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Entire solutions of a functional equation

Dedicated to Professor Zenon Moszner
on the occasion of his 70th birthday

Abstract. The aim of the present note is to show that all entire solutions $f : \mathbb{C} \longrightarrow \mathbb{C}$, of order less than 4, of the equation

$$|f(s+it)f(s-it)| = |f(s)^2 - f(it)^2|, \quad s,t \in \mathbb{R}$$

are given by $f(z) = az$ and $f(z) = a \sin bz$, where a, b are arbitrary complex constants. This is a partial solution of the problem posed by Themistocles M. Rassias [3] (see also [4]).

1. It is known that all entire solutions of the functional equation

$$f(z+w)f(z-w) = f(z)^2 - f(w)^2, \quad z,w \in \mathbb{C} \tag{1}$$

are of the form

$$\text{(i)} \quad f(z) = az \qquad \text{or} \qquad \text{(ii)} \quad f(z) = a \sin bz,$$

where $a, b \in \mathbb{C}$ (cf. e.g. [1]). If we replace in (1) the complex variables z and w by s and it ($s, t \in \mathbb{R}$) and we take the absolute values in the resulting equation, then we obtain the functional equation

$$|f(s+it)f(s-it)| = |f(s)^2 - f(it)^2|. \tag{2}$$

So functions (i) and (ii) are entire solutions of (2). We conjecture that there are no other entire solutions of this equation. We can prove only the following:

THEOREM 1

The only entire solutions of order ρ, $\rho < 4$, of equation (2) *are given by* (i) *and* (ii).

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2. Of course, if f satisfies equation (2) then $f(0) = 0$. It is clear that the function

$$F(z) := \overline{f(\overline{z})} \tag{3}$$

is entire if and only if f is entire. We denote by Ω the set of all entire functions f such that their power series expansions $\sum_{n=0}^{\infty} a_n z^n$ have only real coefficients. To prove the theorem we need the following lemmas.

LEMMA 1 (cf. [6])

Assume that an entire function f has only a finite number of zeros located at the points $z_1, z_2, \ldots, z_n$. Then all entire solutions g which satisfy

$$|g(s)| = |f(s)| \quad \textit{for all } s \in \mathbb{R} \tag{4}$$

are given by

$$g(z) = f(z)\frac{(z-w_1)\cdots(z-w_n)}{(z-z_1)\cdots(z-z_n)}e^{ip(z)}, \tag{5}$$

where $p \in \Omega$ and $w_j \in \{z_j, \overline{z_j}\}$, $j \in \{1, \ldots, n\}$.

Let $f \neq 0$ be an entire function having a zero of multiplicity $m \geqslant 0$ at 0 and let

$$z_1, z_2, \ldots$$

be the sequence of all zeros of f different from 0, where k-fold zero is supposed to be repeated k times in the sequence. We suppose that the zeros are ordered according to increasing absolute values. The Weierstrass Factorization Theorem states that there exists an entire function having zeros at z_n, $n \in \mathbb{N}$ and a zero of multiplicity m at 0 and having no other zeros. If $\{\mu_n\}$ is a sequence of non-negative integers such that the series $\sum_{n=1}^{\infty} |\frac{z}{z_n}|^{\mu_n+1}$ converges uniformly on compact sets, then the product

$$I(z) = z^m \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, \mu_n\right), \tag{6}$$

where $E(z,0) = 1 - z$ and $E(z,\mu) = (1-z)\exp\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^\mu}{\mu}\right)$, $\mu \in \mathbb{N}$, satisfies the above conditions. Moreover the product converges absolutely. It is clear that the quotient $\frac{f}{I}$ is an entire function without zeros (see [5], p. 298).

LEMMA 2 (cf. [6])

Assume that an entire function $f \neq 0$ has an m-fold zero at 0 and another zeros are the elements of the sequence $\{z_n\}$. Then all entire functions g satisfying equation (4) are of the form

$$g(z) = f(z)\frac{J(z)}{I(z)}e^{ip(z)},$$

where $p \in \Omega$, I is given by formula (6) *whereas J has the form*

$$J(z) = z^m \prod_{n=1}^{\infty} E\left(\frac{z}{w_n}, \mu_n\right), \tag{7}$$

where $w_n \in \{z_n, \overline{z_n}\}$, $n \in \mathbb{N}$.

LEMMA 3 (cf. [6])

Under hypotheses of Lemma 2 *all entire functions g satysfying the equation*

$$|f(it)| = |g(it)| \quad \textit{for all } t \in \mathbb{R}$$

are of the form

$$g(z) = f(z)\frac{J(z)}{I(z)}e^{q(z)+ir(z)},$$

where $q, r \in \Omega$, q, r are odd and even, respectively, whereas I and J are given by formulas (6) *and* (7), *respectively, where $w_n \in \{z_n, -\overline{z_n}\}$, $n \in \mathbb{N}$.*

LEMMA 4

If f is an entire solution of (2) *then f^2 is even and $f(0) = 0$.*

Proof. The second fact is obvious. Put

$$\varphi = f^2.$$

By (2)

$$\begin{aligned}|\varphi(s) - \varphi(it)| &= |f(s+it)f(s-it)| = |f(s-(-it))f(s+(-it))| \\ &= |\varphi(s) - \varphi(-it)|\end{aligned}$$

for all $s, t \in \mathbb{R}$. Squaring the both sides of the equality $|\varphi(s) - \varphi(it)| = |\varphi(s) - \varphi(-it)|$ yields

$$(\varphi(s) - \varphi(it))(\Phi(s) - \Phi(-it)) = (\varphi(s) - \varphi(-it))(\Phi(s) - \Phi(it)),$$

where $\Phi(z) := \overline{\varphi(\overline{z})}$. According to the Identity Theorem we obtain

$$(\varphi(z) - \varphi(w))(\Phi(z) - \Phi(-w)) = (\varphi(z) - \varphi(-w))(\Phi(z) - \Phi(w)) \tag{8}$$

for all $z, w \in \mathbb{C}$. Since $\varphi(0) = \Phi(0) = 0$, relation (8) implies $\varphi(w)\Phi(-w) = \varphi(-w)\Phi(w)$, whence by (8)

$$\varphi(z)[\Phi(w) - \Phi(-w)] = \Phi(z)[\varphi(w) - \varphi(-w)]. \tag{9}$$

If $\varphi(s) = \varphi(-s)$ for $s \in \mathbb{R}$ then $\varphi(z) = \varphi(-z)$ for all $z \in \mathbb{C}$ and Lemma 4 follows. Thus suppose that for some $s_0 \in \mathbb{R}$, $\varphi(s_0) \neq \varphi(-s_0)$. It is enough to show that this case does not occur. Setting $w = s_0$ in (9) we infer

$$\Phi(z) = c\varphi(z) \quad \text{for all } z \in \mathbb{C}, \tag{10}$$

where $c := \frac{\Phi(s_0)-\Phi(-s_0)}{\varphi(s_0)-\varphi(-s_0)}$. We observe that the complex numbers $\varphi(s_0) - \varphi(-s_0)$

and $\Phi(s_0) - \Phi(-s_0)$ are conjugate so $|c| = 1$. Since f is a solution of equation (2), we have

$$\begin{aligned} |\varphi(s) - \varphi(it)|^2 &= |\varphi(s+it)\varphi(s-it)| = |\varphi(s+it)\overline{\varphi(s-it)}| \\ &= |\varphi(s+it)\Phi(s+it)|. \end{aligned} \tag{11}$$

By (10) we have also

$$|\varphi(z)\Phi(z)| = |\varphi(z)|^2.$$

Thus (11) goes into

$$\begin{aligned} (\varphi(s) - \varphi(it))\,(\Phi(s) - \Phi(-it)) &= |\varphi(s+it)|^2 \\ &= \varphi(s+it)\Phi(s-it) \end{aligned}$$

for $s, t \in \mathbb{R}$. Again by the Identity Theorem we obtain

$$(\varphi(z) - \varphi(w))\,(\Phi(z) - \Phi(-w)) = \varphi(z+w)\Phi(z-w)$$

for all $z, w \in \mathbb{C}$, whence by (10) we get

$$(\varphi(z) - \varphi(w))\,(\varphi(z) - \varphi(-w)) = \varphi(z+w)\varphi(z-w). \tag{12}$$

Write

$$\pi(z) = \frac{1}{2}\left[\varphi(z) - \varphi(-z)\right], \quad z \in \mathbb{C}.$$

Now we observe that π satisfies equation (1). In fact, by (12)

$$\begin{aligned} \pi(z+w)\pi(z-w) &= \frac{1}{4}[\varphi(z+w) - \varphi(-z-w)][\varphi(z-w) - \varphi(-z+w)] \\ &= \frac{1}{4}[\varphi(z+w)\varphi(z-w) - \varphi(z+w)\varphi(-z+w) \\ &\qquad - \varphi(-z-w)\varphi(z-w) + \varphi(-z-w)\varphi(-z+w)] \\ &= \frac{1}{4}[(\varphi(z) - \varphi(w))\,(\varphi(z) - \varphi(-w)) \\ &\qquad - (\varphi(w) - \varphi(z))\,(\varphi(w) - \varphi(-z)) \\ &\qquad - (\varphi(-w) - \varphi(-z))\,(\varphi(-w) - \varphi(z)) \\ &\qquad + (\varphi(-z) - \varphi(-w))\,(\varphi(-z) - \varphi(w))] \\ &= \frac{1}{4}\left[\varphi(z)^2 - \varphi(w)^2 - \varphi(-w)^2 + \varphi(-z)^2 \right. \\ &\qquad \left. + 2\varphi(w)\varphi(-w) - 2\varphi(z)\varphi(-z)\right] \\ &= \pi(z)^2 - \pi(w)^2. \end{aligned}$$

Thus π satisfies (1), whence (i) $\pi(z) = az$ or (ii) $\pi(z) = a\sin bz$, where a, b are some complex constants such that $a \neq 0$ and $b \neq 0$. Define an even entire function by

$$\pi_1(z) = \frac{1}{2}\left[\varphi(z) + \varphi(-z)\right].$$

When (i) holds, $\varphi(z) = \pi_1(z) + az$. Differentiating both sides of the equality $f(z)^2 = \pi_1(z) + az$ and setting $z = 0$ in the resulting equality we obtain $\pi_1'(0) = -a$, which is impossible since π_1 is even and $a \neq 0$. Similarly, in the case (ii) differentiating the equality $f(z)^2 = \pi_1(z) + a\sin bz$ yields the condition $\pi_1'(0) = -ab$, which again leads us to a contradiction.

Let $a_1 := f'(0)$. It is clear that $F'(0) = \overline{a_1}$, where F is given by (3). The methods used in the next lemma are due to Hiroshi Haruki (cf. e.g. [2]).

Lemma 5

If f is an entire solution of equation (2), *F is given by* (3) *and $a_1 = f'(0)$, then f is odd and the equality*

$$|a_1|^2 f(2z)F(2z) = 4f(z)F(z)f'(z)F'(z) \tag{13}$$

holds in $\mathbb{C}$. Moreover, if $f \neq 0$, then $a_1 \neq 0$.

Proof. We may assume that $f \neq 0$. Since f is a solution of (2), we have

$$f(s+it)\overline{f(s+it)}f(s-it)\overline{f(s-it)} = \left(f(s)^2 - f(it)^2\right)\left(\overline{f(s)^2} - \overline{f(it)^2}\right)$$

for all $s, t \in \mathbb{R}$. Hence by (3), Lemma 4 and the eveness of F^2 (f^2 is even if and only if so is F^2) we obtain

$$f(s+it)F(s-it)f(s-it)F(s+it) = \left(f(s)^2 - f(it)^2\right)\left(F(s)^2 - F(it)^2\right),$$

whence

$$f(z+w)F(z-w)f(z-w)F(z+w) = \left(f(z)^2 - f(w)^2\right)\left(F(z)^2 - F(w)^2\right)$$

for $z, w \in \mathbb{C}$. Dividing the last equality by $(z+w)^2(z-w)^2$ and passing to the limit as $z \longrightarrow w$ we obtain

$$f'(0)F'(0)f(2w)F(2w) = 4f(w)F(w)f'(w)F'(w). \tag{14}$$

Since the ring of all entire functions has no divisors of zero and $f \neq 0$, by (14), it follows that $a_1 \neq 0$.

To end the proof it suffices to establish the oddness of the function f. We can find a disc D centered at 0 such that the function $\psi(z) := \frac{f(-z)}{f(z)}$ is holomorphic in it (both functions $f(z)$ and $f(-z)$ have an 1-fold zero at 0, so $\psi(0) = -1$). The equality $\psi^2 = 1$ in D results from Lemma 4. Consequently $\psi = 1$ or $\psi = -1$ in D, i.e., $f(z) = f(-z)$ or $f(-z) = -f(z)$ in D. In virtue of the Identity Theorem f is even or odd. Since $a_1 = f'(0) \neq 0$, f must be odd.

LEMMA 6

If f is an entire solution of equation (2) then there exists an entire function p belonging to Ω such that

$$2f'(z)f(z) = a_1 f(2z)e^{ip(z)}, \tag{15}$$

where $a_1 = f'(0)$.

Proof. Setting in formula (13) $z = s$, $s \in \mathbb{R}$, we get

$$|a_1|^2 f(2s)\overline{f(2s)} = 4f(s)\overline{f(s)}f'(s)\overline{f'(s)},$$

i.e.,

$$|a_1 f(2s)| = 2|f(s)f'(s)|. \tag{16}$$

At first, suppose that f has a finite number of zeros. Assuming that z_0 is a zero of f we obtain that so is $2z_0$ or $2\overline{z_0}$, according to Lemma 5. Again by this lemma, $4z_0$ or $4\overline{z_0}$ is a zero of f and so on. Thus in this case f must have exactly one zero $z_0 = 0$, of multiplicity 1. In virtue of Lemma 1 we can find $p \in \Omega$ such that

$$2f(z)f'(z) = a_1 f(2z)e^{ip(z)}.$$

Now assume that $f \neq 0$ has an infinite number of zeros. Let all zeros of the function $f(2z)$ different from zero be elements of the sequence $\{z_n\}$. Applying Lemma 2 we can find $p \in \Omega$ such that

$$2f(z)f'(z) = a_1 f(2z)\frac{J(z)}{I(z)}e^{ip(z)}, \tag{17}$$

where I and J are given by (6) and (7), respectively, $m = 1$, $w_n \in \{z_n, \overline{z_n}\}$, $n \in \mathbb{N}$. Taking $z = it$ ($t \in \mathbb{R}$) in formula (13) we obtain

$$|a_1 f(2it)| = 2|f(it)f'(it)|.$$

Lemma 3 says that there exist $u_n \in \{z_n, -\overline{z_n}\}$ and $q, r \in \Omega$ such that

$$2f(z)f'(z) = a_1 f(2z)\frac{J_1(z)}{I(z)}e^{q(z)+ir(z)},$$

where I is given by (6) ($m = 1$) and

$$J_1(z) = z\prod_{n=1}^{\infty} E\left(\frac{z}{u_n}, \mu_n\right).$$

The products J and J_1 would be different only if there exists a zero of one of them that is not a zero of the other. But the both products have the same zeros, namely the zeros of the function $2ff'$, so $J = J_1$. Therefore, without loss of generality, we may assume that $w_n = u_n$ for every n. We claim that

$w_n = z_n$ for all $n \in \mathbb{N}$. Suppose the contrary: $w_n \neq z_n$ for some n. Then $w_n = \overline{z_n}$ and $z_n \notin \mathbb{R}$. On the other hand, since $w_n = u_n = -\overline{z_n}$, we would obtain $z_n = 0$, which is impossible. Consequently $I(z) = J(z)$, whence (15) follows. Finally, let us observe that $f = 0$ satisfies equation (15) with an arbitrary entire function p.

LEMMA 7
All odd entire solutions of the equation

$$a_1 f(2z) = 2f(z)f'(z) \tag{18}$$

such that $f(0) = 0$ *and* $f'(0) = a_1$ *are of the form*

$$\text{(i)} \quad f(z) = a_1 z \qquad \textit{or} \qquad \text{(ii)} \quad f(z) = a \sin bz,$$

where a *and* b *are complex constants such that* $ab = a_1$.

Proof. Let f be an odd entire solution of (18). We may assume that $f \neq 0$. Then $f'(0) = a_1 \neq 0$. Write

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}. \tag{19}$$

We have by (18)

$$\begin{aligned} a_1 \sum_{n=0}^{\infty} a_{2n+1} (2z)^{2n+1} &= \sum_{n=0}^{\infty} a_1 a_{2n+1} 2^{2n+1} z^{2n+1} \\ &= 2 \left(\sum_{n=0}^{\infty} (2n+1) a_{2n+1} z^{2n} \right) \left(\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \right). \end{aligned}$$

Equating the coefficients of z^{2n+1} $(n = 0, 1, 2, \ldots)$ we infer that

$$2^{2n+1} a_1 a_{2n+1} = 2 \sum_{k=0}^{n} (2k+1) a_{2k+1} a_{2(n-k)+1}. \tag{20}$$

It is easy to see that the coefficient a_3 may be arbitrarily chosen. Let $n = 2$. By (20)

$$2^5 a_1 a_5 = 2(a_1 a_5 + 3a_3^2 + 5a_5 a_1).$$

Hence

$$(2^5 - 12)\, a_1 a_5 = 6a_3^2.$$

Thus

$$a_5 = \frac{6^2}{5!} \frac{a_3^2}{a_1}. \tag{21}$$

By induction we shall show that

$$a_{2k+1} = \frac{6^k}{(2k+1)!}\frac{a_3^k}{a_1^{k-1}} \quad \text{for all } k \in \mathbb{N},\ k \geqslant 2. \tag{22}$$

For $k = 2$, (22) reduces to (21). Assume that (22) is true for $k \in \{2, \ldots, n\}$, where $n \geqslant 2$ is a positive integer. We have by (20)

$$\begin{aligned} &2^{2n+3} a_1 a_{2n+3} \\ &= 2\left[a_1 a_{2n+3} + \sum_{k=1}^{n}(2k+1)a_{2k+1}a_{2n+3-2k} + (2n+3)a_{2n+3}a_1\right] \\ &= 2\left[(2n+4)a_1 a_{2n+3} + \sum_{k=1}^{n}(2k+1)\frac{6^k}{(2k+1)!}\frac{a_3^k}{a_1^{k-1}}\frac{6^{n+1-k}}{(2n+3-2k)!}\frac{a_3^{n+1-k}}{a_1^{n-k}}\right]. \end{aligned}$$

Hence

$$\begin{aligned} &\left[2^{2n+3} - 4(n+2)\right] a_1 a_{2n+3} \\ &\quad = 2\frac{a_3^{n+1}}{a_1^{n-1}}6^{n+1}\sum_{k=1}^{n}\frac{1}{(2k)!}\frac{1}{(2n+3-2k)!} \\ &\quad = 2\frac{a_3^{n+1}6^{n+1}}{a_1^{n-1}}\frac{1}{(2n+3)!}\sum_{k=1}^{n}\binom{2n+3}{2k} \\ &\quad = \frac{a_3^{n+1}6^{n+1}}{a_1^{n-1}}\frac{1}{(2n+3)!} \\ &\qquad \times\left[\sum_{j=0}^{2n+3}\binom{2n+3}{j} + \sum_{j=0}^{2n+3}\binom{2n+3}{j}(-1)^j - 2 - 2(2n+3)\right] \\ &\quad = \frac{a_3^{n+1}6^{n+1}}{a_1^{n-1}}\frac{1}{(2n+3)!}\left[2^{2n+3} - 4n - 8\right]. \end{aligned}$$

Consequently

$$a_{2n+3} = \frac{6^{n+1}a_3^{n+1}}{(2n+3)!a_1^n},$$

i.e., we have obtained (22) for $k = n+1$. The induction completes the proof of (22).

We have proved that every odd entire solution different from zero of (18) is given by formula (19), where a_{2n+1} are expressed by (22) for $k = n \geqslant 2$. The coefficient a_3 may be chosen arbitrarily. If $a_3 = 0$ then $f(z) = a_1 z$, i.e., f is of the form (i). Of course, this function is a solution of equation (18). Consider the case $a_3 \neq 0$ and take complex numbers a and b such that

$$a_1 = ab \quad \text{and} \quad a_3 = -\frac{1}{6}ab^3.$$

Then $a_{2n+1} = \frac{1}{(2n+1)!}a(-1)^n b^{2n+1}$ for all $n \in \mathbb{N}$. Consequently

$$f(z) = abz + \sum_{n=1}^{\infty} a_{2n+1}z^{2n+1} = abz + a\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}(bz)^{2n+1}$$
$$= a \sin bz,$$

i.e., the function f is of the form (ii). Since $ab = a_1$, f clearly satisfies equation (18).

LEMMA 8

Assume that an odd entire function $f \neq 0$ is a solution of equation (15), *where $f(0) = 0$, $a_1 = f'(0)$ and p is an entire function. Then p is even, $e^{ip(0)} = 1$ and $p''(0) = 0$.*

Proof. Since f is odd, f' is even. Thus the function $e^{ip(z)}$ must be even. Differentiating the equality $e^{ip(z)} = e^{ip(-z)}$ we obtain $p'(z) = -p'(-z)$, whence the eveness of p follows. Thus $p^{(k)}(0) = 0$ for all odd positive integers. Differentiating (15) and setting $z = 0$ we see that $e^{ip(0)} = 1$. Write

$$q(z) = e^{ip(z)}, \quad h = 2ff' = (f^2)' \quad \text{and} \quad g(z) = a_1 f(2z).$$

Since $p'(0) = 0$ to prove that $p''(0) = 0$ it is enough to show that $q''(0) = 0$. Differentiating the equality $h = qg$ three times we get

$$h'''(z) = g'''(z)q(z) + 3g''(z)q'(z) + 3g'(z)q''(z) + g(z)q'''(z).$$

Setting $z = 0$ and applying conditions $g(0) = g''(0) = 0$, $g'''(0) = h'''(0) = 8a_1 f'''(0)$, $g'(0) = 2a_1^2$ we get

$$8a_1 f'''(0) = 8a_1 f'''(0) + 6a_1^2 q''(0),$$

whence $q''(0) = 0$ follows.

3. Proof of the Theorem

Let $f \neq 0$ be an entire solution of equation (2). By Lemmas 4 and 5 we know that $f(0) = 0$, $f'(0) = a_1 \neq 0$ and f is odd. Assume that the order ρ of f belongs to the interval $[0, 4)$. Write

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

It is easily seen that $M(r, f(2\cdot)) = M(2r, f)$ and $M(r, f^2) = (M(r, f))^2$. Since

$$\rho = \limsup_{r\to\infty} \frac{\log(\log M(r,f))}{\log r}$$

(cf. [5], p. 320), we see that the orders of $f(2\cdot)$ and f^2 are also equal to ρ. It is known that orders of an entire function and its derivative are the same (cf. [5], p. 323). Consequently the orders of functions $a_1 f(2\cdot)$ and $2ff' = (f^2)'$ are equal to ρ. By Lemma 6 there exists an entire function p such that f is a solution of equation (15), i.e.,

$$\frac{2f'(z)f(z)}{a_1 f(2z)} = e^{ip(z)}.$$

Hence the order of the function $e^{ip(z)}$ does not exceed ρ (see [5, Theorem (10.19), Ch. 7]). Hadamard's Theorem states that p is a polynomial of degree $\leqslant \rho$ (see [5, Theorem (10.1), Ch. 7]). Consequently the degree of p is less than 4. In virtue of Lemma 8, p is even and $p''(0) = 0$, so the polynomial p must be a constant. On the other hand $e^{ip(0)} = 1$. Thus f is an odd entire solution of equation (18). Now our theorem results from Lemma 7.

References

[1] H. Haruki, *Studies on certain functional equations from the standpoint of analytic function theory*, Sci. Rep. Osaka Univ. **14** (1965), 1-40.

[2] H. Haruki, *On the equivalence of Hille's and Robinson's functional equations*, Ann. Polon. Math. **28** (1973), 261-264.

[3] Th.M. Rassias, *Problem 30*, Aequationes Math. **47** (1994), 320.

[4] Th.M. Rassias, *Problem 15*, Aequationes Math. **56** (1998), 309.

[5] S. Saks, A. Zygmund, *Analytic Functions*, Monografie Matematyczne, Vol. 28, Warszawa – Wrocław, 1952.

[6] A. Smajdor, W. Smajdor, *Entire solutions of the Hille-type functional equation*, Functional Equations and Inequalities (ed. by Themistocles M. Rassias), Kluwer Academic Publishers, to appear.

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