = h(x_1 + \mu x_1) =_{(22)}= h(x_1) + h(x_2)$.

Case 2: $-1 < \mu < 0$. We put $z := (1 + \mu)x_1$, $\lambda := -\frac{\mu}{\mu+1}$. Then $z \in \operatorname{lin}\{s\} \cup \operatorname{lin}\{t\}$, $\lambda \in \mathbb{R}_+^*$, $\lambda z = -\mu x_1)$, and $h(z + \lambda z) =_{(22)}= h(z) + h(\lambda z)$, i.e., $h((1 + \mu)x_1 - \mu x_1) = h((1 + \mu)x_1) + h(-\mu x_1) =_{(9)}= h((1 + \mu)x_1) - h(\mu x_1)$, i.e., $h(x_1) = h(x_1 + \mu x_1) - h(\mu x_1) =_{(27)}= h(x_1 + x_2) - h(x_2)$, so $h(x_1) + h(x_2) = h(x_1 + x_2)$.

Case 3: $\mu \leqslant -1$. We put $w := -x_1$ and $\lambda := -1 - \mu$. Then $w \in \operatorname{lin}\{s\} \cup \operatorname{lin}\{t\}$. $\lambda \in \mathbb{R}_+$, and $\lambda w = (-1-\mu)(-x_1) = x_1 + \mu x_1 =_{(27)}= x_1 + x_2$. Furthermore $h(w + \lambda w) =_{(22)}= h(w) + h(\lambda w)$, i.e., $h(-x_1 + x_1 + x_2) = h(-x_1) + h(x_1 + x_2)$, so $h(x_2) =_{(9)}= -h(x_1) + h(x_1 + x_2)$, and the assertion holds again, i.e., (26) is ensured.

7) The assumption in part 2) leads to the following contradiction:

$$\begin{aligned} a &=_{(5)}= F(u,v) =_{(8)}= h(u) + h(v) - h(u+v) \\ &=_{(13),(14),(15)}= h\left(\tfrac{1}{5}s\right) + h\left(\tfrac{2}{5}t\right) = h\left(\tfrac{2}{5}s\right) + h\left(-\tfrac{1}{5}t\right) - h\left(\tfrac{3}{5}s\right) - h\left(\tfrac{1}{5}t\right) \\ &= h\left(\tfrac{1}{5}s\right) + h\left(\tfrac{2}{5}s\right) - h\left(\tfrac{3}{5}s\right) + h\left(\tfrac{2}{5}t\right) + h\left(-\tfrac{1}{5}t\right) - h\left(\tfrac{1}{5}t\right) =_{(26)}= 0 + 0 = 0, \end{aligned}$$

i.e., $a = 0$, contrary to the definition of the element a of G. The proof of the Theorem is complete.

REMARK 5

Parts 2), 3), and 4) of the proof of the Theorem create a situation in which (cf. parts 5) and 6)) a method of determination of the odd orthogonally additive mappings on the $(04')$–orthogonality space $(X, \perp)$ can be applied with the necessary care ([3], p. 35/36, Definition 1; p. 36, Example B; p. 38/39, proof of Theorem 5).

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